INvariants for a Class
of Torsion-Free Abelian Groups

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Abstract. In this note we present a complete set of quasi-isomorphism invariants for strongly indecomposable abelian groups of the form $G = G(A_1, \ldots, A_n)$. Here $A_1, \ldots, A_n$ are subgroups of the rationals $Q$ and $G$ is the kernel of $f: A_1 \oplus \cdots \oplus A_n \to Q$, where $f(a_1, \ldots, a_n) = \sum a_i$. The invariants are the collection of numbers $\text{rank} \cap \{G[\sigma] \mid \sigma \in M\}$, where $M$ ranges over all subsets of the type lattice generated by $\{\text{type}(A_i)\}$. Our results generalize the classical result of Baer for finite rank completely decomposable groups, as well as a result of F. Richman on a subset of the groups of the form $G(A_1, \ldots, A_n)$.

The history of the study of abelian groups is replete with attempts to find complete sets of numerical invariants for various subclasses. In the torsion-free case, the initial success of Baer in 1937 in classifying the completely decomposable groups [Ba], was followed by a long period in which very few satisfactory results on invariants were obtained. Moreover, the work of Beaumont and Pierce [BP] on torsion-free abelian groups of rank two indicated that the classification problem would be extremely difficult, even in the finite rank case. Indeed, many of the torsion-free results after Baer served only to demonstrate the pathological nature of this class. However, a subclass first studied systematically by Butler [Bu-1] in 1965 has proved to be reasonably tractable. This is the class of pure subgroups of finite rank completely decomposable groups, henceforth called Butler groups. Butler groups are varied enough to admit direct sum decomposition pathology [A-1] and arbitrary finite dimensional $Q$-algebras as quasi-endomorphism rings [BB], yet they admit multiple characterizations [A-3] and [AV-1], a useful duality [R-1, AV-5], and even “arbitrary” endomorphism rings in special cases [AV-4]. In addition, there is a category equivalence between the quasi-homomorphism category of Butler groups with typeset in a fixed finite lattice $T$ of types and the category of $Q$-representations of the partially ordered set of join-irreducible elements of $T$ [Bu-2 and Bu-3]. The paper [A-2] provides a summary of some important consequences of this equivalence.

In studying the class of Butler groups it is easier to work in the quasi-homomorphism category where there is a Krull-Schmidt Theorem [A-1].
particular, it is sufficient to classify the indecomposables in this category, i.e., the
strongly indecomposable Butler groups. In this note we present a complete set of
quasi-isomorphism invariants for strongly indecomposable groups of the form
\( G = G(A_1, \ldots, A_n) \). Here \( A_1, \ldots, A_n \) are subgroups of \( Q \) and the Butler group
\( G \) is the kernel of \( f: A_1 \oplus \cdots \oplus A_n \to Q \), where \( f(a_1, \ldots, a_n) = a_1 + \cdots + a_n \).
We will denote by \( \Omega \) the class of all groups isomorphic to some group of the
form \( G(A_1, \ldots, A_n) \). This class was first studied by Richman [R-2], who pro-
vided quasi-isomorphism invariants in the case where the \( \text{type}(A_i \cap A_j) \)'s are
incomparable for distinct pairs of indices \( i, j \). The class \( \Omega \) contains the finite
rank completely decomposable groups and our invariants generalize the original
invariants of Baer, as well as those of Richman.

