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ON PROJECTIONS IN POWER SERIES SPACES AND THE EXISTENCE OF BASES

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ABSTRACT. Mityagin posed the problem, whether complemented subspaces of nuclear infinite type power series spaces have a basis. A related more general question was asked by Pełczyński. It is well known for a complemented subspace E of a nuclear infinite type power series space, that its diametral dimension can be represented by $\Delta E = \Delta\Lambda_\infty(\alpha)$ for a suitable sequence α with $\alpha_j \geq \ln(j+1)$. In this article we prove the existence of a basis for E in case that $\alpha_j \geq j$ and $\sup \frac{\alpha_{2j}}{\alpha_j} < \infty$.

It was shown by Mityagin, that complemented subspaces of nuclear finite type power series spaces always have a basis, and he asked, whether the same is valid for infinite type (cf. [**3**, **4**, **5**]). Dubinsky and Vogt [2] obtained a positive solution for some nuclear power series spaces $\Lambda_\infty(\alpha)$, namely they assumed that the set of all finite limit points of $\{\frac{\alpha_i}{\alpha_j} : i, j \in \mathbf{N}\}$ is bounded. Results for some other special cases are stated below. Pełczyński [6] posed the more general problem, whether complemented subspaces of nuclear Fréchet spaces with basis again have a basis. Both problems are open up to now.

Every complemented subspace E of a nuclear infinite type power series space has the same diametral dimension as a power series space $\Lambda_\infty(\alpha)$ for a suitable sequence α with $\alpha_j \geq \ln(j+1)$ (cf. Terzioglu [8]). Considering isomorphisms between spaces of analytic functions Zaharjuta [15] conjectured that E has a basis for stable α (this means $\sup_j \frac{\alpha_{2j}}{\alpha_j} < \infty$). This will be proved in the present note in case $\alpha_j \geq j$. Other positive solutions have been obtained if one of the three following assumptions is satisfied:

(1) $\Lambda_\infty(\alpha)$ is a complemented subspace of E (cf. Vogt [10]).

(2) There is a tame projection onto E (cf. Dubinsky/Vogt [2]).

(3) E is isomorphic to $E \oplus E$ (cf. Wagner [14], see also [11]).

The present proof uses result no. 1 of Vogt [10]. The main tool is the construction of a basis and of a projection in E by a permuted Gram-Schmidt orthonormalization.

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Notation. See also Dubinsky [1] and Terzioglu [7]. Let α denote an exponens sequence, this is a nondecreasing, unbounded sequence of positive numbers.

A power series space of infinite type is defined by

$$\Lambda_\infty(\alpha) := \left\{ x = (x_i)_{i \notin \mathbf{N}} : \|x\|_k := \sum_{i=1}^{\infty} |x^i| \exp(k\alpha_i) < \infty \quad \text{all } k \notin \mathbf{N} \right\}.$$

For $\alpha_j = \ln(j+1)$ is $\Lambda_\infty(\alpha) = s$ and for $\alpha_j = (j)^{1/n}$ is $\Lambda_\infty(\alpha) \cong H(\mathbf{C}^n)$, especially $\Lambda_\infty(\mathbf{N}) \cong H(\mathbf{C})$.

Let E be a Fréchet space with a fundamental system of seminorms $\| \; \|_k$, $k \notin \mathbf{N}$. The corresponding neighbourhoods of zero are denoted by U_k.

$$E \text{ has } DN \text{ iff } \exists p \forall k \exists m, C : x \notin E \|x\|_k^2 \leq C \|x\|_m \|x\|_p .$$

$$E \text{ has } \Omega \text{ iff } \forall p \exists k \forall m \exists n, C : rU_k < Cr^n U_m + C/rU_p .$$

In the nuclear case the condition DN is characteristic for the subspaces of s (cf. Vogt [9]) and Ω for the quotients of s (cf. Vogt/Wagner [12]).

The Diametral Dimension ΔE is defined by

$$\Delta E := \left\{ (x_i)_{i \notin \mathbf{N}} : \forall k \exists m : \lim_i |x_i| d_i(U_m, U_k) = 0 \right\},$$

where the Kolmogorov diameters are $d_i(V, U) := \inf\{d > 0 : V \subset dU + L$ with $L \subset E$ and its dimension $\leq i\}$. The main part of this note is the proof of the following proposition.

