PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 2, February 1989

κ-METRIZABLE SPACES, STRATIFIABLE SPACES AND METRIZATION

J. SUZUKI, K. TAMANO, AND Y. TANAKA

(Communicated by Dennis K. Burke)

ABSTRACT. It is shown that every κ-metrizable CW-complex is metrizable. Examples are given showing that a stratifiable κ-metrizable space and an additively κ-metrizable space need not be metrizable.

1. INTRODUCTION

Let X be a topological space, $\mathscr{F}$ a family of closed subsets of X. A nonnegative real valued function $\phi: X \times \mathscr{F} \to R$ is called an *annihilator* for $\mathscr{F}$ [3] if $\phi(x, F) = 0$ if and only if $x \in F$. Various important classes of generalized metric spaces can be characterized by means of annihilators. κ-metrizable spaces [13], stratifiable spaces [2, 1] and continuously perfectly normal spaces [17] are such spaces. An interesting problem naturally arises:

General Problem. What condition of an annihilator on a space X implies the metrizability of X?

Several answers are known. For example, P. Zenor [17] proved that a space X is metrizable if and only if it has a monotone bicontinuous annihilator for the family 2^X of all closed subsets of X.

Recently T. Isiwata obtained the following metrization theorem which also can be translated into words of annihilators:

Theorem [9]. *A space is metrizable if it is stratifiable and additively κ-metrizable.*

We are concerned with the following two questions:

Question 1 [14]. Is a space metrizable if it is stratifiable and κ-metrizable?

Question 2. Is a space metrizable if it is additively κ-metrizable?

Lašnev spaces (=images of metric spaces under closed continuous mappings) and CW-complexes are typical examples of stratifiable spaces. The second author [14] showed that a κ-metrizable Lašnev space is metrizable. We show that

Received by the editors March 27, 1986 and, in revised form, March 24, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54E35, 54E20.
Key words and phrases. κ-metrizable space, stratifiable space, annihilator, metrization.

a κ-metrizable CW-complex is metrizable, which is another partial positive answer to Question 1. But Question 1 itself has a negative answer. Indeed we construct a space which is stratifiable and κ-metrizable but is not metrizable.

Answering Question 2, we show that the Sorgenfrey line, which is a Lindelöf nonmetrizable space, is additively κ-metrizable.

All spaces are assumed to be regular T_1. The letter N and the letter R denote the set of natural numbers and the set of real numbers respectively.

2. Preliminaries

Let X be a space and $\mathscr{F}$ a family of closed subsets of X. A nonnegative real valued function $\phi: X \times \mathscr{F} \to R$ is an *annihilator* for $\mathscr{F}$ if it satisfies that $\phi(x, F) = 0$ if and only if $x \in F$.

For a family $\mathscr{F}$ of closed subsets of X, we use 2^X, the family of all closed subsets of X; and $R[X]$, the family of all regular closed subsets of X. We consider the following properties of an annihilator ϕ:

An annihilator ϕ is *continuous* if $\phi(\cdot, F)$ is continuous in a variable x for each $F \in \mathscr{F}$. An annihilator ϕ is *bicontinuous* if $\phi(x, F)$ is continuous on the product space $X \times \mathscr{F}$ of X and the space $\mathscr{F}$ with the Vietoris topology (=finite topology=exponential topology). An annihilator ϕ is *monotone* if $H \subset F$ implies $\phi(x, H) \geq \phi(x, F)$ for each $x \in X$. An annihilator ϕ is *linearly additive* if for each subfamily $\mathscr{H}$ of $\mathscr{F}$ which is linearly ordered by inclusion, $\phi(x, \operatorname{cl}\bigcup\mathscr{H}) = \inf\{\phi(x, F): F \in \mathscr{H}\}$. An annihilator ϕ is *additive* if for each subfamily $\mathscr{H}$ of $\mathscr{F}$, $\phi(x, \operatorname{cl}\bigcup\mathscr{H}) = \inf\{\phi(x, F): F \in \mathscr{H}\}$.

It should be noted that the distance function $d: X \times 2^X \to R$ of a metric space X satisfies all of the above properties.