Additional work on the class \( \Omega \) appears in [AV-2 and AV-3]. In [AV-2] the
authors introduced the concept of quasi-representing graph for groups of the
form \( G = G(A_1, \ldots, A_n) \). A quasi-representing graph for \( G \) is a subgraph,
\( U \), of the complete graph \( \hat{U} \) with vertices \( 1, \ldots, n \) (or, where convenient,
\( A_1, \ldots, A_n \)) and edges \( ij \) labelled by \( \text{type}(A_i \cap A_j) \), which is obtained by
iteration of the following algorithm: if a graph contains a circuit \( c \) with all
the edges of \( c \) labelled by types \( \geq \tau \) and at least one edge labelled by \( \tau \), then
remove an edge of \( c \) labelled by \( \tau \). Each edge \( ij \) of \( U \) can be identified with
the pure subgroup \( G_{ij} = G \cap (A_i \oplus A_j) \) of \( G \). Moreover, there is a natural
quasi-epimorphism \( \bigoplus_{i,j \in U} G_{ij} \to G \) which is a balanced projective cover in the
quasi-homomorphism category (see [AV-3]). As a consequence, up to quasi-
ismorphism, the group \( G = G(A_1, \ldots, A_n) \), may be regarded as the direct
sum of the edges of \( U \) modulo the circuits of \( U \) (by a circuit in \( U \) we mean
a sequence of vertices \( i_1, \ldots, i_m, i_{m+1} = i_1 \) where \( i_1 \) is the only repeated
vertex and \( i_j i_{j+1} \in U \) for \( 1 \leq j \leq m \)). This viewpoint was utilized in [AV-3]
to classify, up to quasi-isomorphism, the \( CT \)-groups, those groups of the form
\( G(A_1, \ldots, A_n) \) having a quasi-representing graph with pairwise incomparable
labels of edges. The complete set of invariants in this case was the collection
of numbers \( r_G(\sigma, \tau) = \text{rank}(G(\tau) + G[\sigma])/G[\sigma] \), where \( \sigma \) and \( \tau \) range over all
pairs of types.

We will need one technical lemma on quasi-representing graphs.

**Lemma 1.** Let \( T \) be a quasi-representing graph for \( G = G(A_1, \ldots, A_n) \) and
suppose \( i, j, k \) are distinct vertices of \( T \) with \( \text{type}(A_j \cap A_k) \leq \text{type}(A_i) \). If \( jk \)
is an edge of \( T \), then there is a path \( p \) in \( T \setminus \{jk\} \) connecting either \( j \) or \( k \)
to \( i \). If \( p \) connects \( j \) (respectively, \( k \)) to \( i \), then \( jk \) may be replaced by \( ik \)
(respectively, \( ij \)) to obtain a new quasi-representing graph for \( G \).

**Proof.** By [AV-3, Lemma 2.2] there is a path \( p \) in \( T \) with edges labelled by
types greater than or equal to \( \tau = \text{type}(A_j \cap A_k) \) which connects the edge \( jk \)
to the vertex \( i \). Assume, for the sake of simplicity, that \( p \) connects the vertex \( k \)
to the vertex \( i \) (and does not contain the edge \( jk \). The condition \( \tau \leq \text{type}(A_i) \)
implies \( \tau \leq \text{type}(A_i \cap A_j) = \text{the label on the edge } ij \). It follows that \( ij \) does
not belong to \( T \) since in that case \( ij, jk, p \) would form a circuit in \( T \) with
edges labelled by types \( \geq \tau \) and containing an edge labelled by \( \tau \). Moreover, adding the edge \( ij \) to \( T \) must create a circuit \( c \) in \( T \) consisting of edges labelled by types greater than or equal to \( \sigma = \text{type}(A_i \cap A_j) \) by the algorithm given above for constructing quasi-representing graphs. It follows that \( \sigma = \tau \), since if \( jk \notin c \), \( c \setminus ij \cdot jk \cdot p \) contains a circuit in \( T \) with edges labelled by types \( \geq \tau \). Thus, the algorithm implies that the edge \( ij \) may be added and the edge \( jk \) deleted from \( T \) to form a new quasi-representing graph for \( G \) (also see [AV-2]).

