PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 2, February 1989

INVARIANTS FOR A CLASS OF TORSION-FREE ABELIAN GROUPS

D. ARNOLD AND C. VINSONHALER

(Communicated by Warren J. Wong)

ABSTRACT. In this note we present a complete set of quasi-isomorphism invariants for strongly indecomposable abelian groups of the form $G = G(A_1, \ldots, A_n)$. Here $A_1, \ldots, A_n$ are subgroups of the rationals Q and G is the kernel of $f: A_1 \oplus \cdots \oplus A_n \rightarrow Q$, where $f(a_1, \ldots, a_n) = \Sigma a_i$. The invariants are the collection of numbers $\operatorname{rank} \cap \{G[\sigma] \mid \sigma \in M\}$, where M ranges over all subsets of the type lattice generated by $\{\operatorname{type}(A_i)\}$. Our results generalize the classical result of Baer for finite rank completely decomposable groups, as well as a result of F. Richman on a subset of the groups of the form $G(A_1, \ldots, A_n)$.

The history of the study of abelian groups is replete with attempts to find complete sets of numerical invariants for various subclasses. In the torsion-free case, the initial success of Baer in 1937 in classifying the completely decomposable groups [Ba], was followed by a long period in which very few satisfactory results on invariants were obtained. Moreover, the work of Beaumont and Pierce [BP] on torsion-free abelian groups of rank two indicated that the classification problem would be extremely difficult, even in the finite rank case. Indeed, many of the torsion-free results after Baer served only to demonstrate the pathological nature of this class. However, a subclass first studied systematically by Butler [Bu-1] in 1965 has proved to be reasonably tractable. This is the class of pure subgroups of finite rank completely decomposable groups, henceforth called *Butler groups*. Butler groups are varied enough to admit direct sum decomposition pathology [A-1] and arbitrary finite dimensional Q-algebras as quasi-endomorphism rings [BB], yet they admit multiple characterizations [A-3] and [AV-1], a useful duality [R-1, AV-5], and even "arbitrary" endomorphism rings in special cases [AV-4]. In addition, there is a category equivalence between the quasi-homomorphism category of Butler groups with typeset in a fixed finite lattice T of types and the category of Q-representations of the partially ordered set of join-irreducible elements of T [Bu-2 and Bu-3]. The paper [A-2] provides a summary of some important consequences of this equivalence.

In studying the class of Butler groups it is easier to work in the quasi-homomorphism category where there is a Krull-Schmidt Theorem [A-1]. In

Received by the editors December 15, 1987 and, in revised form, March 21, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20K15.

particular, it is sufficient to classify the indecomposables in this category, i.e. the strongly indecomposable Butler groups. In this note we present a complete set of quasi-isomorphism invariants for strongly indecomposable groups of the form $G = G(A_1, \dots, A_n)$. Here $A_1, \dots A_n$ are subgroups of Q and the Butler group G is the kernel of $f: A_1 \oplus \cdots \oplus A_n \to Q$, where $f(a_1, \dots, a_n) = a_1 + \cdots + a_n$. We will denote by Ω the class of all groups isomorphic to some group of the form $G(A_1, \dots, A_n)$. This class was first studied by Richman [R-2], who provided quasi-isomorphism invariants in the case where the $\text{type}(A_i \cap A_j)$'s are incomparable for distinct pairs of indices i, j. The class Ω contains the finite rank completely decomposable groups and our invariants generalize the original invariants of Baer, as well as those of Richman.

Additional work on the class Ω appears in [AV-2 and AV-3]. In [AV-2] the authors introduced the concept of quasi-representing graph for groups of the form $G = G(A_1, \dots, A_n)$. A *quasi-representing graph* for G is a subgraph, U, of the complete graph $\hat{U}$ with vertices $1, \dots, n$ (or, where convenient, $A_1, \dots, A_n$) and edges ij labelled by $\text{type}(A_i \cap A_j)$, which is obtained by iteration of the following algorithm: if a graph contains a circuit c with all the edges of c labelled by types $\geq \tau$ and at least one edge labelled by τ, then remove an edge of c labelled by τ. Each edge ij of U can be identified with the pure subgroup $G_{ij} = G \cap (A_i \oplus A_j)$ of G. Moreover, there is a natural quasi-epimorphism $\bigoplus_{ij \in U} G_{ij} \to G$ which is a balanced projective cover in the quasi-homomorphism category (see [AV-3]). As a consequence, up to quasi-isomorphism, the group $G = G(A_1, \dots, A_n)$, may be regarded as the direct sum of the edges of U modulo the circuits of U (by a *circuit* in U we mean a sequence of vertices $i_1, \dots, i_m, i_{m+1} = i_1$ where i_1 is the only repeated vertex and $i_j i_{j+1} \in U$ for $1 \leq j \leq m$). This viewpoint was utilized in [AV-3] to classify, up to quasi-isomorphism, the *CT-groups*, those groups of the form $G(A_1, \dots, A_n)$ having a quasi-representing graph with pairwise incomparable labels of edges. The complete set of invariants in this case was the collection of numbers $r_G(\sigma, \tau) = \text{rank}(G(\tau) + G[\sigma])/G[\sigma]$, where σ and τ range over all pairs of types.