By the use of these definitions, let us define various generalized metric spaces:

A space X is *κ-metrizable* if it has a continuous, monotone and linearly additive annihilator for $R[X]$, which is called a *κ-metric.* A space with an additive κ-metric is called *additively κ-metrizable.* A space X is *stratifiable* if it has a continuous and monotone annihilator for 2^X. A space X is *continuously perfectly normal* if it has a bicontinuous annihilator for 2^X.

3. κ-metrizable CW-complexes

In this section we give partial positive answers to Question 1 in the introduction. We start with the definition of the well-known examples S and S_2. The *sequential fan* S is the quotient space obtained from the topological sum of countably many convergent sequences by identifying all the limit points. The *Arens' space* $S_2 = (N \times N) \cup N \cup \{\infty\}$ is the space with each point of $N \times N$ isolated. A basis of neighborhoods of $n \in N$ consists of all sets of the form $\{n\} \cup \{(m, n): m \geq k\}$. And U is a neighborhood of ∞ if and only if $\infty \in U$ and U is a neighborhood of all but finitely many $n \in N$.

A space X is *sequential* if a set $A \subset X$ is closed if and only if no sequence in A converges to a point not in A. A space X is *Fréchet* if for every $A \subset X$ and every $x \in \operatorname{cl} A$, there exists a sequence $\{x_n : n \in N\}$ in A converging to x. A space X is *strongly Fréchet* [12] (=countably bi-sequential [11]) if whenever $\{A_n : n \in N\}$ is a decreasing sequence of sets in X and $x \in \bigcap\{\operatorname{cl} A_n : n \in N\}$, there exists an $x_n \in A_n$ such that the sequence $\{x_n : n \in N\}$ converges to x.

Combining [10, Corollary 2.3] and [15, Theorem 1.5], we have

Lemma 3.1. *Let X be a hereditarily normal sequential space. Then X is strongly Fréchet if and only if X contains no copy of S and no copy of S_2.*

A space X is *monotonically normal* [6] if to each pair (H, K) of disjoint closed subsets of X, one can assign an open set $D(H, K)$ such that

(a) $H \subset D(H, K) \subset \operatorname{cl} D(H, K) \subset X - K$; and
(b) If $H \subset H'$ and $K \supset K'$, then $D(H, K) \subset D(H', K')$. The function D is called a *monotone normality operator* for X. Stratifiable spaces are known to be monotonically normal. The following lemma is easily proved:

Lemma 3.2. *Let $\mathscr{F}$ be a family of closed subsets of a space X which is closed under finite unions. Suppose that $\phi: X \times \mathscr{F} \to R$ is a monotone annihilator. Then the following are equivalent:*

(a) *for each increasing countable sequence $\mathscr{H}$ in $\mathscr{F}$ and a point $x \in \operatorname{cl} \bigcup \mathscr{H}$, $\inf\{\phi(x, F) : F \in \mathscr{H}\} = 0$;*
(b) *for each countable subfamily $\mathscr{H}$ of $\mathscr{F}$, a point $x \in \operatorname{cl} \bigcup \mathscr{H}$ and $\varepsilon > 0$, there exists a finite subfamily $\mathscr{K}$ of $\mathscr{H}$ such that $\phi(x, \bigcup \mathscr{K}) < \varepsilon$.*

Lemma 3.3. *Let X be a monotonically normal sequential space. Suppose that there is a monotone annihilator $\phi: X \times R[X] \to R$ for $R[X]$ satisfying* (a) *in Lemma* 3.2. *Then X is strongly Fréchet.*

Proof. Since every monotonically normal space is hereditarily normal, by Lemma 3.1, we need only show that X contains no copy of S and no copy of S_2. The second author [14] essentially proved that X contains no copy of S. So it is sufficient to prove that X contains no copy of S_2. Suppose the contrary. We may assume that $S_2 = (N \times N) \cup N \cup \{\infty\}$ is a subset of X. Let D be a monotone normality operator for X. Define $B_{mn} = \operatorname{cl} D(\{(m, n)\}, \{\infty\})$. Then $\{B_{mn} : m, n \in N\}$ is a family of regular closed subsets of X satisfying the following:

(a) for each $n \in N$, $n \in \operatorname{cl}(\bigcup\{B_{mn} : m \in N\})$; and
(b) for each function $f: N \to N$,

$$\infty \notin \operatorname{cl}\left(\bigcup\{B_{mn} : m, n \in N, m \leq f(n)\}\right).$$

It is easy to see (a). The property (b) follows from the property of the monotone normality operator D. Note that $\infty \notin \operatorname{cl}\{(m, n) : m, n \in N, m \leq f(n)\}$.

