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COMPLETELY BOUNDED LINEAR EXTENSIONS OF OPERATOR-VALUED FUNCTIONS ON *-SEMIGROUPS

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ABSTRACT. Let G be a unital *-semigroup [7, p. 1] in a unital (complex) C^*-algebra such that the linear span of G is norm dense in it. Extending the results of [6], we have completely bounded linear extension theorems of operator-valued functions on G. Applying extension theorems, we have that each regular bounded operator measure has the form $V_1^* F(\cdot) V_2$, where V_1 and V_2 are linear operators and F is a selfadjoint spectral operator measure.

1. INTRODUCTION

Stinespring [8] defined completely positive linear maps on C^*-algebras and then generalized Naimark's dilation theorem for positive operator-valued measures. Arveson [1] further broadened the connection between completely positive maps and dilation theory of operator algebras. Recently, Paulsen [3] has shown that much of the theory of completely positive maps can be quite easily extended to a considerably broader class of maps, the completely bounded maps. The general dilation theorem of Sz.-Nagy [9] says that an operator-valued function on a *-semigroup is dilatable if and only if it is positive definite and satisfies the boundedness condition. F. H. Szafraniec [7] gives the equivalent conditions of boundedness. In this paper we show that the two conditions of [6, (7) and (8)] are equivalent to the simpler conditions which we call the M-property. Using the M-property, we extend the work of Z. Sebesty′en [6] for the completely positive case, which gives a generalization of the theorems of Stinespring [8] and Sz.-Nagy [9] to the completely bounded case. Let M_n denote the C^*-algebra of complex $n \times n$ matrices. Let A and B be C^*-algebras and let $L: A \to B$ be a bounded linear map. If for the maps

$$L \otimes I_n : A \otimes M_n \to B \otimes M_n ,$$

one has that $\sup \|L \otimes I_n\|$ is finite, then L is called completely bounded. The map L is called positive provided that $L(a)$ is positive whenever a is positive,

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and is called completely positive if $L \otimes I_n$ is positive for all n. We define a map $L^*: A \to B$ by $L^*(a) = L(a^*)^*$. Given $S \subseteq L(H)$, we let S' denote its commutant.

2. Extension theorems

Throughout this section let A be a unital C^*-algebra, let G be a unital *-semigroup in A such that the linear span of G is norm dense in A, and let $L(H)$ be the algebra of all bounded linear operators on a Hilbert space H.

Definition 2.1. A map $f: G \to L(H)$ has the M-property provided there exists $M > 0$ such that

$$\frac{1}{M}(f(a_i)^* f(a_j)) \leq (f(a_i^* a_j)) \quad \text{and} \quad \left\| \sum_i c_i f(a_i) \right\| \leq M \left\| \sum_i c_i a_i \right\|,$$

for any finite sequences $\{a_i\} \subset G$ and complex numbers $\{c_i\}$.

Definition 2.2. The numerical radius of an operator T in $L(H)$ is defined by

$$W(T) = \sup\{|\langle T\xi, \xi\rangle| : \xi \in H, \|\xi\| = 1\}.$$

Proposition 2.3. *Let $f: G \to L(H)$ be a map. The following two statements are equivalent:*

(i) *f has the M-property;*

(ii) *f satisfies the following two conditions* [6, (7) *and* (8)]:

$$\text{(7)} \qquad \frac{1}{M}\left\| \sum_i f(a_i)\xi_i \right\|^2 \leq \sum_{i,j} \langle f(a_i^* a_j)\xi_j, \xi_i\rangle, \text{ and}$$

$$\text{(8)} \qquad \sum_{i,j} \bar{c}_i c_j \langle f(a_i^* a_j)\xi, \xi\rangle \leq M\|\xi\|^2 \left\| \sum_i c_i a_i \right\|^2 \qquad (\xi \in H)$$

for any finite sequences $\{\xi_i\} \subset G$ and complex numbers $\{c_i\}$.