We will need one technical lemma on quasi-representing graphs.

Lemma 1. *Let T be a quasi-representing graph for $G = G(A_1, \dots, A_n)$ and suppose i, j, k are distinct vertices of T with $\text{type}(A_j \cap A_k) \leq \text{type}(A_i)$. If jk is an edge of T, then there is a path p in $T \setminus \{jk\}$ connecting either j or k to i. If p connects j (respectively, k) to i, then jk may be replaced by ik (respectively, ij) to obtain a new quasi-representing graph for G.*

Proof. By [AV-3, Lemma 2.2] there is a path p in T with edges labelled by types greater than or equal to $\tau = \text{type}(A_j \cap A_k)$ which connects the edge jk to the vertex i. Assume, for the sake of simplicity, that p connects the vertex k to the vertex i (and does not contain the edge jk). The condition $\tau \leq \text{type}(A_i)$ implies $\tau \leq \text{type}(A_i \cap A_j) =$ the label on the edge ij. It follows that ij does not belong to T since in that case ij, jk, p would form a circuit in T with edges labelled by types $\geq \tau$ and containing an edge labelled by τ. Moreover, adding the edge ij to T must create a circuit c in T consisting of edges labelled by types greater than or equal to $\sigma = \text{type}(A_i \cap A_j)$ by the algorithm given above for constructing quasi-representing graphs. It follows that $\sigma = \tau$, since if $jk \notin c$, $c \backslash ij, jk, p$ contains a circuit in T with edges labelled by types $\geq \tau$. Thus, the algorithm implies that the edge ij may be added and the edge jk deleted from T to form a new quasi-representing graph for G (also see [AV-2]).

If G is any torsion-free abelian group (hereafter, simply group) and σ is a type (or a rank-1 group), then $G[\sigma] = \bigcap\{\ker f \mid f : G \to Q$ and $\text{type}(f(G)) \leq \sigma\}$. If M is a set of types, $G[M] = \bigcap\{G[\sigma] \mid \sigma \in M\}$, and $r_G[M] = \text{rank}\, G[M]$. If $G = G(A_1, \dots, A_n)$, then we may assume $(A_1, \dots, A_n)$ is *adjusted*, that is, $1 \in A_i$ for each i and $\text{type}(A_i \cap A_j) \leq \text{type}(A_k \cap A_l)$ implies $A_i \cap A_j \subset A_k \cap A_l$. There is no loss of generality in this assumption as each $G(A_1, \dots, A_n)$ is quasi-isomorphic to some $G(B_1, \dots, B_n)$ with $(B_1, \dots, B_n)$ adjusted [AV-2]. Also, there is no loss of generality in assuming each n-tuple $(A_1, \dots, A_n)$ is *trimmed*, that is, the natural projection of $G(A_1, \dots, A_n)$ into A_i is onto for each i (see [R-2]). Frequent use is made of the rank-1 subgroups $G_{ij} = G \cap (A_i \oplus A_j)$ of G and the elements $g_{ij} = (1, -1) \in G_{ij}$, noting that $g_{ij} = -g_{ji}$. All undefined notation and terminology can be found in [A-1 or AV-1].

Following one additional definition, our first two theorems will show that it suffices to classify the strongly indecomposable groups of the form $G(A_1, \dots, A_n)$.

A quasi-representing graph T for $G = G(A_1, \dots, A_n)$ is called *reduced* provided either $\text{rank}\, G = 1$ or any two edges of T belong to a circuit in T and if ij and kl are edges of T with label $ij \leq$ label kl, then ij belongs to a circuit of T not containing kl.

Theorem 2 [AV-3]. *Let $G = G(A_1, \dots, A_n)$ and assume $(A_1, \dots, A_n)$ is adjusted. Then $G = G_1 \oplus \cdots \oplus G_k$, where for each i, G_i is of the form $G(B_1, \dots, B_m)$ for some $m = m(i) \geq 2$ and $B_1, \dots, B_m \subset Q$, and each quasi-representing graph for G_i is reduced.*

Theorem 3. *Let $G = G(A_1, \dots, A_n)$. The following are equivalent.*

(a) *G is strongly indecomposable.*

(b) *Each quasi-representing graph for G is reduced.*

(c) *$G/G[A_i]$ has rank one for $1 \leq i \leq n$.*

(d) $\text{End}(G)$ *is isomorphic to a subring of Q.*

Proof. The implication (a) → (b) follows from Theorem 2, while (d) → (a) by a well-known result of J. D. Reid [Re]. To show (b) → (c), recall [AV-2,3] that each type σ defines an equivalence relation on the vertices of a quasi-representing graph T for G by $A_i \approx A_j$ if $i = j$ or A_i and A_j are connected by a path in T all of whose edges are labelled by types not less than or equal to