If \( G \) is any torsion-free abelian group (hereafter, simply group) and \( \sigma \) is a type (or a rank-1 group), then \( \overline{G[\sigma]} = \bigcap\{\ker f \mid f: G \to \mathbb{Q} \text{ and } \text{type}(f(G)) \leq \sigma\} \). If \( M \) is a set of types, \( \overline{G[M]} = \bigcap\{\overline{G[\sigma]} \mid \sigma \in M\} \), and \( r_G[M] = \text{rank} G[M] \). If \( G = G(A_1, \ldots, A_n) \), then we may assume \((A_1, \ldots, A_n)\) is adjusted, that is, \( 1 \in A_i \) for each \( i \) and \( \text{type}(A_i \cap A_j) \leq \text{type}(A_k \cap A_l) \) implies \( A_i \cap A_j \subset A_k \cap A_l \). There is no loss of generality in this assumption as each \( G(A_1, \ldots, A_n) \) is quasi-isomorphic to some \( G(B_1, \ldots, B_n) \) with \((B_1, \ldots, B_n)\) adjusted [AV-2]. Also, there is no loss of generality in assuming each \( n \)-tuple \((A_1, \ldots, A_n)\) is trimmed, that is, the natural projection of \( G(A_1, \ldots, A_n) \) into \( A_i \) is onto for each \( i \) (see [R-2]). Frequent use is made of the rank-1 subgroups \( G_{ij} = G \cap (A_i \oplus A_j) \) of \( G \) and the elements \( g_{ij} = (1, -1) \in G_{ij} \), noting that \( g_{ij} = -g_{ji} \). All undefined notation and terminology can be found in [A-1 or AV-1].

Following one additional definition, our first two theorems will show that it suffices to classify the strongly indecomposable groups of the form \( G(A_1, \ldots, A_n) \).

A quasi-representing graph \( T \) for \( G = G(A_1, \ldots, A_n) \) is called reduced provided either \( \text{rank} G = 1 \) or any two edges of \( T \) belong to a circuit in \( T \) and if \( ij \) and \( kl \) are edges of \( T \) with label \( ij \leq \text{label } kl \), then \( ij \) belongs to a circuit of \( T \) not containing \( kl \).

**Theorem 2** [AV-3]. Let \( G = G(A_1, \ldots, A_n) \) and assume \((A_1, \ldots, A_n)\) is adjusted. Then \( G = G_1 \oplus \cdots \oplus G_k \), where for each \( i \), \( G_i \) is of the form \( G(B_1, \ldots, B_m) \) for some \( m = m(i) \geq 2 \) and \( B_1, \ldots, B_m \subset \mathbb{Q} \), and each quasi-representing graph for \( G_i \) is reduced.

**Theorem 3.** Let \( G = G(A_1, \ldots, A_n) \). The following are equivalent.

(a) \( G \) is strongly indecomposable.
(b) Each quasi-representing graph for \( G \) is reduced.
(c) \( G/[A_i] \) has rank one for \( 1 \leq i \leq n \).
(d) \( \text{End}(G) \) is isomorphic to a subring of \( \mathbb{Q} \).

**Proof.** The implication (a) \( \rightarrow \) (b) follows from Theorem 2, while (d) \( \rightarrow \) (a) by a well-known result of J. D. Reid [Re]. To show (b) \( \rightarrow \) (c), recall [AV-2,3] that each type \( \sigma \) defines an equivalence relation on the vertices of a quasi-representing graph \( T \) for \( G \) by \( A_i \approx A_j \) if \( i = j \) or \( A_i \) and \( A_j \) are connected by a path in \( T \) all of whose edges are labelled by types not less than or equal to
The condition "in $T$" is redundant. Indeed, if $ij \notin T$ is labelled by a type $\tau \notin \sigma$, then by Theorem 1.6(c) in [AV-2], $i$ and $j$ are connected by a path in $T$ all of whose edges are labelled by a type $\geq \tau$, hence $\notin \sigma$. By Lemma 1.4 of [AV-2], if $i \neq j$, then $A_i \approx A_j$ if and only if $G_{ij} \cap \sum_{k \in S} G_{kl} \neq 0$, where $S = \{kl \in T \mid \text{type}(A_k \cap A_l) \notin \sigma \}$. If $\sigma = \text{type}(A_i)$, then the corresponding equivalence classes of vertices are $V_1 = \{A_i\}, V_2, \ldots, V_m$, where $m \geq 2$ and the rank of $G/G[A_i]$ is $m - 1$ [AV-2, Corollary 2.1]. Moreover, if $2 \leq k \neq l \leq m$, then type$(A_r \cap A_s) \leq \sigma$ if $A_r \in V_k$ and $A_s \in V_l$. It follows from Lemma 1 that $G$ has a quasi-representing graph with no edge connecting vertices in $V_k$ and $V_l$ for $2 \leq k \neq l \leq m$ since any such edge could be replaced by an edge having $A_i$ as a vertex. Repeated applications of this observation yield a quasi-representing graph $T$ with the property that if any edge of $T$ does not have vertices lying in a single $V_k$, then that edge must have $A_i$ as a vertex. Such a graph is not reduced unless $m = 2$. Since (b) is our hypothesis, $m = 2$ and rank$(G/G[A_i]) = 1$.