Proposition 1. *For an exponens sequence α with $\sup_i \frac{\alpha_{2i}}{\alpha_i} < \infty$ and $\alpha_i \geq i$ for all $i \notin \mathbf{N}$ every quotient E of $\Lambda_\infty(\alpha)$ with DN and with $\Delta E = \Delta\Lambda_\infty(\alpha)$ has a complemented subspace isomorphic to $\Lambda_\infty(\alpha)$.*

Proof. Let $U_k \supset U_{k+\mu}$, $k \notin \mathbf{N}$, Hilbert balls and a fundamental system of neighbourhoods of zero in E so that for all k there are m, C with $\|x\|_k^2 \leq C\|x\|_m \|x\|_0$ for all $x \notin E$.

The surjection from $\Lambda_\infty(\alpha)$ onto E is denoted by q and there is a basis $(e_i)_{i \notin \mathbf{N}}$ in $\Lambda_\infty(\alpha)$ with $\|q(e_i)\|_0 \leq \exp(-\alpha_i)$. For every $k \notin \mathbf{N}$ there is an m_k so that $\|q(e_i)\|_k \leq \exp(m_k \alpha_i)$ for all i.

First we want to show the following lemma.

Lemma 1. *There is a number N such that for every $i \notin \mathbf{N}$ and every operator $T \notin L(\Lambda_\infty(\alpha), E)$ with* $\dim(\text{range}(T)) < i$ *we find a $j \notin \mathbf{N}$ with* $\|q(e_j) - T(e_j)\|_0 \geq \exp(-N\alpha_i)$.

Proof. If the lemma were false there would be a sequence $(i_n)_{n \notin \mathbf{N}}$ and $T_n \notin L(\Lambda_\infty(\alpha), E)$ with $\dim(\text{range}(T_n)) < i_n$ so that for all $j \notin \mathbf{N}$ $\|q(e_j) - T(e_j)\|_0 \leq \exp(-n\alpha_{i_n})$. Hence there is an m_0 with $U_{m_0} < \exp(-n\alpha_{i_n})U_0 + \text{range}(T_n)$.

Since E has property DN, for every k there are m, C with $d_{i_n}(U_m, U_k) \leq d_{i_n}(U_{m_0}, U_0)$ (e.g. see Terzioglu [8] condition (7)) and with $\exp(n\alpha_{i_n})d_{i_n}(U_m, U_k) \leq C$. Since $\Delta E = \Delta\Lambda_\infty(\alpha)$, there is a K with $n\alpha_{i_n} \leq K\alpha_{i_n}$ for all $n \notin \mathbf{N}$, hence the lemma must be true.

Lemma 2. *There is an orthonormal system* $(f_i)_{i \notin \mathbf{N}}$ *in* $(E, \|\ \|_0)$ *and a sequence of positive numbers* $(\mu_i)_{i \notin \mathbf{N}}$, *such that for all* $j \notin \mathbf{N}$

(a) $\|(id - P_{j-1})qe_i\|_0 \leq \frac{1}{\mu_j}$ *for all* i *and for* $P_\nu(x) := \sum_{k=1}^{\nu}(f_k, x)_0 f_k$.

(b) $\mu_j \leq \exp(N\alpha_j)$

(c) $\|f_j\|_k \leq \mu_j 2^j \exp(m_k N\alpha_j)$ *for all* k.

Proof. We want to prove Lemma 2 by induction over j. Since

$$0 \leq \lim_i \|(id - P_{j-1})qe_i\|_0 \leq \lim_i \|qe_i\|_0 = 0,$$

there is an n_j with $\|(id - P_{j-1})qe_{n_j}\|_0 = \sup_i \|(id - P_{j-1})qe_i\|_0$. Let

$$\mu_j := \frac{1}{\|(id - P_{j-1})qe_{n_j}\|_0} \qquad \text{and} \qquad f_j := \mu_j(id - P_{j-1})qe_{n_j}.$$

Then f_j is orthogonal to $f_1, f_2, \ldots, f_{j-1}$, $\|f_j\|_0 = 1$ and (a) is valid. Since $\exp(-N\alpha_j) \leq \sup_i \|(id - P_{j-1})qe_i\|_0 = \frac{1}{\mu_j}$, (b) is satisfied.

