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ON A DOMAIN CHARACTERIZATION OF SCHRÖDINGER OPERATORS WITH GRADIENT MAGNETIC VECTOR POTENTIALS AND SINGULAR POTENTIALS

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ABSTRACT. Of concern are the minimal and maximal operators on $L^2(\mathbf{R}^n)$ associated with the differential expression

$$\tau_Q = \sum_{j=1}^{n} (i\partial/\partial x_j + q_j(x))^2 + W(x)$$

where $(q,\dots,q_n) = \operatorname{grad} Q$ for some real function W on $\mathbf{R}^n$ and W satisfies $c|x|^{-2} \le W(x) \le C|x|^{-2}$. In particular, for $Q = 0$, τ_Q reduces to the singular Schrödinger operator $-\Delta + W(x)$. Among other results, it is shown that the maximal operator (associated with the τ_Q) is the closure of the minimal operator, and its domain is precisely

$$\operatorname{Dom}\left(\sum_{j=1}^{n} (i\partial/\partial x_j + q_j(x))^2\right) \cap \operatorname{Dom}(W),$$

provided that $C \ge c > -n(n-4)/4$.

1. INTRODUCTION

Consider the formal differential expression

$$\tau_c = H_0 + V = -\Delta + c/|x|^2$$

acting on functions on $\mathbf{R}^n$; here Δ is the Laplacian and c is a real constant. The *minimal* and *maximal operators* associated with τ_c, H_{cm} and H_{cM}, are given by τ_c acting on the domains

$$\mathscr{D}(H_{cm}) = C_0^\infty(\mathbf{R}^n\backslash\{0\}) = \{u \in C^\infty(\mathbf{R}^n)\colon u \text{ has compact support in } \mathbf{R}^n\backslash\{0\}\},$$
$$\mathscr{D}(H_{cM}) = \{u \in L^2(\mathbf{R}^n)\colon c|x|^{-2}u \in L^1_{\text{loc}}(\mathbf{R}^n),\ \tau_c u \in L^2(\mathbf{R}^n)\}.$$

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(Here and in the sequel, "m" stands for minimal and "M" for maximal.) These operators, viewed as operators on $L^2(\mathbf{R}^n)$, have many remarkable properties. (Cf. [7]; see also [1, 8].) In particular we have:

(i) H_{cm} is semibounded if and only if $H_{cm} \geq 0$ if and only if $c \geq -[(n-2)/2]^2$.

(ii) H_{cm} is essentially selfadjoint if and only if $c \geq -n(n-4)/4 = 1-[(n-2)/2]^2$.

(iii) $\overline{H_{cm}} = H_{cM}$ if and only if $c \geq -n(n-4)/4$.

The remarkable aspect of (i)–(iii) is that these properties depend on the values of c rather than the form of the potential. This is a hightly unusual occurrence in perturbation theory, and it shows that $c/|x|^2$ cannot be thought of as a small perturbation of $-\Delta$ if $c \neq 0$.

In our earlier paper [5] we extended these and other related results by taking advantage of scaling properties. More precisely, let $H_0 = -\Delta$, $V(x) = c/|x|^2$, and $\lambda > 0$. The unitary scaling operator $U(\lambda)$ is defined by

$$(U(\lambda)f)(x) = \lambda^{n/2} f(x) \quad \text{for } f \in L^2(\mathbf{R}^n),\ x \in \mathbf{R}^n.$$

Then

$$U(\lambda)AU(\lambda)^{-1} = \lambda^{-2}A$$

holds for both $A = H_0$ and $A = V$ (i.e., A is multiplication by $V(x)$). Thus $-\Delta$ and $V(x)$ both *scale like* λ^{-2}. It turns out that the same is true of $\sum_{j=1}^n (i\partial/\partial x_j + \alpha x_j/|x|^2)^2$, and this fact formed part of the heuristic background for [5]. On the other hand, the vector whose jth component is $\alpha x_j |x|^{-2}$ (for $\alpha \in \mathbf{R}$) is the gradient of $\alpha \log|x|$. Operators of the form

$$\sum_{j=1}^{n} (i\partial/\partial x_j + q_j(x))^2 + c|x|^2, \tag{1}$$

where $(q, \ldots q_n) = \operatorname{grad} Q$, turn out to have the same properties as H_{cm}. Such a $(q, \ldots, q_n)$ will be termed a *gradient magnetic vector potential*. Properties of the operator (1) will be discussed in the sequel.

