A LIE PROPERTY IN GROUP RINGS

ANTONIO GIAMBRUNO AND SUDARSHAN K. SEHGAL

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ABSTRACT. Let $A$ be an additive subgroup of a group ring $R$ over a field $K$. Denote by $[A, R]$ the additive subgroup generated by the Lie products $[a, r] = ar - ra$, $a \in A$, $r \in R$. Inductively, let $[A, R_n] = [[A, R_{n-1}], R]$. We prove that $[A, R_n] = 0$ for some $n \Rightarrow [A, R]R$ is a nilpotent ideal.

INTRODUCTION

Let $A$ be an additive subgroup of a ring $R$. We define $[A, R]$ to be the additive subgroup of $R$ generated by all Lie products $[a, r] = ar - ra$, $a \in A$, $r \in R$. Inductively, we set

$$[A, R, R, \ldots, R]_{n+1} = [A, R_{n+1}] = [[A, R_n], R].$$

The main result of this note is

Theorem. Let $R = KG$ be the group algebra of a group $G$ over a field $K$ and $A$ an additive subgroup of $R$. Then $[A, R_n] = 0$ for some $n \Rightarrow [A, R]R$ is an (associative) nilpotent ideal of $R$.

Taking $A = KG$, we conclude that if $KG$ is Lie nilpotent, then

$$[KG, KG]KG = \Delta(G')KG$$

is nilpotent where $\Delta(G')$ is the augmentation ideal of the derived group $G'$. This implies that $G$ is abelian if $\chi(K)$, the characteristic of $K$ is 0 and $G'$ is a finite $p$-group if $\chi(K) = p > 0$. This is a theorem of Passi, Passman and Sehgal [3, 5, p. 151]. To prove our theorem we need two results due to Gupta, Levin and Passman.

Lemma 1 [1]. The $n$th term $G_n$ of the lower central series of $G$ is contained in

$$1 + [R, R, \ldots, R]$$

where $R = KG$. 

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Lemma 2 [4, p. 202]. If $KG$ satisfies a nondegenerate multilinear generalized polynomial identity then \( \phi \), the FC subgroup of $G$, is of finite index and has finite derived group $\phi'$. 

In the next section we collect some preliminary results and prove the theorem in §3.

2. SOME PRELIMINARY RESULTS

In this section $R$ is an arbitrary ring with 1 and $A$ is an additive subgroup of $R$. We denote by $C_R(A) = \{ x \in R | xa = ax \text{ for all } a \in A \}$, the centralizer of $A$ in $R$. We write \([x, y, z] = [[x, y], z]\) for $x, y, z \in R$.

Lemma 3. (i) For $k \geq 1$, $[A, R_k]R = R[A, R_k]$ so that $[A, R_k]R$ is the two-sided ideal generated by $[A, R_k]$.

(ii) For $a, b, x, y \in R$ we have \([a, by, x] = [a, b, x]y + [a, b][y, x] + b[a, y, x] + [b, x][a, y].\)


(iv) For $b \in C_R(A), [A, R][b, R] \leq [A, R, R]R$.

Proof. (i) Let $a \in [A, R_{k-1}], x, y \in R$. Then

\[ [a, x]y = (ax - xa)y = a(xy) - (xy)a - x(ay - ya) = [a, xy] - x[a, y] \]

and

\[ x[a, y] = [a, xy] - [a, x]y. \]

(ii) is clear.

To obtain (iii) in the expression for $[b, x][a, y]$ in (ii), we take $a \in A, b \in [R, R], x, y \in R$ to conclude


\[ \leq [A, R, R]R \quad \text{(by the Jacobi identity and (i))}. \]

(iv) We first observe that for $a \in A, r \in R, [a, r]b = [a, rb]$ and thus $[A, R]b \leq [A, R]$. Now, taking $x \in [A, R], r \in R$ we have

\[ x[b, r] = [xb, r] - [x, r]b \]

which implies


The next lemma is an extension of a result of Jennings [2] to our situation. The proof parallels that of Jennings.

Lemma 4. Suppose that $[A, R_n] = 0$ for some $n \geq 1$. Then $[A, R]R$ is a nil ideal and $[A, R, R]R$ is a nilpotent ideal.