So whenever $m \leq f(n)$, we have $D(\{(m,n)\}, \{\infty\}) \subset D(\text{cl}\{(m,n)\colon m, n \in N, m \leq f(n)\}, \{\infty\})$, which implies

$$\bigcup\{B_{mn}\colon m, n \in N, m \leq f(n)\} \subset \text{cl}\, D(\text{cl}\{(m,n)\colon m, n \in N, m \leq f(n)\}, \{\infty\}) \subset X - \{\infty\}.$$

Hence $\infty \notin \text{cl}(\bigcup\{B_{mn}\colon m, n \in N, m \leq f(n)\})$.

Now one can choose a sequence $\{\mathscr{C}_n\colon n \in N\}$ of finite subfamilies of $\{B_{mn}\colon m, n \in N\}$ satisfying that

(c) $\phi(\infty, \bigcup\mathscr{C}_n) < 1/n$; and
(d) if $B_{ij} \in \mathscr{C}_n$, then $j \geq n$.

Indeed by (a) $\infty \in \text{cl}(\bigcup\{B_{ij}\colon i, j \in N, j \geq n\})$. So it follows from Lemma 3.2 that there is a finite subfamily $\mathscr{C}_n$ of $\{B_{ij}\colon i, j \in N, j \geq n\}$ such that $\phi(\infty, \bigcup\mathscr{C}_n) < 1/n$.

Define $\mathscr{C} = \bigcup\{\mathscr{C}_n\colon n \in N\}$. By (c) and the monotone property of ϕ, we have $\phi(\infty, \text{cl}\bigcup\mathscr{C}) = 0$. On the other hand, by (d), for each n, $\mathscr{C} \cap \{B_{in}\colon i \in N\}$ is finite. Hence by (b), $\infty \notin \text{cl}\bigcup\mathscr{C}$. Since ϕ is an annihilator, $\phi(\infty, \text{cl}\bigcup\mathscr{C}) \neq 0$, a contradiction.

Theorem 3.4. *Let X be a monotonically normal sequential space. If X is κ-metrizable, then X is strongly Fréchet.*

Proof. A κ-metric for X satisfies the condition of Lemma 3.3. Hence X is strongly Fréchet.

A space X is a said to be *dominated by* a closed cover $\mathscr{F}$ if whenever a subset A of X has a closed intersection with every element of some $\mathscr{F}' \subset \mathscr{F}$ which covers A, then A is closed in X. As is well known, every CW-complex is dominated by a cover of compact metric subsets.

Lemma 3.5. *A strongly Fréchet space X is metrizable if one of the following properties holds:*

(a) [11] *X is a Lašnev space;*
(b) *X is dominated by a closed cover of metric subsets.*

Proof. We show (b). By [16], every Fréchet space dominated by a closed cover of metric subsets is a Lašnev space. Hence by (a), X is metrizable.

Theorem 3.6. *A κ-metrizable space is metrizable if one of the following properties listed below holds:*

(a) [14] *X is a Lašnev space;*
(b) *X is dominated by a closed cover of metric subsets;*
(c) *X is a* CW-*complex.*

Proof. (c) follows from (b). We show (b). Suppose that X is dominated by a closed cover of metric subsets. It is easy to check that X is sequential. By [1], every space dominated by a closed cover of stratifiable subspaces is stratifiable.

So X is stratifiable, hence X is monotonically normal. Combining Theorem 3.4 and Lemma 3.5, we see that X is metrizable.

