PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 2, February 1989

# A LIE PROPERTY IN GROUP RINGS

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(Communicated by Donald S. Passman)

ABSTRACT. Let $A$ be an additive subgroup of a group ring $R$ over a field $K$. Denote by $[A, R]$ the additive subgroup generated by the Lie products $[a, r] = ar - ra$, $a \in A$, $r \in R$. Inductively, let $[A, R_n] = [[A, R_{n-1}], R]$. We prove that $[A, R_n] = 0$ for some $n \Rightarrow [A, R]R$ is a nilpotent ideal.

## INTRODUCTION

Let $A$ be an additive subgroup of a ring $R$. We define $[A, R]$ to be the additive subgroup of $R$ generated by all Lie products $[a, r] = ar - ra$, $a \in A$, $r \in R$. Inductively, we set

$$[A, \underbrace{R, R, \dots, R}_{n+1}] = [A, R_{n+1}] = [[A, R_n], R].$$

The main result of this note is

**Theorem.** *Let $R = KG$ be the group algebra of a group $G$ over a field $K$ and $A$ an additive subgroup of $R$. Then $[A, R_n] = 0$ for some $n \Rightarrow [A, R]R$ is an (associative) nilpotent ideal of $R$.*

Taking $A = KG$, we conclude that if $KG$ is Lie nilpotent, then

$$[KG, KG]KG = \Delta(G')KG$$

is nilpotent where $\Delta(G')$ is the augmentation ideal of the derived group $G'$. This implies that $G$ is abelian if $\chi(K)$, the characteristic of $K$ is 0 and $G'$ is a finite $p$-group if $\chi(K) = p > 0$. This is a theorem of Passi, Passman and Sehgal [3, 5, p. 151]. To prove our theorem we need two results due to Gupta, Levin and Passman.

**Lemma 1** [1]. *The nth term $G_n$ of the lower central series of $G$ is contained in*

$$1 + [\underbrace{R, R, \dots, R}_{n}]R \quad \textit{where } R = KG.$$

Received by the editors December 2, 1987 and, in revised form, February 22, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 16A27, 16A68.
This research is supported by NSERC, Canada and MPI, Italy.

**Lemma 2** [4, p. 202]. *If $KG$ satisfies a nondegenerate multilinear generalized polynomial identity then $\phi$, the FC subgroup of $G$, is of finite index and has finite derived group $\phi'$.*

In the next section we collect some preliminary results and prove the theorem in §3.

## 2. Some preliminary results

In this section $R$ is an arbitrary ring with 1 and $A$ is an additive subgroup of $R$. We denote by $C_R(A) = \{x \in R | xa = ax \text{ for all } a \in A\}$, the centralizer of $A$ in $R$. We write $[x, y, z] = [[x, y], z]$ for $x, y, z \in R$.

**Lemma 3.** (i) *For $k \geq 1$, $[A, R_k]R = R[A, R_k]$ so that $[A, R_k]R$ is the two-sided ideal generated by $[A, R_k]$.*

(ii) *For $a, b, x, y \in R$ we have*

$$[a, by, x] = [a, b, x]y + [a, b][y, x] + b[a, y, x] + [b, x][a, y].$$

(iii) $[R, R, R][A, R] \leq [A, R, R]R$.

(iv) *For $b \in C_R(A)$, $[A, R][b, R] \leq [A, R, R]R$.*

*Proof.* (i) Let $a \in [A, R_{k-1}]$, $x, y \in R$. Then

$$[a, x]y = (ax - xa)y = a(xy) - (xy)a - x(ay - ya) = [a, xy] - x[a, y]$$

and

$$x[a, y] = [a, xy] - [a, x]y.$$

(ii) is clear.

To obtain (iii) in the expression for $[b, x][a, y]$ in (ii), we take $a \in A$, $b \in [R, R]$, $x, y \in R$ to conclude

$$[R, R, R][A, R] \leq [A, R, R]R + [A, [R, R]]R + R[A, R, R]$$
$$\leq [A, R, R]R \quad \text{(by the Jacobi identity and (i))}.$$

(iv) We first observe that for $a \in A$, $r \in R$, $[a, r]b = [a, rb]$ and thus $[A, R]b \leq [A, R]$. Now, taking $x \in [A, R]$, $r \in R$ we have

$$x[b, r] = [xb, r] - [x, r]b$$

which implies

$$[A, R][b, R] \leq [[A, R]b, R] + [A, R, R]R \leq [A, R, R]R.$$

The next lemma is an extension of a result of Jennings [2] to our situation. The proof parallels that of Jennings.