Proof. We first claim

\[ (*) \quad \text{If } [A, R_n] = 0 \text{ for some } n > 2 \text{ then } ([A, R_{n-1}]R)^2 = 0. \]
Let $u, v \in [A, R_{n-3}]$ and $x, y \in R$. Then by the Jacobi identity

$$-[u, x], [v, y] = [v, [y, [u, x]]] + [y, [[u, x], v]]$$

$$= [[u, x], y, v] - [[u, x], v, y] \in [A, R_n] = 0.$$ 

This means that $[A, R_{n-2}]$ is commutative. Now, pick $a, b \in [A, R_{n-2}]$, $x, y \in R$, and use Lemma 3(h) to conclude

$$[a, by, x] = [a, b, x]y + b[a, y, x] + [b, x][a, y].$$

Since $[A, R_n] = 0$, we obtain $[b, x][a, y] = 0$ i.e. $([A, R_{n-1}])^2 = 0$. Due to the fact that $[A, R_{n-1}]$ is central we may conclude that $([A, R_{n-1}])^2 = 0$ establishing our claim (*).

To prove our lemma by induction on $n$, we may assume that $n \geq 2$. First let $n = 2$ i.e. $[A, R, R] = 0$. We have to show that $[A, R]R$ is a nil ideal. Since $[A, R]$ is central it is enough to prove that $[A, R]$ is nil. In Lemma 3(ii), take $a \in A$, $x \in R$, $x = y$, $a = b$; we get $[a, x]^2 = 0$, proving the lemma for $n = 2$. Now, let $n > 2$. Writing $R = R/[A, R_{n-1}]R$ and $\overline{A}$ for the homomorphic image of $A$ etc., we have $[\overline{A}, R_{n-1}]R = 0$. Therefore, due to the induction hypothesis $[\overline{A}, R]R$ is nil and $[\overline{A}, R, R]R$ is nilpotent. Hence due to (*) we conclude that $[A, R]R$ is nil and $[A, R, R]R$ is nilpotent as desired.

Remark. The statement of our theorem is not true for arbitrary rings. This was conjectured by Jennings [2] and confirmed by Gupta and Levin [1] by means of an example. We record another easy example to the same effect.

Example. Let $E$ be the exterior algebra on a countable infinite-dimensional vector space $V$ over a field of characteristic not 2. Let

$$V = \text{Span}\{v_1, v_2, \ldots\}, \quad E = \text{Span}\{v_{i_1}v_{i_2} \cdots v_{i_m}|i_1 < i_2 < \cdots < i_m\}.$$ 

We have $[E, E] = \text{Span}\{v_{i_1}v_{i_2} \cdots v_{i_m}|i_1 < i_2 < \cdots < i_{2m}\}$ and $[E, E, E] = 0$. It is clear that $[E, E]$ has elements of arbitrarily large index of nilpotency; for instance, $a = v_1v_2 + v_3v_4 + \cdots + v_{2m-1}v_{2m}$ has index of nilpotency $m + 1$.

3. Proof of theorem

In this section we shall fix $R = KG$, $A$ the given additive subgroup of $R$, $\phi$ the FC-subgroup of $G$. We shall denote by $\Delta(G, N) = (\Delta N)KG$ the kernel of the homomorphism $KG \to K(G/N)$ where $\Delta(N)$ is the augmentation ideal of the normal subgroup $N$. We separate a special case as

Lemma 5. Suppose that $\phi$ the FC-subgroup of $G$ is abelian and of finite index. Then $[A, R_n] = 0$ for some $n \Rightarrow [A, R] = 0$.