Proof. It is obvious that [6, (7)] is equivalent to

$$\frac{1}{M}(f(a_i)^* f(a_j)) \leq (f(a_i^* a_j)).$$

Since

$$M\left\| \sum_i c_i a_i \right\|^2 \geq W\left(\sum_{i,j} \bar{c}_i c_j f(a_i^* a_j) \right)$$

$$\geq \sum_{i,j} \left\langle f(a_i^* a_j)\frac{c_j \xi}{\|\xi\|}, \frac{c_i \xi}{\|\xi\|} \right\rangle \geq \frac{1}{M}\left\| \sum_i c_i f(a_i)\frac{\xi}{\|\xi\|} \right\|^2,$$

where $\xi \neq 0$ and ξ is in H. We have

$$M\left\| \sum_i c_i a_i \right\|^2 \geq \frac{1}{M}\left\| \sum_i c_i f(a_i) \right\|^2.$$

Hence (ii) $\Rightarrow$ (i).

$$W\left(\sum_{i,j}\bar{c}_i c_j f(a_i^* a_j)\right) = \left\|\sum_{i,j}\bar{c}_i c_j f(a_i^* a_j)\right\|$$
$$< M\left\|\sum_{i,j}\bar{c}_i c_j a_i^* a_j\right\| = M\left\|\sum_i c_i a_i\right\|^2.$$

Hence (i) $\Rightarrow$ (ii).

Z. Sebesty′en [6, Theorem 3] proves the following theorem.

Theorem 2.4. *Let $f: G \to L(H)$. The following three statements are equivalent:*

(i) *f has the M-property;*
(ii) *f has a completely positive linear extension $\phi: A \to L(H)$;*
(iii) *there exist a Hilbert space K, a *-homomorphism $\pi: A \to L(K)$, and a linear operator $W: H \to K$ such that $f(\cdot) = W^*\pi|_G(\cdot)W$ and* $\operatorname{span}\pi(A)WH$ *is dense in K.*

The following theorem extends Theorem 2.4.

Theorem 2.5. *Let $f: G \to L(H)$. The following statements are equivalent:*

(i) *$f = \phi_1 - \phi_2 + i(\phi_3 - \phi_4)$, where ϕ_i has the M-property ($i = 1, 2, 3, 4$);*
(ii) *f has a completely bounded linear extension $\phi: A \to L(H)$;*
(iii) *there exist maps ψ_1 and ψ_2 with the M-property such that the map*

$$\begin{pmatrix}\psi_1 & f\\ f^* & \psi_2\end{pmatrix}: G \to L(H)\otimes M_2$$

defined by

$$\begin{pmatrix}\psi_1 & f\\ f^* & \psi_2\end{pmatrix}(g) = \begin{pmatrix}\psi_1(g) & f(g)\\ f(g^*)^* & \psi_2(g)\end{pmatrix}$$

has the M-property;
(iv) *there exist a Hilbert space K, a *-homomorphism $\pi: A \to L(K)$ and linear operators $V: H \to K$, $T: K \to K$ such that*

$$f(\cdot) = V^*T\pi|_G(\cdot)V \text{ with } T \in \pi(A)';$$

(v) *there exist a Hilbert space $\widetilde{K}$, a *-homomorphism $\tilde{\pi}: A \to L(\widetilde{K})$, and linear operators V_1, V_2 such that $f(\cdot) = V_1^*\tilde{\pi}|_G(\cdot)V_2$.*

Proof. Applying Theorem 2.4, we have (i) $\Rightarrow$ (ii). Applying [10, Satz 4.5] or [3, Theorem 2.6], we have (ii) $\Rightarrow$ (i). Applying [3, Theorem 2.5], there exist completely positive maps ψ_1 and ψ_2 such that the map

$$\begin{pmatrix}\psi_1 & \phi\\ \phi^* & \psi_2\end{pmatrix}: A \to L(H)\otimes M_2$$

is completely positive. Hence the map $\left(\begin{smallmatrix}\psi_1 & f\\ f^* & \psi_2\end{smallmatrix}\right): G \to L(H)\otimes M_2$ has the M-property. Thus, we have (ii) $\Rightarrow$ (iii). Applying [5, Theorem 2.10], there exist a Hilbert space K, a *-homomorphism $\pi: A \to L(K)$ and linear operators $V: H \to K$, $T: K \to K$ such that $\phi = V^* T \pi V$ with $T \in \pi(A)'$. Hence (ii) $\Rightarrow$ (iv). Since the map

$$\begin{pmatrix} \psi_1 & f \\ f^* & \psi_2 \end{pmatrix} : G \to L(H) \otimes M_2$$

has the M-property, by Theorem 2.4, it has a completely positive linear extension $\phi: A \to L(H) \otimes M_2$. By the Stinespring Theorem, there exist a Hilbert space $\widetilde{K}$, a *-homomorphism $\tilde{\pi}: A \to L(\widetilde{K})$, and a linear operator $\widetilde{V}: H \oplus H \to \widetilde{K}$ such that

$$\begin{pmatrix} \psi_1 & f \\ f^* & \psi_2 \end{pmatrix} (\cdot) = \widetilde{V}^* \tilde{\pi}(\cdot) \widetilde{V}.$$