To show (c) $\rightarrow$ (d), note that each $G[A_i]$ is a fully invariant subgroup of $G$, so each endomorphism $f$ of $G$ induces a rational multiplication (by $a_i \in \mathbb{Q}$) on $G/G[A_i]$. In particular, if $g_{ij} = (1, -1) \in G \cap (A_i \oplus A_j)$, then $f(g_{ij} + G[A_i]) = a_i g_{ij} + G[A_i]$. This implies that $f(g_{ij}) = a_i g_{ij} + x_{ij}$ for some $x_{ij} \in G[A_j]$. Similarly, $f(g_{ji}) = a_j g_{ji} + x_{ji}$ for some $x_{ji} \in G[A_j]$. However, $g_{ij} = -g_{ji}$. Moreover, $x_{ij} \in G[A_i] \subset (A_i \oplus \cdots \oplus A_n)[A_i]$ implies that, as an element of $A_i \oplus \cdots \oplus A_n$, $x_{ij}$ has zero component in $A_i$. Similarly, $x_{ji}$ has zero component in $A_j$. The equation $a_i g_{ij} + x_{ij} = -(a_j g_{ji} + x_{ji})$ then implies $a_i = a_j$, and $x_{ij} + x_{ji} = 0$, in view of the definition of $G(A_1, \ldots, A_n)$. We conclude $x_{ij} = -x_{ji} \in G[A_i] \cap G[A_j]$.

Write $a = a_i$, $1 \leq i \leq n$, and let $i, j, k$ be distinct. Then $0 = f(0) = f(g_{ij} + g_{jk} + g_{ki}) = f(g_{ij} + f(g_{jk}) + f(g_{ki}) = a_i g_{ij} + x_{ij} + a_j g_{jk} + x_{jk} + a_k g_{ki} + x_{ki} = x_{ij} + x_{jk} + x_{ki}$. Therefore, $x_{ij} = -(x_{jk} + x_{ki}) \in G[A_k]$. Since $k$ was arbitrary, $x_{ij} \in \bigcap \{G[A_k] \mid 1 \leq k \leq n\} = 0$, and the endomorphism $f$ is multiplication by $a \in \mathbb{Q}$. This completes the proof.

The above two theorems show that each adjusted $G(A_1, \ldots, A_n)$ decomposes into strongly indecomposable summands of the same form, whose endomorphism rings are principal ideal domains. As noted in the introduction, there is a Krull-Schmidt theorem for these groups in the quasi-homomorphism category, so that we need only classify the strongly indecomposable ones. This we do with two lemmas followed by the main theorem.

**Lemma 4.** Let $G = G(A_1, \ldots, A_n)$ be strongly indecomposable and $M \subset \{A_1, \ldots, A_n\}$. Then $G[M] = G \cap \bigoplus \{A_i \mid A_i \notin M\}$. In particular, for all $i \neq j$, $G_{ij} = \bigcap \{G[A_k] \mid k \neq i, k \neq j\}$.