2. Background

Clearly

$$\mathscr{D}(H_{cM}) \supset \mathscr{D}(H_0) \cap \mathscr{D}(V).$$

It was recently discovered [**2**, **6**, 12], that the converse containment holds if and only if $c > -n(n-4)/4$. (Cf. also [3, **5**, 7, 9, 10].) Thus to (i)–(iii) we can add

(iv) $\mathscr{D}(H_{cM}) = \mathscr{D}(H_0) \cap \mathscr{D}(V) = \{u \in W^{2,2}(\mathbf{R}^n) \colon |x|^{-2}u \in L^2(\mathbf{R})\}$ for

$$c > c_0(n) = \begin{cases} \frac{3}{4} & \text{if } n = 1 \text{ or } 3, \\ 1 & \text{if } n = 2, \\ 0 & \text{if } n \geq 4. \end{cases}$$

In particular, (iii) shows that (iv) holds only if H_{cm} is essentially selfadjoint, in which case $\overline{H_{cm}} = H_{cM}$ is selfadjoint.

In fact, from [12] we can rewrite the criterion for (iv) in a more general form as follows.

Proposition 1. *Let H_{cM}, H_0, and V be as above. Then the following four statements are equivalent.*

(a) $\mathscr{D}(H_{cM}) = \mathscr{D}(H_0) \cap \mathscr{D}(V)$.

(b) *There exist constants $a \geq 0$, $b \geq 0$ such that*

$$\|Vu\| \leq a\|H_{cM}u\| + b\|u\|$$

holds for all $u \in \mathscr{D}(H_{cM})$.

(c) *There exists a constant $a \geq 0$ such that*

$$\|Vu\| \leq a\|H_{cM}u\|$$

holds for all $u \in \mathscr{D}(H_{cM})$.

(d) $\mathscr{D}(H_{cM}) \subset \mathscr{D}(V)$ *and* $\|V(H_{cM} + I)^{-1}\| < \infty$.

In [5] we extended this to

$$\tau_{c\alpha} = H_{0\alpha} + V = \sum_{j=1}^{n}(i\partial/\partial x_j + \alpha x_j/|x|^2)^2 + c/|x|^2$$

(for α, $c \in \mathbf{R}$) acting on functions on $\mathbf{R}^n$. As before, let $H_{c\alpha m}$ and $H_{c\alpha M}$ denote the minimal and maximal operators associated with $\tau_{c\alpha}$ on $L^2(\mathbf{R}^n)$, i.e.,

$$\mathscr{D}(H_{c\alpha m}) = C_0^\infty(\mathbf{R}^n\backslash\{0\}),$$

$$\mathscr{D}(H_{c\alpha M}) = \{u \in L^2(\mathbf{R}^n):\ c|x|^{-2}u \in L'_{\mathrm{loc}}(\mathbf{R}^n)',\ \tau_{c\alpha}u \in L^2(\mathbf{R}^n)\}.$$

Then Proposition 1 holds with H_{cM} replaced by $H_{c\alpha M}$ for all $\alpha \in \mathbf{R}$.

In [5] we also pointed out the obstacle to (iv) holding for all positive c in dimensions 2 and 3. Let $\mathscr{H}_0 = L^2(\mathbf{R}^n)$ and let $\mathscr{H}_1$ be the closure in $\mathscr{H}_0$ of

$$\left\{ f \in L^2(\mathbf{R}^n) \cap C(\mathbf{R}^n): \int_{|x|=r} f(x)\, dS_x = 0 \text{ for each } r > 0 \right\},$$

i.e. $\mathscr{H}_1$, consists of the functions in $\mathscr{H}_0$ having *spherical means zero*. For $l = 0, 1, 2, \ldots$ let