**Lemma 4.** *Suppose that $[A, R_n] = 0$ for some $n \geq 1$. Then $[A, R]R$ is a nil ideal and $[A, R, R]R$ is a nilpotent ideal.*

*Proof.* We first claim

( * ) If $[A, R_n] = 0$ for some $n > 2$ then $([A, R_{n-1}]R)^2 = 0$.

Let $u, v \in [A, R_{n-3}]$ and $x, y \in R$. Then by the Jacobi identity

$$\begin{aligned} -[[u, x], [v, y]] &= [v, [y, [u, x]]] + [y, [[u, x], v]] \\ &= [[u, x], y, v] - [[u, x], v, y] \in [A, R_n] = 0. \end{aligned}$$

This means that $[A, R_{n-2}]$ is commutative. Now, pick $a, b \in [A, R_{n-2}]$, $x, y \in R$, and use Lemma 3(ii) to conclude

$$[a, by, x] = [a, b, x]y + b[a, y, x] + [b, x][a, y].$$

Since $[A, R_n] = 0$, we obtain $[b, x][a, y] = 0$ i.e. $([A, R_{n-1}])^2 = 0$. Due to the fact that $[A, R_{n-1}]$ is central we may conclude that $([A, R_{n-1}]R)^2 = 0$ establishing our claim $(*)$.

To prove our lemma by induction on $n$, we may assume that $n \geq 2$. First let $n = 2$ i.e. $[A, R, R] = 0$. We have to show that $[A, R]R$ is a nil ideal. Since $[A, R]$ is central it is enough to prove that $[A, R]$ is nil. In Lemma 3(ii), take $a \in A$, $x \in R$, $x = y$, $a = b$; we get $[a, x]^2 = 0$, proving the lemma for $n = 2$. Now, let $n > 2$. Writing $\overline{R} = R/[A, R_{n-1}]R$ and $\overline{A}$ for the homomorphic image of $A$ etc., we have $[\overline{A}, \overline{R}_{n-1}] = 0$. Therefore, due to the induction hypothesis $[\overline{A}, \overline{R}]\overline{R}$ is nil and $[\overline{A}, \overline{R}, \overline{R}]\overline{R}$ is nilpotent. Hence due to $(*)$ we conclude that $[A, R]R$ is nil and $[A, R, R]R$ is nilpotent as desired.

*Remark.* The statement of our theorem is not true for arbitrary rings. This was conjectured by Jennings [2] and confirmed by Gupta and Levin [1] by means of an example. We record another easy example to the same effect.

**Example.** Let $E$ be the exterior algebra on a countable infinite-dimensional vector space $V$ over a field of characteristic not 2. Let

$$V = \text{Span}\{v_1, v_2, \dots\}, \qquad E = \text{Span}\{v_{i_1} v_{i_2} \cdots v_{i_m} | i_1 < i_2 < \cdots < i_m\}.$$

We have $[E, E] = \text{Span}\{v_{i_1} v_{i_2} \cdots v_{i_{2m}} | i_1 < i_2 < \cdots < i_{2m}\}$ and $[E, E, E] = 0$. It is clear that $[E, E]$ has elements of arbitrarily large index of nilpotency; for instance, $a = v_1 v_2 + v_3 v_4 + \cdots + v_{2m-1} v_{2m}$ has index of nilpotency $m + 1$.

## 3. Proof of theorem

In this section we shall fix $R = KG$, $A$ the given additive subgroup of $R$, $\phi$ the FC-subgroup of $G$. We shall denote by $\Delta(G, N) = (\Delta N)KG$ the kernel of the homomorphism $KG \to K(G/N)$ where $\Delta(N)$ is the augmentation ideal of the normal subgroup $N$. We separate a special case as

**Lemma 5.** *Suppose that $\phi$ the FC-subgroup of $G$ is abelian and of finite index. Then $[A, R_n] = 0$ for some $n \Rightarrow [A, R] = 0$.*