4. A NONMETRIZABLE STRATIFIABLE κ-METRIZABLE SPACE

The following definition of distance functions for the real line R and the plane R^2 is adopted throughout the remainder of this paper:

Let E be R or R^2, d_0 the usual distance function E. Define the distance function d by $d(x, y) = \min\{1, d_0(x, y)\}$. The distance function $d(x, A)$ from a point x to a subset A of E is defined by letting $d(x, A) = \inf\{d(x, y): y \in A\}$ if $A \neq \varnothing$, and $d(x, \varnothing) = 1$. For a pair, A, B of subsets of E, we define $\overline{d}(A, B) = \sup\{d(x, B): x \in A\}$ if $A \neq \varnothing$, and $\overline{d}(\varnothing, B) = 1$. It is well known that the functions d and $\overline{d}$ have the following properties:

(a) for a fixed A of E, the distance $d(x, A)$ is a continuous function on X;
(b) $A \subset B$ then $d(x, A) \geq d(x, B)$;
(c) for every family $\mathscr{A}$ of subsets of E, $d(x, \mathrm{cl}\bigcup\mathscr{A}) = \inf\{d(x, A): A \in \mathscr{A}\}$; and
(d) $d(x, A) \leq d(x, B) + \overline{d}(B, A)$.

Example 4.1. There is a nonmetrizable first countable space Z which has a continuous, monotone and linearly additive annihilator ϕ for 2^Z.

Remark 4.2. The space Z is stratifiable and κ-metrizable. Indeed, since ϕ is a continuous monotone annihilator for 2^Z, Z is stratifiable. On the other hand, if we restrict ϕ to $R[Z]$, ϕ becomes a κ-metric on Z. Note that linear additivity cannot be replaced by additivity, since every stratifiable additively κ-metrizable space is metrizable by Isiwata's theorem in the introduction.

Proof of Example 4.1. The space Z is a variant of the example of a stratifiable nonmetrizable space described in [2, Example 9.2]. Several variants are well known. An example similar to Z can be found in [4, Example 4.2]. Let $Z = \{(x, y) \in R^2: y \geq 0\}$, i.e., the closed upper half-plane. Define $x(z) = x$, $y(z) = y$ for each $z = (x, y) \in Z$, $B_n(z) = \{z' \in Z: d(z, z') < 1/n\}$ for each $z \in Z$, $n \in N$, and $L_x = \{z \in Z: x(z) = x, y(z) > 0\}$ for each $x \in R$. Let Z have the following topology: the points of $\{z \in Z: y(z) > 0\}$ are isolated, and a basic neighborhood of $z = (x, 0)$ is the set $U_n(z) = B_n(z) - L_x$. It can be easily shown that Z is a nonmetrizable first countable regular T_1-space.

Now let us construct the desired annihilator. Define $e(z, A) = \sup\{d(z, A - L_x): x \in R\}$ for each $z \in Z$ and $A \subset Z$. Define an annihilator $\phi: Z \times 2^Z \to R$ as the following:

$$\phi(z, F) = \begin{cases} 0 & \text{if } z \in F, \\ \max\{e(z, F), y(z)\} & \text{if } z \notin F. \end{cases}$$

We claim that ϕ is a continuous, monotone and linearly additive annihilator for 2^Z. For each $z \in Z$ and nonempty set $A \subset Z$, there is a point $a(z, A) \in \mathrm{cl}_{R^2} A$ such that $d(z, A) = d(z, \mathrm{cl}_{R^2} A) = d(z, a(z, A))$.

Claim 1. $e(z, A) = d(z, A - L_{x(a(z,A))})$.

Proof. We need only show that $d(z, A - L_x) \leq d(z, A - L_{x(a(z,A))})$ for each $x \in R$ with $x \neq x(a(z, A))$. Since $x \neq x(a(z, A))$, we have $a(z, A) \in \mathrm{cl}_{R^2}(A - L_x)$. Hence

$$d(z, A - L_x) \leq d(z, a(z, A)) = d(z, A) \leq d(z, A - L_{x(a(z,A))}).$$

Claim 2. ϕ is a monotone annihilator.

Proof. Obviously ϕ is monotone. So we show that ϕ is an annihilator. Suppose that $z \notin F$. If $y(z) > 0$, then $\phi(z, F) \geq y(z) > 0$. If $y(z) = 0$, then there is a basic neighborhood $U_n(z)$ of z with $U_n(z) \cap F = \varnothing$. Then $B_n(z) \cap (F - L_{x(z)}) = \varnothing$. Hence $\phi(z, F) \geq e(z, F) \geq 1/n > 0$.

Claim 3. ϕ is continuous.