$$M_l = L^2([0, \infty)) \oplus H^l(S^{n-1}),$$

so that the spherical harmonic decomposition of $L^2(\mathbf{R}^n)$ [13, p. 138ff.] becomes

$$\mathscr{H}_0 = L^2([0, \infty)) \otimes L^2(S^{n-1}) = L^2([0, \infty)) \otimes \bigoplus_{l=0}^{\infty} H^l(S^{n-1}) \equiv \bigoplus_{l=0}^{\infty} M_l,$$

and we have

$$\mathscr{H}_1 = M_0^\perp = \bigoplus_{l=1}^{\infty} M_l.$$

It was shown in [5] that a careful examination of the proof in [13] shows that for $c > 0$ and $j = 0, 1$, there is a constant $a(j, n)$ such that

$$\|Vu\| \leq a(j, n)\|H_{cM}u\| \tag{2}$$

holds for all $u \in \mathscr{D}(H_{cM}) \cap \mathscr{H}_j$ and all $c > c_j(n)$, where $c_0(n)$, is as before (in (iv)) and

$$c_1(n) = \begin{cases} \frac{3}{4} & \text{if } n = 1, \\ 0 & \text{if } n \geq 4. \end{cases}$$

Thus the obstruction to (iv) holding in dimension 2 and 3 is the subspace M_0 of radial functions.

3. Gradient magnetic vector potentials

One of our goals here is to generalize the above results to the minimal and maximal operators, H_{Qm} and H_{QM}, associated with the differential expression

$$\tau_Q = H(Q) + W = \sum_{j=1}^{n}(i\partial/\partial x_j + q_j(x))^2 + W(x)$$

where $(q_1, \dots, q_n) = \operatorname{grad} Q$ for some real function Q in $W^{1,1}_{\text{loc}}(\mathbf{R}^n)$ and where the real function $W \in L^1_{\text{loc}}(\mathbf{R}^n \backslash \{0\})$ satisfies $W(x) \geq c/|x|^2$ for all x and some $c > -n(n-4)/4$. The connection between τ_Q and the Schrödinger operator $-\Delta + W(x)$ is made clear by the following result.

Proposition 2. *Let $A_j = i\partial/\partial x_j + q_j(x)$ for $j = 1, \dots, n$ where $(q_1, \dots, q_n) = \operatorname{grad} Q$ for some real function Q in $W^{1,1}_{\text{loc}}(\mathbf{R}^n)$. Let U be the unitary operator of multiplication by $e^{iQ(x)}$. Then for $j \in \{1, \dots, n\}$, $A_j = UB_jU^{-1}$ where $B_j = i\partial/\partial x_j$ acts on $\mathscr{D}(B_j) = \{u \in L^2(\mathbf{R}^n)$: the distributional derivative $\partial u/\partial x_j$ is in $L^2(\mathbf{R}^n)\}$.*

Proof. A straightforward computation gives

$$UB_jU^{-1}u = (i\partial/\partial x_j + q_j(x))u$$

for any $u \in \mathscr{D}(A_j) = e^{iQ}\mathscr{D}(B_j)$. □

The point of the above computation is that the unitary operator U is *independent* of j.

Note that for $Q \equiv 0$, the minimal operator H_{om} is known to be essentially selfadjoint [11]. Also, $\overline{H_{om}} = H_{oM}$. Combining these observations with the facts that $H(Q) = \sum_{j=1}^n A_j^2$ and $UWU^{-1} = W$ leads to the following result.

Corollary 3. *Let Q be as in Proposition 2. Then H_{Qm} is essentially selfadjoint, $\overline{H_{Qm}} = H_{QM}$, and $\mathscr{D}(H_{QM}) = e^{iQ}\mathscr{D}(H_{oM})$.*