*Proof.* Let $m = (G: \phi)$. Then fix a right transversal $\{1 = x_1, x_2, \dots, x_m\}$ of $\phi$ in $G$. Write $(\phi, x_i) = \{a^{-1} x_i^{-1} a x_i | a \in \phi\}$ and $S_i = \Delta(\phi, x_i)$ for $i = 2, \dots, m$ and $S = S_2 S_3 \cdots S_m$. It is well known [5, p. 145, 4 or 3] that there is an imbedding $\rho: KG \to M_m(K\phi)$ of $KG$ into $m \times m$ matrices over the commutative group ring $K\phi$. Moreover, since $K\phi$ imbeds in $M_m(K\phi)$ as scalar matrices,

$$M_m(K\phi) \geq K\phi \cdot \rho(KG) \geq M_m(S).$$

We may clearly assume that $A \neq 0$ so that $K\phi\rho(A) \neq 0$. Since $K\phi$ are scalar matrices, we have

$$[K\phi\rho(A), K\phi\rho(KG), \ldots, K\phi\rho(KG)] = (K\phi)^n \rho([A, KG, KG, \ldots, KG]) = 0.$$

Let $0 \neq u \in \rho(A)$ and write $u = \sum_{i,j} x_{ij} e_{ij}$ with $x_{ij} \in K\phi$, where the $e_{ij}$ are the usual matrix units. Let $i \neq j$ be fixed; then for $s_1, \ldots, s_n \in S$ we have

$$\begin{aligned} 0 &= [u, s_1 e_{ii}, s_2 e_{jj}, s_3 e_{jj}, \ldots, s_n e_{jj}] \\ &= s_1 s_2 \cdots s_n \left[ \sum_{l=1}^{n} x_{li} e_{li} - \sum_{t=1}^{n} x_{it} e_{it}, e_{jj}, \ldots, e_{jj} \right] \\ &= \cdots = \pm s_1 s_2 \cdots s_n x_{ij} e_{ij} \pm s_1 s_2 \cdots s_n x_{ji} e_{ji}. \end{aligned}$$

Thus $s_1 s_2 \cdots s_n x_{ij} = 0$ and since $s_i$ are arbitrary elements of $S$ which is a product of augmentation ideals of infinite groups, it follows that $x_{ij} = 0$ for all $i \neq j$. Thus $u = \sum_{i=1}^{m} x_{ii} e_{ii}$. Suppose that $x_{ll} \neq 0$. Then for $j \neq l$, we get

$$\begin{aligned} 0 &= [u, s_1 e_{lj}, s_2 e_{ll}, \ldots, s_n e_{ll}] = s_1 s_2 \cdots s_n [(x_{ll} - x_{jj}) e_{lj}, e_{ll}, \ldots, e_{ll}] \\ &= s_1 s_2 \cdots s_n (x_{ll} - x_{jj}) e_{lj}. \end{aligned}$$

This implies that $s_1 s_2 \cdots s_n (x_{ll} - x_{jj}) = 0$ for all $s_i \in \Delta(\phi, x_i)$. It follows that $x_{ll} = x_{jj}$ for all $j = 1, 2, \ldots, m$. Thus $u$ is a scalar matrix. Hence, $0 = [\rho(A), \rho(KG)] = \rho[A, KG]$ and so $[A, KG] = 0$.

We are now ready to give the

*Proof of the Theorem.* We know by Lemma 4 that $[A, R]R$ is a nil ideal and $[A, R, R]R$ is a nilpotent ideal. If $\chi(K) = 0$, $R$ contains no nonzero nil ideals [4, p. 47]; so $[A, R]R = 0$ and we are done. Therefore, we suppose that $\chi(K) = p > 0$. If $G_3$ is the third term of the lower central series of $G$ then by Lemma 1 we have $\Delta(G_3) \leq [R, R, R]R$ and so it follows that $\Delta(G, G_3) = \Delta(G_3)R \leq [R, R, R]R$. Since by Lemma 3(iii), $[R, R, R]R[A, R] = [R, R, R][A, R]R \leq [A, R, R]R$ we get

$$\Delta(G_3)R[A, R] \leq [A, R, R]R$$

which is a nilpotent ideal. So $\Delta(G_3)R[A, R]$ is a nilpotent ideal of $R$.