Proof. We show that $\phi(z, F)$ is continuous at a point $z \in Z$. Since the points of $\{z \in Z: y(z) > 0\}$ are isolated, we may assume that $y(z) = 0$.

Case 1. $z \in F$. Suppose that $\varepsilon > 0$ is given. Choose $n \in N$ with $1/n < \varepsilon$. Note that $z \in F - L_x$ for each $x \in R$. Therefore if $z' \in U_n(z)$, then $e(z', F) < \varepsilon$, and obviously $y(z') < 1/n$. Hence $\phi(z', F) < \varepsilon$.

Case 2. $z \notin F$. Suppose that $\varepsilon > 0$ is given. Since $d(\cdot, A)$ is continuous with respect to the usual Euclidean topology, there exists $n \in N$ with $1/n < \varepsilon$ such that

(1) $U_n(z) \cap F = \varnothing$; and
(2) if $z' \in U_n(z)$, then $|d(z', F - L_{x(a(z,F))}) - d(z, F - L_{x(a(z,F))})| < \varepsilon$.

Suppose that $z' \in U_n(z)$. We show that $|\phi(z', F) - \phi(z, F)| < \varepsilon$. Since $1/n < \varepsilon, |y(z') - y(z)| < \varepsilon$. So by (1), we need only show that $|e(z', F) - e(z, F)| < \varepsilon$. By Claim 1 and (2), we have $e(z', F) = d(z', F - L_{x(a(z',F))}) \geq d(z', F - L_{x(a(z,F))}) > d(z, F - L_{x(a(z,F))}) - \varepsilon = e(z, F) - \varepsilon$. It remains to show that $e(z', F) < e(z, F) + \varepsilon$. First suppose that $x(a(z, F)) = x(a(z', F))$. Then

$$\begin{aligned} e(z', F) &= d(z', F - L_{x(a(z',F))}) = d(z', F - L_{x(a(z,F))}) \\ &< d(z, F - L_{x(a(z,F))}) + \varepsilon = e(z, F) + \varepsilon. \end{aligned}$$

Next suppose that $x(a(z, F)) \neq x(a(z', F))$. Then, since $a(z, F) \in \mathrm{cl}_{R^2}(F - L_{x(a(z',F))})$, we have $e(z', F) = d(z', F - L_{x(a(z',F))}) \leq d(z', a(z, f)) \leq d(z', z) + d(z, a(z, F)) < \varepsilon + d(z, F) \leq \varepsilon + d(z, F - L_{x(a(z,F))}) = \varepsilon + e(z, F)$. The proof of Claim 3 is completed.

Claim 4. ϕ is linearly additive.

Proof. Let $\{F_\alpha\}$ be an increasing family of closed subsets of X. First we show that

(3) $e(z, \mathrm{cl}_Z(\bigcup_\alpha F_\alpha)) = \inf_\alpha e(z, F_\alpha)$.
By the monotone property of $e(z, \cdot)$, we need only show that $\inf_\alpha e(z, F_\alpha) \leq e(z, \mathrm{cl}_Z(\bigcup_\alpha F_\alpha))$.
Case 1. Suppose that for each α, there exists, $\beta > \alpha$ with $x(a(z, F_\alpha)) \neq x(a(z, F_\beta))$. Then since $a(z, F_\alpha) \in \mathrm{cl}_{R^2}(F_\alpha - L_{x(a(z, F_\beta))})$, $d(z, F_\alpha) = d(z, F_\alpha - L_{x(a(z, F_\beta))})$. Therefore by Claim 1, $e(z, F_\beta) = d(z, F_\beta - L_{x(a(z, F_\beta))}) \leq d(z, F_\alpha - L_{x(a(z, F_\beta))}) = d(z, F_\alpha)$. Thus $\inf_\alpha e(z, F_\alpha) \leq \inf_\alpha d(z, F_\alpha) = d(z, \bigcup_\alpha F_\alpha) = d(z, \mathrm{cl}_Z(\bigcup_\alpha F_\alpha)) \leq d(z, \mathrm{cl}_Z(\bigcup_\alpha F_\alpha) - L_{x(a(z, \mathrm{cl}_Z(\bigcup_\alpha F_\alpha)))}) = e(z, \mathrm{cl}_Z(\bigcup_\alpha F_\alpha))$.