Proof. Let $m = (G: \phi)$. Then fix a right transversal $\{1 = x_1, x_2, \ldots, x_m\}$ of $\phi$ in $G$. Write $(\phi, x_i) = \{a^{-1}x_i^{-1}ax|a \in \phi\}$ and $S_i = \Delta(\phi, x_i)$ for $i = 2, \ldots, m$ and $S = S_2S_3 \cdots S_m$. It is well known [5, p. 145, 4 or 3] that there is an imbedding $\rho: KG \to M'_m(K\phi)$ of $KG$ into $m \times m$ matrices over the
commutative group ring $K\phi$. Moreover, since $K\phi$ imbeds in $M_m(K\phi)$ as scalar matrices,

$$M_m(K\phi) \geq K\phi \cdot \rho(KG) \geq M_m(S).$$

We may clearly assume that $A \neq 0$ so that $K\phi \rho(A) \neq 0$. Since $K\phi$ are scalar matrices, we have

$$[K\phi \rho(A), K\phi \rho(KG), \ldots, K\phi \rho(KG)] = (K\phi)^n \rho([A, KG, KG, \ldots, KG]) = 0.$$  

Let $0 \neq u \in \rho(A)$ and write $u = \sum_{i,j} x_{ij} e_{ij}$ with $x_{ij} \in K\phi$, where the $e_{ij}$ are the usual matrix units. Let $i \neq j$ be fixed; then for $s_1, \ldots, s_n \in S$ we have

$$0 = [u, s_1 e_{ii}, s_2 e_{jj}, s_3 e_{jj}, \ldots, s_n e_{jj}]$$

$$= s_1 s_2 \cdots s_n \left[ \sum_{i=1}^n x_{ii} e_{ii} - \sum_{i=1}^n x_{ii} e_{ij} \cdot e_{jj} \cdot \ldots \cdot e_{jj} \right]$$

$$= \cdots = \pm s_1 s_2 \cdots s_n x_{ij} e_{ij} \pm s_1 s_2 \cdots s_n x_{ji} e_{ij}.$$  

Thus $s_1 s_2 \cdots s_n x_{ij} = 0$ and since $s_i$ are arbitrary elements of $S$ which is a product of augmentation ideals of infinite groups, it follows that $x_{ij} = 0$ for all $i \neq j$. Thus $u = \sum_{i=j} x_{ii} e_{ii}$. Suppose that $x_{ii} \neq 0$. Then for $j \neq i$, we get

$$0 = [u, s_1 e_{ij}, s_2 e_{jj}, \ldots, s_n e_{jj}] = s_1 s_2 \cdots s_n [(x_{ii} - x_{ij}) e_{ij}, e_{ii}, \ldots, e_{ii}]$$

$$= s_1 s_2 \cdots s_n (x_{jj} - x_{ij}) e_{ij}.$$  

This implies that $s_1 s_2 \cdots s_n (x_{jj} - x_{ij}) = 0$ for all $i \in \Delta(\phi, x_i)$. It follows that $x_{jj} = x_{ij}$ for all $j = 1, 2, \ldots, m$. Thus $u$ is a scalar matrix. Hence, $0 = [\rho(A), \rho(KG)] = \rho[A, KG]$ and so $[A, KG] = 0$.

We are now ready to give the

**Proof of the Theorem.** We know by Lemma 4 that $[A, R]R$ is a nil ideal and $[A, R, R]R$ is a nilpotent ideal. If $\chi(K) = 0$, $R$ contains no nonzero nil ideals [4, p. 47]; so $[A, R]R = 0$ and we are done. Therefore, we suppose that $\chi(K) = p > 0$. If $G_3$ is the third term of the lower central series of $G$ then by Lemma 1 we have $\Delta(G_3) \leq [R, R, R]R$ and so it follows that $\Delta(G, G_3) = \Delta(G_3)R \leq [R, R, R]R$. Since by Lemma 3(iii), $[R, R, R][A, R] = [R, R, R][A, R] \leq [A, R, R]R$ we get

$$\Delta(G_3)R[A, R] \leq [A, R, R]R$$

which is a nilpotent ideal. So $\Delta(G_3)R[A, R]$ is a nilpotent ideal of $R$.