Now, let $\pi: KG \to K(G/G_3)$ be the natural map. If the theorem holds for nilpotent class two groups, then, we will have

$$\pi([A, R]R) = [\pi(A), \pi(R)]\pi(R) \text{ is a nilpotent ideal}.$$

So for some $m \geq 1$, $([A, R]R)^m \leq \Delta(G_3)R$ and

$$([A, R]R)^m [A, R] \leq \Delta(G_3)R[A, R], \quad \text{a nilpotent ideal}.$$

So $([A, R]R)^{m+1}$ and hence $[A, R]R$ is a nilpotent ideal, the desired conclusion. Thus we may assume that $G$ is nilpotent of class two, and so, $G'$ is central. Since $[A, R_n] = 0$, for $a \in A$, $[a, x_1, \dots, x_n]$ is a multilinear generalized polynomial identity for $R$. Also, if for all $a \in A$, $[a, x_1]x_2 \cdots x_n$ is a generalized polynomial identity for $R$, then since $1 \in R$, we get $[A, R] = 0$. The theorem is proved in this case. Thus, we may assume that for some $a \in A$, $[a, x_1, \dots, x_n]$ is a nondegenerate generalized polynomial identity for $R$. It follows by Lemma 2 that $(G\colon \phi) < \infty$ and $|\phi'| < \infty$. Let $P$ be the Sylow $p$-subgroup of $\phi'$. Since $\Delta(G, P)$ is nilpotent, factoring by $P$, it is enough to prove the theorem when $\phi'$ is a $p'$-group. We use induction on $|\phi'|$. If $|\phi'| = 1$, $\phi$ is abelian and we are done by Lemma 5. Thus we assume $|\phi'| > 1$. Since $[A, R_n] = 0$, then also $[A, R_{p^m}] = 0$ where $p^m$ is the smallest power of $p$ greater than $n$. Thus for $x \in R$, $a \in A$, we get

$$0 = [a, \underbrace{x, x, \dots, x}_{p^m}] = ax^{p^m} - x^{p^m}a.$$

Therefore, for all $x \in R$, $x^{p^m} \in C_R(A)$. Now, if for all $g \in \phi$, $g^{p^m}$ belongs to the centre of $\phi$ then it follows by Schur's theorem [5, p. 39] that $\phi'$ is a $p$-group contrary to our assumption that $\phi'$ is a nontrivial $p'$-group. Thus there is a $g \in \phi$ such that $g^{p^m} \notin C(\phi)$, the centre of $\phi$. Pick $h \in \phi$ such that $g^{p^m}h - hg^{p^m} \neq 0$. Since $g^{p^m} \in C_R(A)$, we have, by Lemma 3(iv), $[A, R][g^{p^m}, h] \leq [A, R, R]R$. Thus letting $z = h^{-1}g^{-p^m}hg^{p^m}$ we get

$$\begin{aligned}[A, R]R(1 - z) &= [A, R]Rh^{-1}g^{-p^m}g^{p^m}h(1 - z) = [A, R]R[g^{p^m}, h] \\ &= R[A, R][g^{p^m}, h] \leq [A, R, R]R.\end{aligned}$$

Since the last ideal is nilpotent by Lemma 4, we have that $[A, R]R(1 - z)$ is a nilpotent ideal. Since $z$ is central in $G$, $G$ being nilpotent of class two, we have

$$([A, R]R)^k(1 - z)^k = 0 \quad \text{for some } k \geq 1.$$

Suppose $k > 1$ and let $l_R(1 - z)$ be the left annihilator of $(1 - z)$ in $R$. Then

$$([A, R]R)^k(1 - z)^{k-1} \in R(1 - z) \cap l_R(1 - z) = R(1 - z) \cap R\hat{z} = 0,$$

$\hat{z} = (1 + z + \cdots + z^{|z|-1})$. By iterating this process we conclude that

$$([A, R]R)^k(1 - z) = 0.$$

Factoring $G$ by $\langle z \rangle$ we conclude by the inductive hypothesis that

$$([A, R]R)^t \leq \Delta(G, \langle z \rangle) = R(1 - z).$$

Therefore, $([A, R]R)^{kt}(1 - z) = 0$ and $([A, R]R)^{kt} \leq R(1 - z)$. It follows that $([A, R]R)^{kt} \leq R(1 - z) \cap R\hat{z} = 0$ as desired.

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