Case 2. Suppose that there exists α_0 such that $x(a(x, F_\alpha)) = x(a(z, F_{\alpha_0})) = x_0$ for each $\alpha \geq \alpha_0$. Then

$$\inf_\alpha e(z, F_\alpha) = \inf_{\alpha \geq \alpha_0} e(z, F_\alpha) = \inf_{\alpha \geq \alpha_0} d(z, F_\alpha - L_{x_0})$$
$$= d(z, \bigcup_\alpha F_\alpha - L_{x_0}) = d\left(z, \mathrm{cl}_Z\left(\bigcup_\alpha F_\alpha - L_{x_0}\right)\right).$$

Since L_{x_0} is clopen in Z, $\mathrm{cl}_Z(\bigcup_\alpha F_\alpha - L_{x_0}) = \mathrm{cl}_Z(\bigcup_\alpha F_\alpha) - L_{x_0}$. Therefore

$$\inf_\alpha e(z, F_\alpha) = d\left(z, \mathrm{cl}_Z\left(\bigcup_\alpha F_\alpha\right) - L_{x_0}\right)$$
$$\leq d\left(z, \mathrm{cl}_Z\left(\bigcup_\alpha F_\alpha\right) - L_{x(a(z, \mathrm{cl}_Z \bigcup_\alpha F_\alpha))}\right) = e\left(z, \mathrm{cl}_Z\left(\bigcup_\alpha F_\alpha\right)\right).$$

That completes the proof of (3).

To show the linear additivity of ϕ, by the monotone property of $\phi(z, \cdot)$, we need only show that $\inf_\alpha \phi(z, F_\alpha) \leq \phi(z, \mathrm{cl}_Z(\bigcup_\alpha F_\alpha))$.
Case 1. Suppose that $z \in \mathrm{cl}_Z(\bigcup_\alpha F_\alpha)$. Without loss of generality, we may assume that $z \notin F_\alpha$ for each α and $y(z) = 0$. For any $\varepsilon > 0$, there exist F_α and two points z', $z'' \in F_\alpha$ such that $d(z, z') < \varepsilon$, $d(z, z'') < \varepsilon$, and $x(z') \neq x(z'')$. Then since $F_\alpha - L_{x(a(z, F_\alpha))}$ contains z' or z'', we have $e(z, F_\alpha) = d(z, F_\alpha - L_{x(a(z, F_\alpha))}) < \varepsilon$. Since $y(z) = 0$, $\phi(z, F_\alpha) < \varepsilon$. Thus $\phi(z, \mathrm{cl}_Z(\bigcup_\alpha F_\alpha)) = \inf_\alpha \phi(z, F_\alpha) = 0$.
Case 2. Suppose that $z \notin \mathrm{cl}_Z(\bigcup_\alpha F_\alpha)$. Then by (3),

$$\phi\left(z, \mathrm{cl}_Z\left(\bigcup_\alpha F_\alpha\right)\right) = \max\left\{e\left(z, \mathrm{cl}_Z\left(\bigcup_\alpha F_\alpha\right)\right), y(z)\right\} = \max\{\inf_\alpha e(z, F_\alpha), y(z)\}$$
$$= \inf_\alpha \max\{e(z, F_\alpha), y(z)\} = \inf_\alpha \phi(z, F_\alpha).$$

That completes the proof of Example 4.1.

5. Additively κ-metrizable spaces

We shall say that an annihilator $\phi\colon X \times \mathscr{F} \to R$ has the *semi-closure condition* if for each point $x \in X$ and a subfamily $\mathscr{H}$ of $\mathscr{F}$ with $x \in \mathrm{cl} \bigcup \mathscr{H}$, we have $\inf\{\phi(x, F): F \in \mathscr{H}\} = 0$. For an annihilator $\phi: X \times \mathscr{F} \to R$, define $z(x, y) = \sup\{\phi(x, F): y \in F \in \mathscr{F}\}$ for each x, $y \in X$. Let $S_n(x) = \{y \in X: z(x, y) < 1/n\}$. Recall that a family $\mathscr{F}$ of closed subsets of X is a *base for closed sets* if every closed subset of X is an intersection of members of $\mathscr{F}$. A space (X, τ) is a *β-space* [7] if there is a function $g: X \times N \to \tau$ such that $x \in g(x, n)$, and if $x \in g(x_n, n)$, then the set $\{x_n: n \in N\}$ has a cluster point in X. A space (X, τ) is a *γ-space* [8] if there is a function $g: X \times N \to \tau$ such that $x \in g(x, n)$, and if $y_n \in g(x, n)$ and $x_n \in g(y_n, n)$, then the sequence $\{x_n: n \in N\}$ converges to x.