**Proof.** For each $i \neq j$, denote $M_{ij} = \{A_k \mid k \neq i, k \neq j\}$. By Theorem 3, for each $k$ the $A_k$-equivalence classes consist of $\{A_k\}$ and $\{A_i \mid k \neq i\}$. Therefore, if $k \neq i$ and $k \neq j$, then $A_i$ and $A_j$ are in the same $A_k$-equivalence
class and thus, $G_{ij} \cap \sum \{G_{lm} \mid \text{type}(G_{lm}) \notin \text{type}(A_k)\} \neq 0$ (refer to the proof of Theorem 3). It follows that $G_{ij} \subset G[A_k]$ whenever $k \neq i$ and $k \neq j$, so that $G_{ij} \subset \bigcap \{G[A_k] \mid k \neq i, k \neq j\} = G[M_{ij}]$. If $M \subset M_{ij}$, then $G_{ij} \subset G[M_{ij}] \subset G[M]$. We conclude that $\sum \{G_{ij} \mid A_i, A_j \notin M\} = G \cap \bigoplus \{A_i \mid A_i \notin M\} \subset G[M]$. On the other hand, it is clear from the projection maps that any element of $G[M]$, written as an $n$-tuple $(a_1, \ldots, a_n)$ in $A_1 \oplus \cdots \oplus A_n$, must have $a_i = 0$ for all $k$ such that $A_k \in M$. Therefore, $G[M] \subset G \cap \bigoplus \{A_i \mid A_i \notin M\}$.

**Lemma 5.** Let $G = G(A_1, \ldots, A_n)$ and $H = G(B_1, \ldots, B_n)$ be strongly indecomposable and assume that $r_G[M] = r_H[M]$ for each finite set of types $M$. If $M_{ij} = \{A_k \mid k \neq i, k \neq j\}$, then,

(a) $X_{ij} = H[M_{ij}]$ has rank one and type $X^\tau = \text{type}(G)$.

(b) If $S$ is a subset of $\{kl \mid 1 \leq k \leq l \leq n\}$, then $G_{ij} \cap \sum \{G_{kl} \mid kl \in S\} = 0$ if and only if $X_{ij} \cap \sum \{X_{kl} \mid kl \in S\} = 0$.

**Proof.** (a) For a fixed $ij$, denote $M = M_{ij}$. By Lemma 4, $1 = r_G[M] = r_H[M]$. Thus, there is a (unique) pure rank-1 subgroup $X_{ij} = H[M_{ij}]$ of $H$. Note that $X_{ij} \cap H[A_i] = 0$ since $\text{rank}(H[M_{ij}] \cap H[A_i]) = r_H[M_{ij}] = \text{rank}(H[M_{ij} \cup \{A_i\}]) = 0$. Similarly, $X_{ij} \cap H[A_j] = 0$. We conclude $\text{type}(X_{ij}) \leq \min\{\text{type}(A_i), \text{type}(A_j)\} = \text{type}(G)$. Furthermore, if $\tau = \text{type}(G)$, then $G(\tau) = G[M]$ where $M = \{\sigma \in \text{typeset} G \mid \tau \notin \sigma\}$ (see AV-1). Thus, $1 = \text{rank}(G(\tau) \cap G[M]) = \text{rank}(G[M_{ij} \cup M]) = \text{rank}(H[M_{ij} \cup M]) = \text{rank}(H(\tau) \cap H[M])$. This shows that $\text{type}(X_{ij}) = \tau$, and therefore that $\text{type}(X_{ij}) = \text{type}(G)$. (b) Let $S$ be a subset of $\{ij \mid 1 \leq i \neq j \leq n\}$. We first show that $\{G_{ij} \mid ij \in S\}$ independent in $G$ implies $\{X_{ij} \mid ij \in S\}$ independent in $H$. Apply induction on $|S|$, the case $|S| = 1$ being trivial. In general, $\{G_{ij} \mid ij \in S\}$ independent implies that there is an index $k$ which appears in exactly one element $kl$ of $S$ [AV-2, Lemma 1.4]. Let $S' = S \setminus \{kl\}$. By induction $\{X_{ij} \mid ij \in S'\}$ is independent in $H$. By the choice of $k$, $X_{ij} \subset H[A_k]$ for all $ij \in S'$. Since $X_{kl} \cap H[A_k] = 0$ as above, $X_{kl}$ is independent of $\{X_{ij} \mid ij \in S'\}$. This proves the “only if” portion of (b).