We use the following estimates to show (c):

$$\begin{aligned}
\|(id - P_{j-1})qe_{n_j}\|_k &\leq \|q(e_{n_j})\|_k + \sum_{\nu=1}^{j-1} |(f_\nu, qe_{n_j})_0|\, \|f_\nu\|_k \\
&\leq \|q(e_{n_j})\|_k + \sum_{\nu=1}^{j-1} |(f_\nu, (id - P_{\nu-1})qe_{n_j})_0|\, \|f_\nu\|_k \\
&\leq \exp(m_k\alpha_{n_j}) + \sum_{\nu=1}^{j-1} \frac{1}{\mu_\nu}\mu_\nu 2^\nu \exp(m_k N\alpha_\nu) \\
&\leq \exp(m_k\alpha_{n_j}) + \exp(m_k N\alpha_j)(2^j - 1).
\end{aligned}$$

Hence it is sufficient to show $\alpha_{n_j} \leq N\alpha_j$. But this fact follows from $\exp(-N\alpha_j) \leq \|(id - P_{j-1})qe_{n_j}\|_0 \leq \|qe_{n_j}\|_0 \leq \exp(-\alpha_{n_j})$.

Lemma 3. *There is a projection* P *in* E *with* range(P) *isomorphic to* $\Lambda_\infty(\alpha)$.

Proof. Let $n_k := (m_k N + N + 2)C$ with $\sup_j \frac{\alpha_{2j}}{\alpha_j} \leq C$. Now we construct an orthonormal sequence $(g_j)_{j \notin \mathbf{N}}$ in E with respect to $(\ ,\ )_0$ so that g_j is orthogonal to $g_1, g_2, \ldots, g_{j-1}$, $qe_1, qe_2, \ldots, qe_j$ and $g_j = \sum_{i=1}^{2j} |x_i| f_i$ with $\sum_{j=1}^{2j} |x_i|^2 = 1$. Then

$$\begin{aligned}
\|g_j\|_k &\leq \sum_{i=1}^{2j} |x_i|\, \|f_i\|_k \leq 2j \max_{i \leq 2j} \mu_i 2^i \exp(m_k N\alpha_i) \\
&\leq \exp(n_k\alpha_j)
\end{aligned}$$

and

$$\left\|\left(g_j\,,q\left(\sum_{i=1}^{\infty}x_ie_i\right)\right)_0 g_j\right\|_k \le \sum_{i=j+1}^{\infty}|x_i(g_j\,,qe_i)_0|\,||g_j||_k$$
$$\le \sum_{i=j+1}^{\infty}|x_i|\exp(-\alpha_j)\exp(n_k\alpha_j)$$
$$\le \exp(-\alpha_j)\sum_{i=1}^{\infty}|x_i|\exp(n_k\alpha_i) \le \exp(-\alpha_j)||x||_{n_k}\,.$$

Hence $P(x) := \sum_{j=1}^{\infty}(g_j\,,x)_0 g_j$ defines a projection in E and $(g_j)_{j\notin \mathbf{N}}$ is a basis in the range of P. For $a_{j,k} := ||g_j||_k$ is $\lambda(A) \cong \text{range}(P)$ a Köthe sequence space with DN, Ω and $\Delta\lambda(A) \supset \Delta E = \Delta\Lambda_\infty(\alpha)$. Since $||g_j||_0 = 1$ and $||g_j||_k \le \exp(n_k\alpha_j)$ we obtain $\Delta\lambda(A) \subset \Delta\Lambda_\infty(\alpha)$, hence $\Lambda_\infty(\alpha) \cong \lambda(A) \cong \text{range}(P)$.

Now we apply Proposition 1 to show our main result.