Theorem 5.1. *Let $\mathscr{F}$ be a base for closed sets of X. Then an annihilator $\phi: X \times \mathscr{F} \to R$ satisfies the semi-closure condition if and only if $\{\operatorname{int} S_n(x): n \in N\}$ is a local neighborhood base at x for each point $x \in X$.*

Proof. Refer to the proof of Isiwata [9] for an additive κ-metric space.

Theorem 5.2. *Let $\mathscr{F}$ be a base for closed sets of a space (X, τ). Suppose that there is a continuous annihilator $\phi: X \times \mathscr{F} \to R$ with the semi-closure condition. Then X is a γ-space.*

Proof. By [5, Theorem 10.6], we need only show that there exists a function $g: X \times N \to \tau$ such that $x \in g(x, n)$, and if K is a compact subset of X and U is an open set of X containing K, then $\bigcup\{g(x, n): x \in K\} \subset U$ for some n. Define $g(x, n) = \operatorname{int} S_n(x)$. By Theorem 5.1, $x \in g(x, n)$. Suppose that K is compact and U is an open set containing K. Since K is compact and $\mathscr{F}$ is a base for closed sets of X, there exists a finite subfamily $\{F_1, F_2, \dots, F_k\}$ of $\mathscr{F}$ such that $K \subset X - \bigcap\{F_i: i = 1, 2, \dots, k\} \subset U$. Since ϕ is an annihilator, for each $x \in K$, $\max\{\phi(x, F_i): i = 1, 2, \dots, k\} > 0$. Using the continuity of $\phi(x, F_i)$ and the compactness of K, we can take $n \in N$ such that $\max\{\phi(x, F_i): i = 1, 2, \dots, k\} > 1/n$ for any $x \in K$. Since $F_i \supset X - U$ for each $i = 1, 2, \dots, k$, if $y \in X - U$, then $z(x, y) \geq \max\{\phi(x, F_i): i = 1, 2, \dots, k\} > 1/n$. Hence $y \notin S_n(x)$. Thus $S_n(x) \subset U$. The proof is completed.

It is easy to see that an additive κ-metric has the semi-closure condition. Hodel [8] showed that if a space is a β-space and a γ-space, then it is developable. It is known that a stratifiable space is a paracompact β-space and a paracompact developable space is metrizable. So we have

Theorem 5.3 [9]. *An additively κ-metrizable β-space is developable. In particular, an additively κ-metrizable stratifiable space is metrizable.*

Example 5.4. The Sorgenfrey line has an additive κ-metric but is not metrizable.

Proof. The Sorgenfrey line X is the set of all real numbers with the base consisting of all intervals $[x, y)$. We use the distance function d on R defined in §4. We define an annihilator $\phi: X \times R[X] \to R$ by letting $\phi(x, F) = d(x, F \cap [x, \infty))$ for each point $x \in X$ and a regular closed set F of X. We claim that ϕ is a monotone, continuous and additive annihilator. It is easy to check that ϕ is a monotone and additive annihilator, because the distance function d has the property (b) and (c) in §4. It remains to check the continuity of ϕ.

Fix any regular closed set F of X and a point $x \in X$. We show the continuity of $\phi(x, F)$ at x.

Case 1. $F \cap [x, \infty) = \emptyset$. Then $U = [x, \infty)$ is a neighborhood of x and for each $y \in U$, $\phi(y, F) = 1$ since $F \cap [y, \infty) = \emptyset$.