For the converse, we show that if $S$ is a subset of $\{ij \mid 1 \leq i \neq j \leq n\}$, then $\sum_{ij \in S} G_{ij} \cap G_{kl} \neq 0$ implies $\sum_{ij \in S} X_{ij} \cap X_{kl} \neq 0$. Without loss of generality we may assume that, given a double index $kl$, $S$ is minimal with respect to $\sum_{ij \in S} G_{ij} \cap G_{kl} \neq 0$. In this case, either $S = \{kl\}$, or $kl \neq S$ and $S \cup \{kl\} = c$ forms a circuit in the complete graph with vertices $1, \ldots, n$ and edges $ij$ connecting $i$ and $j$ [AV-2, Lemma 1.4]. In the first instance the desired conclusion is immediate. In the second, $\sum_{ij \in S} G_{ij} = \bigoplus_{ij \in S} G_{ij}$ is a full subgroup of (has torsion index in) $G[M]$, where $M = \{A_m \mid m$ is not a vertex of $c\}$. To see this, note that for each $ij \in S$, $G_{ij} \subset G[M]$ by Lemma 4, that the sum $\sum G_{ij}$ is direct since $S$ is minimal, and that $\bigoplus G_{ij}$ is full in $G[M]$ since $G[M] = G \cap \bigoplus \{A_i \mid A_i \notin M\}$ by Lemma 4. Now, by the first part of the proof and the definition of $X_{ij}$, $\sum_{ij \in S} X_{ij} = \bigoplus X_{ij}$ is a subgroup.
of $H[M]$. However, $r_G[M] = r_H[M]$, so that $\bigoplus_{ij \in S} X_{ij}$ is a full subgroup of $H[M]$. Again by definition, $X_{kl} \subset H[M]$, and the lemma follows.

**Theorem 6.** Let $G = G(A_1, \ldots, A_n)$ and $H = G(B_1, \ldots, B_n)$ be strongly indecomposable. Then $r_G[M] = r_H[M]$ for each finite set of types $M$ if and only if $G$ and $H$ are quasi-isomorphic.

**Proof.** The "if" direction is clear. For the converse, up to quasi-isomorphism we may, as usual, assume the $n$-tuples $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ are adjusted and trimmed. Recall that $-g_{ij} = g_{ij} = (1, -1) \in G \cap (A_i \oplus A_j)$. We will define a quasi-monomorphism $\varphi: G \to H$ by defining $0 \leq \varphi(g_{ij}) = x_{ij} \in X_{ij}$ (type $G_{ij} = \text{type } X_{ij}$ by Lemma 5) for all indices $ij$, and showing that for all subsets $S$ of indices and all choices of coefficients $a_{ij} \in \mathbb{Q}$,

$$\sum_{ij \in S} a_{ij} g_{ij} = 0 \quad \text{in } G \quad \text{if and only if} \quad \sum_{ij \in S} a_{ij} x_{ij} = 0 \quad \text{in } H.$$

The theorem will then follow by symmetry.

We will use induction on the set of possible indices $ij$ which can appear in $S$, beginning with the first nontrivial case, $S = \{12, 23, 31\}$. Since $g_{12} + g_{23} + g_{31} = 0$, by Lemma 5 there is a choice $0 \neq \varphi(g_{ij}) = x_{ij} \in X_{ij}$ for $ij \in \{12, 23, 31\}$, such that $x_{23} + x_{23} + x_{31} = 0$. In this case the only nontrivial relationship between the $g_{ij}$'s is of the form $a(g_{12} + g_{23} + g_{31}) = 0$ for $a \in \mathbb{Q}$.

Suppose $ax_{12} + bx_{23} + cx_{31} = 0$ and assume, for example, $a \neq b$. Subtracting $a(x_{12} + x_{23} + x_{31}) = 0$ yields $(b - a)x_{23} + (c - a)x_{31} = 0$, which contradicts Lemma 5. Thus, (7) is satisfied for the first step of the induction.

