PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 2, February 1989

A CHARACTERIZATION OF WEAK PSEUDOCONVEXITY

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(Communicated by Irwin Kra)

ABSTRACT. It is proved that a smooth domain D of $\mathbf{C}^n$ is weakly pseudoconvex, if, for every strongly pseudoconvex domain D' with $D \cap D' = \varnothing$ and $E = \overline{D} \cap \overline{D}' \neq \varnothing$, E is totally real.

Let D be a bounded domain of $\mathbf{C}^n$ with C^2 boundary and p a point of ∂D. D is said to be *weakly* (or *strongly*) *pseudoconvex* at p if, for every C^2 function ρ on an open neighborhood U of p in $\mathbf{C}^n$ such that $D \cap U = \{z \in U : \rho(z) < 0\}$ and $d\rho(p) \neq 0$, the Levi form

$$L[\rho;\zeta](p) = \sum_{j\ k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p)\zeta_j\bar{\zeta}_k$$

is nonnegative (or positive, resp.) for any nonzero vector ζ with $\sum_j \frac{\partial \rho}{\partial z_j}(p)\zeta_j = 0$. We say that D is weakly (or strongly) pseudoconvex if D is weakly (or strongly resp.) pseudoconvex at every point of ∂D. A subset T is called a *totally real set*, if it is the zero set of a nonnegative C^2 strongly plurisubharmonic function on an open neighborhood of T. A C^1 submanifold M is totally real if and only if it has no nonzero complex tangent vectors.

It is known that if D is a weakly pseudoconvex domain and D' is a strongly pseudoconvex domain with $D \cap D' = \varnothing$ then $T = \overline{D} \cap \overline{D}'$ is a totally real set, provided T is nonempty (see [1] and [2]). As a converse of this fact, we prove the following theorem.

Theorem. *Let D be a domain with C^3 boundary. If, for every strongly pseudoconvex domain D' such that $D \cap D' = \varnothing$ and $M = \overline{D} \cap \overline{D}'$ is a real C^1 submanifold, M is totally real, then D is weakly pseudoconvex.*

Proof. We assume that $n > 1$, since, in the case $n = 1$, all domains with C^2 boundaries are strongly pseudoconvex. Let $z = (z_1, \dots, z_n)$, $z_j = x_j + iy_j$, $j = 1, \dots, n$, denote the complex coordinates of $\mathbf{C}^n$. Suppose that D is not

Received by the editors January 29, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32F15.
Key words and phrases. Weakly and strongly pseudoconvex domain, Levi form, totally real.

weakly pseudoconvex at p. After a suitable holomorphic change of coordinates, we can assume that p is the origin and that D is locally written as

$$y_n < \phi(z_1, z', x_n), \qquad z' = (z_2, \dots, z_{n-1}),$$

where ϕ is a C^3 function on an open neighborhood N of the origin in the hypersurface $y_n = 0$ whose derivatives of the first order at the origin are all zero and which satisfies

$$\frac{\partial^2 \phi}{\partial z_1 \partial \bar{z}_1}(0, 0, 0) > 0.$$

By expanding ϕ in a Taylor series, we have

$$\phi(z_1, z', x_n) = \phi(z_1, 0, 0) + \sum_{j=2}^{n-1} \frac{\partial \phi}{\partial z_j}(z_1, 0, 0) z_j + \sum_{j=2}^{n-1} \frac{\partial \phi}{\partial \bar{z}_j}(z_1, 0, 0) \bar{z}_j + \frac{\partial \phi}{\partial x_n}(z_1, 0, 0) x_n + O\left(\sum_{j=2}^{n-1} |z_j|^2 + x_n^2\right).$$

We define the function

$$\psi(z_1, z', x_n) = \phi(z_1, 0, 0) + \sum_{j=2}^{n-1} \frac{\partial \phi}{\partial z_j}(z_1, 0, 0) z_j + \sum_{j=2}^{n-1} \frac{\partial \phi}{\partial \bar{z}_j}(z_1, 0, 0) \bar{z}_j + \frac{\partial \phi}{\partial x_n}(z_1, 0, 0) x_n + c\left(\sum_{j=2}^{n-1} |z_j|^2 + x_n^2\right),$$

where c is a positive constant. If c is sufficiently large, then we have $\psi \geq \phi$. The equality holds just when $z_2 = \cdots = z_{n-1} = x_n = 0$. We put $\sigma = \psi(z_1, z', x_n) - y_n$ and $D_1 = \{z : \sigma(z) < 0\}$. Then the intersection $\overline{D} \cap \overline{D}_1$ is the manifold

$$M = \{z : z_2 = \cdots = z_{n-1} = x_n = 0,\ \phi(z_1, z', x_n) = y_n\}.$$

For every vector ζ, we have

$$L[\sigma, \zeta](0) = \frac{\partial^2 \phi}{\partial z_1 \partial \bar{z}_1}(0, 0, 0)|\zeta_1|^2 + 2\,\mathrm{Re}\left[\sum_{j=2}^{n-1} \frac{\partial^2 \phi}{\partial z_1 \partial \bar{z}_j}(0, 0, 0)\zeta_1 \bar{\zeta}_j\right] + \mathrm{Re}\left[\frac{\partial^2 \phi}{\partial z_1 \partial x_n}(0, 0, 0)\zeta_1 \bar{\zeta}_n\right] + c\left[\sum_{j=2}^{n-1} |\zeta_j|^2 + \frac{1}{4}|\zeta_n|^2\right].$$

We write the right member as $F(\zeta) + cG(\zeta)$. Then F and G are continuous on $\Gamma = \{\zeta \in C^n : |\zeta| = 1\}$ and G is nonnegative. When $G(\zeta) = 0$, we have

$$F(\zeta) = \frac{\partial^2 \phi}{\partial z_1 \partial \bar{z}_1}(0, 0, 0)|\zeta_1|^2 > 0.$$

Therefore, we can find a constant c so that $L[\sigma;\zeta](0) > 0$ for every nonzero vector ζ. Hence σ is strongly plurisubharmonic in an open neighborhood of the origin.

Thus we can find an open neighborhood V of the origin and a strongly pseudoconvex domain D' contained in D_1 such that $\partial D' \cap V$ coincides with $\partial D_1 \cap V$. The tangent space of the manifold $M \cap V$ at the origin is the z_1-plane and hence $M \cap V$ is not totally real. This proves the theorem.

We remark that the theorem is also valid for domains of complex manifolds, since all the arguments are quite local.

REFERENCES

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