Case 2. $F \cap [x, \infty) \neq \emptyset$ and $x \notin F$. Define $z = \inf(F \cap [x, \infty))$. Then $x < z$, $\phi(x, F) = d(x, z)$, and for each $\varepsilon > 0$ with $x < x + \varepsilon < z$, $U = [x, x + \varepsilon)$ is a neighborhood of x satisfying that if $y \in U$, then $\phi(y, F) = d(y, F \cap [y, \infty)) = d(y, F \cap [x, \infty) = d(y, z)$. Since $d(x, z) - \varepsilon < d(y, z) \leq d(x, z)$, we have $|\phi(y, F) - \phi(x, F)| < \varepsilon$.

Case 3. $x \in F$. Suppose that $\varepsilon > 0$ is given. Since F is regular closed, there is a point $z \in F$ with $x < z < x + \varepsilon$. Let $U = [x, z)$. If $y \in U$, then $\phi(y, F) = d(y, F \cap [y, \infty)) \geq d(y, z) < \varepsilon$. Since $\phi(x, F) = 0$, $|\phi(y, F) - \phi(x, F)| < \varepsilon$.

Remark 5.5. For the annihilator ϕ of the above example, we cannot replace $R[X]$ by 2^X by the same reason as Remark 4.2. Annihilator ϕ is called *regular* [13] if it satisfies the following triangle axiom:

$\phi(x, F) \leq \phi(x, H) + \overline{\phi}(H, F)$ for every x, F, H, where $\overline{\phi}(H, F) = \sup\{\phi(x, F) : x \in H\}$.

It is easy to see that the annihilator ϕ defined in the above example is regular because the distance function d on R has the property (d) in §4.

REFERENCES

1. C. J. R. Borges, *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1–16.

2. J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. **11** (1961), 105–125.

3. A. N. Dranišnikov, *Simultaneous annihilation of families of closed sets, κ-metrizable and stratifiable spaces*, Soviet Math. Dokl. **19** (1978), 1466–1469.

4. G. Gruenhage, *Continuously perfectly normal spaces and some generalizations*, Trans. Amer. Math. Soc. **224** (1976), 323–338.

5. ——, *Generalized metric spaces*, Handbook of Set-Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984.

6. R. W. Heath, D. J. Lutzer and P. L. Zenor, *Monotonically normal spaces*, Trans. Amer. Soc. **178** (1973), 481–493.

7. R. E. Hodel, *Moore spaces and $w\Delta$-spaces*, Pacific J. Math. **38** (1971), 641–652.

8. ——, *Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points*, Duke Math. J. **39** (1972), 253–263.

9. T. Isiwata, *Metrization of additive κ-metric spaces*, Proc. Amer. Math. Soc. **100** (1987), 164–168.

10. V. Kannan, *Every compact T_5 sequential space is Fréchet*, Fund. Math. **107** (1980), 85–90.

11. E. A. Michael, *A quintuple quotient quest*, General Topology Appl. **2** (1972), 91–138.

12. F. Siwiec, *Sequence-covering and countably bi-quotient mappings*, General Topology Appl. **1** (1971), 143–154.

13. E. V. Ščepin, *On κ-metrizable spaces*, Math. USSR Izvestija **14** (1980), 407–440.

14. K. Tamano, *Closed images of metric spaces and metrization*, Topology Proc. **10** (1985), 177–186.

15. Y. Tanaka, *Metrizability of certain quotient spaces*, Fund. Math. **119** (1983), 157–168.

16. Y. Tanaka and Zhou Hao-xuan, *Spaces dominated by metric subsets*, Topology Proc. **9** (1984), 149–163.

17. P. Zenor, *Some continuous separation axioms*, Fund. Math. **90** (1976), 143–158.

DEPARTMENT OF MATHEMATICS, TOKYO GAKUGEI UNIVERSITY, KOGANEI-SHI, TOKYO 184, JAPAN

FACULTY OF LIBERAL ARTS, SHIZUOKA UNIVERSITY, OHYA, SHIZUOKA 422, JAPAN

DEPARTMENT OF MATHEMATICS, TOKYO GAKUGEI UNIVERSITY, KOGANEI-SHI, TOKYO 184, JAPAN

Current address (J. Suzuki): Ashikaga Institute of Technology, Ashikaga-shi Tochigi 326, Japan

Current address (K. Tamano): Department of Mathematics, Faculty of Engineering, Yokohama National University, Yokohama 240, Japan