Assume $\varphi$ has been defined on $G_{ij}$ for $i, j < m (m \geq 4)$ so that (7) is satisfied. We will show how to define $\varphi$ on $G_{mi} = G_{im}, i < m$. Since $(G_{1m} + G_{m2}) \cap G_{21} \neq 0$, by Lemma 5 we can define $0 \neq \varphi(g_{1m}) = x_{1m}$ and $0 \neq \varphi(g_{m2}) = x_{m2}$ so that $x_{1m} + x_{m2} + x_{21} = 0$. To show that (7) holds for the set of indices $S = \{ij \mid i, j < m \text{ or } ij = 1m \text{ or } m2\}$, suppose $\sum_{ij \in S} a_{ij} x_{ij} = 0$. Without loss of generality we may assume $a_{1m} \neq 0$.

Let $x = \sum_{ij \in S} a_{ij} x_{ij} - a_{1m}(x_{1m} + x_{m2} + x_{21}) = 0$. Note that if $a_{1m} \neq a_{m2}$ the equation $x = 0$ implies $\{\sum_{X_{ij} | i, j < m}\} \cap X_{m2} \neq 0$, an impossibility by Lemma 5. Therefore, $x = \sum_{i, j < m} b_{ij} x_{ij} = 0$, for some choice of $b_{ij}$, so that by induction, $\sum_{ij \in S} a_{ij} g_{ij} - a_{1m}(g_{1m} + g_{m2} + g_{21}) = \sum_{i, j < m} b_{ij} g_{ij} = 0$. It follows that $\sum_{ij \in S} a_{ij} x_{ij} = 0$. A symmetric argument shows that $\sum_{ij \in S} a_{ij} g_{ij} = 0$ implies $\sum_{ij \in S} a_{ij} x_{ij} = 0$.

Suppose $\varphi$ has been defined on $G_{ij}$ for $i, j < m$ and for $i = m, 1 \leq j \leq k-1$, where $3 \leq k < m$. Since $(G_{(k-1)m} + G_{mk}) \cap G_{(k-1)} \neq 0$, by Lemma 5 there exist $0 \neq \varphi(g_{mk}) = x_{mk} \in X_{mk}$ and $a \in \mathbb{Q}$ such that $y = x_{k(k-1)} + ax_{(k-1)m} + x_{mk} = 0$. We first show $a = 1$. If not, $y - (x_{k(k-1)} + x_{(k-1)m} + x_{m1} + x_{1k}) = 0$ implies that $X_{(k-1)m} \cap (X_{mk} + X_{m1} + X_{1k}) \neq 0$, which is impossible by Lemma 5.
Therefore \(a = 1\). To show (7) holds for the augmented set of indices including \(mk\) is a calculation similar to that performed in the previous paragraph. The proof is then complete.

To conclude, we return to the invariants of Baer, Richman and [AV-3]. Note that if \(G\) is completely decomposable, then the number of summands of type \(\tau\) is 
\[
\text{rank}(G^*[\tau]/G[\tau]) = \text{rank} G^*[\tau] - \text{rank} G[\tau],
\]
where \(G^*[\tau] = \bigcap \{G[\sigma] \mid \sigma < \tau\} = G[M]\) for \(M = \{\sigma \mid \sigma < \tau\}\). Thus, the \(r_G[M]\)'s provide the invariants of Baer. On the other hand, if \(G = G(A_1, \ldots, A_n)\) is doubly incomparable, one can check that the Richman invariants, \(\{\text{type}(A_i) \mid 1 \leq i \leq n\}\) are the meet irreducible elements of cotypset \(G = \{\sigma \mid \text{type}(f(G)) = \sigma\text{ for some } f: G \to Q\} = \{\sigma \mid \text{rank}(G^*[\sigma]/G[\sigma]) \neq 0\}\). Therefore these invariants also may be calculated from the \(r_G[M]\)'s. Finally, to calculate the invariants \(r_G(\sigma, \tau) = \text{rank}(G(\tau) + G[\sigma])/G[\sigma] = \text{rank}(G(\tau)/(G(\tau) \cap G[\sigma]))\) of [AV-3], we use the fact noted previously that \(G(\tau) = G[M_\tau]\), where \(M_\tau = \{\sigma \mid \tau \not\leq \sigma\}\). Indeed, it is then routine to check that \(r_G(\sigma, \tau) = r_G[M_\tau] - r_G[M_\tau \cup \{\sigma\}]\).

**References**


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