Now, let $\pi: KG \rightarrow K(G/G_3)$ be the natural map. If the theorem holds for nilpotent class two groups, then, we will have

$$\pi([A, R]R) = [\pi(A), \pi(R)]\pi(R) \text{ is a nilpotent ideal.}$$

So for some $m \geq 1$, $([A, R]R)^m \leq \Delta(G_3)R$ and

$$([A, R]R)^m[A, R] \leq \Delta(G_3)R[A, R].$$

So $([A, R]R)^{m+1}$ and hence $[A, R]R$ is a nilpotent ideal, the desired conclusion. Thus we may assume that $G$ is nilpotent of class two, and so, $G'$ is
central. Since \([A, R_n] = 0\), for \(a \in A\), \([a, x_1, \ldots, x_n]\) is a multilinear generalized polynomial identity for \(R\). Also, if for all \(a \in A\), \([a, x_1]x_2 \cdots x_n\) is a generalized polynomial identity for \(R\), then since \(1 \in R\), we get \([A, R] = 0\).

The theorem is proved in this case. Thus, we may assume that for some \(a \in A\), \([a, x_1, \ldots, x_n]\) is a nondegenerate generalized polynomial identity for \(R\). It follows by Lemma 2 that \((G: \phi) < \infty\) and \(|\phi| < \infty\). Let \(P\) be the Sylow \(p\)-subgroup of \(\phi\). Since \(\Delta(G, P)\) is nilpotent, factoring by \(P\), it is enough to prove the theorem when \(\phi'\) is a \(p'\)-group. We use induction on \(|\phi'|\). If \(|\phi'| = 1\), \(\phi\) is abelian and we are done by Lemma 5. Thus we assume \(|\phi'| > 1\). Since \([A, R_n] = 0\), then also \([A, R_{p^m}] = 0\) where \(p^m\) is the smallest power of \(p\) greater than \(n\). Thus for \(x \in R\), \(a \in A\), we get

\[
0 = [a, x, x, \ldots, x] = ax^{p^m} - x^{p^m}a.
\]

Therefore, for all \(x \in R\), \(x^{p^m} \in C_R(A)\). Now, if for all \(g \in \phi\), \(g^{p^m}\) belongs to the centre of \(\phi\) then it follows by Schur's theorem [5, p. 39] that \(\phi'\) is a \(p'\)-group contrary to our assumption that \(\phi'\) is a nontrivial \(p'\)-group. Thus there is a \(g \in \phi\) such that \(g^{p^m} \notin C(\phi)\), the centre of \(\phi\). Pick \(h \in \phi\) such that \(gh^{p^m} \neq h\). Since \(g^{p^m} \in C_R(A)\), we have, by Lemma 3(iv), \([A, R][g^{p^m}, h] \leq [A, R, R]R\). Thus letting \(z = h^{-1}g^{-p^m}h^{p^m}\) we get

\[
[A, R]R(1 - z) = [A, R]Rh^{-1}g^{-p^m}g^{p^m}h(1 - z) = [A, R][g^{p^m}, h]
\]

\[
= R[A, R][g^{p^m}, h] \leq [A, R, R]R.
\]

Since the last ideal is nilpotent by Lemma 4, we have that \([A, R]R(1 - z)\) is a nilpotent ideal. Since \(z\) is central in \(G\), \(G\) being nilpotent of class two, we have

\[
([A, R]R)^k(1 - z)^k = 0 \quad \text{for some } k \geq 1.
\]

Suppose \(k > 1\) and let \(l_R(1 - z)\) be the left annihilator of \((1 - z)\) in \(R\). Then

\[
([A, R]R)^k(1 - z)^{k-1} \in R(1 - z) \cap l_R(1 - z) = R(1 - z) \cap R\hat{z} = 0,
\]

\(\hat{z} = (1 + z + \cdots + z^{l-1})\). By iterating this process we conclude that

\[
([A, R]R)^k(1 - z) = 0.
\]

Factoring \(G\) by \(\langle z \rangle\) we conclude by the inductive hypothesis that

\[
([A, R]R)^l \leq \Delta(G, \langle z \rangle) = R(1 - z).
\]

Therefore, \(([A, R]R)^kl(1 - z) = 0\) and \(([A, R]R)^kt \leq R(1 - z)\). It follows that \(([A, R]R)^kt \leq R(1 - z) \cap R\hat{z} = 0\) as desired.

References


Department of Mathematics, University of Palermo, Palermo, Italy

Department of Mathematics, University of Alberta, Edmonton, Canada