Theorem 1. *For an exponens sequence with* $\sup_j \frac{\alpha_{2j}}{\alpha_j} < \infty$ *and* $\alpha_j \ge j$ *for all* $j \notin \mathbf{N}$ *a Fréchet space* E *is isomorphic to* $\Lambda_\infty(\alpha)$ *if and only if* E *has the properties* DN, Ω *and* $\Delta E = \Delta\Lambda_\infty(\alpha)$.

Corollary. *A Fréchet space* E *is isomorphic to* $H(\mathbf{C})$ *if and only if* E *has the properties* DN, Ω *and* $\Delta E = \Delta\Lambda_\infty(\mathbf{N})$.

Proof. If E has the properties DN, Ω and $\Delta E = \Delta\Lambda_\infty(\alpha)$, then due to Vogt/Wagner [13] E is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$. On the other hand Proposition 1 shows that $\Lambda_\infty(\alpha)$ is isomorphic to a complemented subspace of E. Hence Vogt [10] yields that E is isomorphic to $\Lambda_\infty(\alpha)$ (cf. result no. 1 stated in the introduction).

In the proof of the theorem we can use instead of Proposition 1 the following modification:

Proposition 1*. *Let* α *be an exponens sequence with* $\sup_j \frac{\alpha_{2j}}{\alpha_j} < \infty$ *and* $\alpha_j \ge j$ *for all* $j \notin \mathbf{N}$, *and let* E *be a subspace of* $\Lambda_\infty(\alpha)$. *If* $\Delta E = \Delta\Lambda_\infty(\alpha)$ *and if* E *is isomorphic to a quotient of* $\Lambda_\infty(\alpha)$, *then there is a complemented subspace of* $\Lambda_\infty(\alpha)$, *which is both contained in* E *and isomorphic to* $\Lambda_\infty(\alpha)$.

Proof. In the proof of Proposition 1, Lemma 3 we construct a g_j orthogonal to $g_1, g_2, \dots, g_{j-1}, qe_1, qe_2, \dots, qe_j$. For a proof of Proposition 1* we merely have to exchange this by g_j orthogonal to $g_1, g_2, \dots, g_{j-1}, e_1, e_2, \dots, e_j$.

Finally we want to point out that it is an interesting problem to prove Proposition 1 or 1* for some other exponens sequences. Especially we want to know, whether it is sufficient to assume $\alpha_j \ge \ln(j+1)$ instead of $\alpha_j \ge j$. This was stated without proof by Zaharjuta [15], for his work we need in particular the cases $\alpha_j = (j)^{1/n}$. Most important are sequences $\alpha_j = a^j$, because then

Proposition 1* would solve the above mentioned prob'lems of Mityagin and Pełczyński.

Theorem 2. *If the claim made in Proposition* 1^* *were true for* $\alpha_j = a^j$ *with an* $a > 1$, *then there would be a complemented subspace of* s *without basis.*

Proof. ω is the set of all sequences endowed with the seminorms $\|(x_i)_{i\in\mathbf{N}}\|_k := \max_{i\leq k}|x_i|$ for all $k \in \mathbf{N}$. For every $a > 1$ and $\alpha_j := a^j$ there is a surjection T from $\Lambda_\infty(\alpha)$ onto ω, so that the kernel of T called K has the properties Ω and $\Delta K = \Delta\Lambda_\infty(\alpha)$. This result is proved by Vogt [8]. Now we want to consider the case that every complemented subspace of s has a basis, hence that K is isomorphic to $\Lambda_\infty(\alpha)$. If Proposition 1* were true for this α, then there would be a projection P in $\Lambda_\infty(\alpha)$, so that the range of P is contained in K and isomorphic to $\Lambda_\infty(\alpha)$. The first condition implies that T restricted to F =: range($id - P$) is still a surjection onto ω. Since $F \oplus \Lambda_\infty(\alpha) \cong F \oplus \text{range}(P) = \Lambda_\infty(\alpha)$, this yields a contradiction, because here $\alpha_j = a^j$ with $a > 1$, so that $\Lambda_\infty(\alpha)$ is not isomorphic to $\Lambda_\infty(\alpha)\oplus\Lambda_\infty(\beta)$ for every exponens sequence β.

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