Let $V_1 h = \widetilde{V}(h \oplus 0)$ and $V_2 h = \widetilde{V}(0 \oplus h)$; we have

$$\begin{aligned} \langle V_1^* \tilde{\pi}(g) V_2 h, k\rangle &= \langle \tilde{\pi}(g) \widetilde{V}(0 \oplus h), V_1 k\rangle = \langle \widetilde{V}^* \tilde{\pi}(g) \widetilde{V}(0 \oplus h), k \oplus 0\rangle \\ &= \left\langle \begin{pmatrix} \psi_1(g) & f(g) \\ f^*(g) & \psi_2(g) \end{pmatrix} (0 \oplus h), (k \oplus 0) \right\rangle = \langle f(g) h, k\rangle, \end{aligned}$$

where $g \in G$ and $h, k \in H$. Hence

$$f(\cdot) = V_1^* \tilde{\pi}|_G(\cdot) V_2.$$

Thus, we have (iii) $\Rightarrow$ (v). (iv) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (ii) are obvious.

3. Applications

Let X be a compact Hausdorff space and B the σ-algebra of Borel subsets of X. An $L(H)$-valued measure on X is a map $E: B \to L(H)$ which is weakly countably additive; that is, if $\{O_i\}$ is a countable collection of disjoint Borel sets with union O, then

$$\langle E(O)x, y\rangle = \sum_i \langle E(O_i)x, y\rangle$$

for all x, y in H. The measure is bounded provided that

$$\|E\| = \sup\{\|E(O)\|: O \in B\} < \infty.$$

The measure is regular, provided that for all x, y in H the complex measure given by $E_{x,y}(\cdot) = \langle E(\cdot)x, y\rangle$ is regular. If E is a regular, bounded $L(H)$-valued measure, then E is called:

(i) spectral, if $E(M \cap N) = E(M) \cdot E(N)$;
(ii) positive, if $E(M) \geq 0$; and
(iii) selfadjoint, if $E(M)^* = E(M)$, for all Borel sets M and N.

Let G be the *-semigroup of all characteristic functions, that is,

$$G = \{\chi_O: O \text{ is a Borel set}\},$$

then the linear span of G is norm dense in the C^*-algebra $M(X)$ of all bounded measurable functions on X. Combining Hadwin [2, Theorem 20], Paulsen [4, pp. 107–110], and Theorem 2.5, we obtain the following theorem.

Theorem 3.1. *Let E be a regular, bounded $L(H)$-valued measure. The following statements are equivalent*:

(i) *E has a Hahn decomposition $E = (E_1 - E_2) + i(E_3 - E_4)$, where*

$$E_i \ (i = 1, 2, 3, 4) \text{ are positive measures on } B;$$

(ii) *there exist positive measures F_1 and F_2 such that*

$$\begin{pmatrix} F_1(\cdot) & E(\cdot) \\ E(\cdot)^* & F_2(\cdot) \end{pmatrix}$$

is positive;

(iii) *there exist a Hilbert space K, a selfadjoint, spectral, $L(K)$-valued measure F on X, and linear operators $V: H \to K$, $T: K \to K$ such that $E(\cdot) = V^*TF(\cdot)V$ with $TF(\cdot) = F(\cdot)T$*;

(iv) *there exist a Hilbert space K', a selfadjoint, spectral, $L(K')$-valued measure F' on X, and linear operators $V_i: H \to K'$ $(i = 1, 2)$ such that $E(\cdot) = V_1^* F'(\cdot) V_2$*;

(v) *there exist a Hilbert space $H_1 \supset H$ and a spectral measure $\widetilde{F}_1$ such that $P\widetilde{F}_1(\cdot)|_H = E(\cdot)$, where P is an orthogonal projection of H_1 onto H.*

Proof. From [2, Theorem 20], we know that (i) and (v) are equivalent. From [4, pp. 107–110], we know that (i) and (iv) are equivalent. Applying Theorem 2.5 with $E(O) = f(\chi_O)$, we have that (i), (ii), and (iii) are equivalent.

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