H. Kalf [6] conjectured that for suitable values of c, $\mathscr{D}(H_{oM}) = \mathscr{D}(H_0) \cap \mathscr{D}(W)$. We shall establish a special case of this conjecture. Namely, we shall verify it for (measurable) potentials W satisfying

$$c_1/|x|^2 \le W(x) \le c_2/|x|^2 + c_3 \tag{3}$$

for any constants c_i satisfying

$$-n(n-4)/4 < c_1 \le c_2, \qquad c_3 \ge 0. \tag{4}$$

Theorem 4. *Let* (3) *and* (4) *hold. Let Q be as in Proposition* 2. *Then there exists a constant a, depending only on c_1, c_2 and c_3, such that*

$$\|Wu\| \le a\|H_{QM}u\| + a\|u\| \tag{5}$$

holds for all $u \in \mathscr{D}(H_{QM})$. Thus

$$\mathscr{D}(H_{QM}) = \mathscr{D}(H(Q)) \cap \mathscr{D}(W)$$

and H_{QM} is selfadjoint.

Proof. It follows from the previous discussion that (5) holds if and only if

$$\|Wu\| \le a\|H_{oM}u\| + b\|u\| \tag{6}$$

holds for all $u \in \mathscr{D}(H_{oM})$. Clearly (6) is equivalent to

$$\|W(H_{oM} + I)^{-1}\| = \|W(H_0 + W + 1)^{-1}\| < \infty.$$

Let $V(x) = c_1/|x|^2$. Then by Proposition 1 and (2) we have

$$\|V(H_{c_1 M} + I)^{-1}\| = \|V(H_0 + V + 1)^{-1}\| < \infty$$

if and only if $c_1 > -n(n-4)/4$.

Note that c_1 can be negative only in dimension five or more. Define cutoff functions

$$U_\nu(x) = U(x) \text{ or } c_1\nu^2,$$

according as $|x| \ge 1/\nu$ or $|x| < 1/\nu$, for $U = V, W$. Then V_ν and W_ν are in $L^\infty(\mathbf{R}^n)$, and $V_\nu \le W_\nu$ holds on $\mathbf{R}^n$. The Trotter product formula (cf. e.g. [4, 8]) implies that

$$\exp\{-t(H_0 + W_n)\}\phi \le \exp\{-t(H_0 + V_n)\}\phi$$

for all $0 \le \phi \in L^2(\mathbf{R}^n)$. Integrating this inequality gives the resolvent inequality

$$(H_0 + W_n + 1)^{-1}\phi \le (H_0 + V_n + 1)^{-1}\phi$$

for all $0 \le \phi \in L^2(\mathbf{R}^n)$. Letting $n \to \infty$ gives

$$(H_0 + W + 1)^{-1}\phi \le (H_0 + V + 1)^{-1}\phi \tag{7}$$

for all such ϕ.

From (3) we deduce, for all $0 \le \phi \in L^2(\mathbf{R}^n)$,

$$\begin{aligned}|W|(H_0 + W + 1)^{-1}\phi \le\ & \max(|c_1|, |c_2|)|x|^{-2}(H_0 + W + 1)^{-1}\phi \\ & + c_3(H_0 + W + 1)^{-1}\phi \\ \le\ & \max(|c_1|, |c_2|)|x|^{-2}(H_0 + V + 1)^{-1}\phi \\ & + c_3(H_0 + V + 1)^{-1}\phi\end{aligned}$$

by (7).

Next recall that for a bounded, positivity preserving operator A we have $|A\psi| \le A|\psi|$ a.e. for each $\psi \in L^2(\mathbf{R}^n)$. Thus we deduce

$$\begin{aligned} \|W(H_0 + W + 1)^{-1}\| \le \max(|c_1|, |c_2|)\| \, |x|^{-2}(H_0 + V + 1)^{-1}\| \\ + c_3\|(H_0 + V + 1)^{-1}\| < \infty. \end{aligned}$$

This completes the proof. □

We remark that when $c_1 > 0$, the approximation argument (involving V_ν, W_ν) becomes unnecessary and the above proof simplifies.

The purpose of c_3 was to write W as $W_1 + W_2$ where

$$c_1/|x|^2 \le W_1(x) \le c_2/|x|^2, \qquad W_2 \in L^\infty(\mathbf{R}^n).$$

The bounded potential W_2 is a small perturbation of $H_0 + W_1$ from the viewpoint of selfadjointness. Various unbounded potentials could take its place.

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