σ. The condition "in T" is redundant. Indeed, if $ij \notin T$ is labelled by a type $\tau \not\leq \sigma$, then by Theorem 1.6(c) in [AV-2], i and j are connected by a path in T all of whose edges are labelled by a type $\geq \tau$, hence $\not\leq \sigma$. By Lemma 1.4 of [AV-2], if $i \neq j$, then $A_i \approx A_j$ if and only if $G_{ij} \cap \sum_{kl \in S} G_{kl} \neq 0$, where $S = \{kl \in T \mid \text{type}(A_k \cap A_l) \not\leq \sigma\}$. If $\sigma = \text{type}(A_i)$, then the corresponding equivalence classes of vertices are $V_1 = \{A_i\}, V_2, \ldots, V_m$, where $m \geq 2$ and the rank of $G/G[A_i]$ is $m-1$ [AV-2, Corollary 2.1]. Moreover, if $2 \leq k \neq l \leq m$, then $\text{type}(A_r \cap A_s) \leq \sigma$ if $A_r \in V_k$ and $A_s \in V_l$. It follows from Lemma 1 that G has a quasi-representing graph with no edge connecting vertices in V_k and V_l for $2 \leq k \neq l \leq m$ since any such edge could be replaced by an edge having A_i as a vertex. Repeated applications of this observation yield a quasi-representing graph T with the property that if any edge of T does not have vertices lying in a single V_k, then that edge must have A_i as a vertex. Such a graph is not reduced unless $m = 2$. Since (b) is our hypothesis, $m = 2$ and $\text{rank}(G/G[A_i]) = 1$.

To show (c) $\rightarrow$ (d), note that each $G[A_i]$ is a fully invariant subgroup of G, so each endomorphism f of G induces a rational multiplication (by $a_i \in Q$) on $G/G[A_i]$. In particular, if $g_{ij} = (1, -1) \in G \cap (A_i \oplus A_j)$, then $f(g_{ij} + G[A_i]) = a_i g_{ij} + G[A_i]$. This implies that $f(g_{ij}) = a_i g_{ij} + x_{ij}$ for some $x_{ij} \in G[A_i]$. Similarly, $f(g_{ji}) = a_j g_{ji} + x_{ji}$ for some $x_{ji} \in G[A_j]$. However, $g_{ij} = -g_{ji}$. Moreover, $x_{ij} \in G[A_i] \subset (A_1 \oplus \cdots \oplus A_n)[A_i]$ implies that, as an element of $A_1 \oplus \cdots \oplus A_n$, x_{ij} has zero component in A_i. Similarly, x_{ji} has zero component in A_j. The equation $a_i g_{ij} + x_{ij} = -(a_j g_{ji} + x_{ji})$ then implies $a_i = a_j$, and $x_{ij} + x_{ji} = 0$, in view of the definition of $G(A_1, \ldots, A_n)$. We conclude $x_{ij} = -x_{ji} \in G[A_i] \cap G[A_j]$.

Write $a = a_i$, $1 \leq i \leq n$, and let i, j, k be distinct. Then $0 = f(0) = f(g_{ij} + g_{jk} + g_{ki}) = f(g_{ij}) + f(g_{jk}) + f(g_{ki}) = ag_{ij} + x_{ij} + ag_{jk} + x_{jk} + ag_{ki} + x_{ki} = x_{ij} + x_{jk} + x_{ki}$. Therefore, $x_{ij} = -(x_{jk} + x_{ki}) \in G[A_k]$. Since k was arbitrary, $x_{ij} \in \bigcap\{G[A_k] \mid 1 \leq k \leq n\} = 0$, and the endomorphism f is multiplication by $a \in Q$. This completes the proof.

The above two theorems show that each adjusted $G(A_1, \ldots, A_n)$ decomposes into strongly indecomposable summands of the same form, whose endomorphism rings are principal ideal domains. As noted in the introduction, there is a Krull-Schmidt theorem for these groups in the quasi-homomorphism category, so that we need only classify the strongly indecomposable ones. This we do with two lemmas followed by the main theorem.

Lemma 4. *Let $G = G(A_1, \ldots, A_n)$ be strongly indecomposable and $M \subset \{A_1, \ldots, A_n\}$. Then $G[M] = G \cap \bigoplus\{A_i \mid A_i \notin M\}$. In particular, for all $i \neq j$, $G_{ij} = \bigcap\{G[A_k] \mid k \neq i, k \neq j\}$.*

Proof. For each $i \neq j$, denote $M_{ij} = \{A_k \mid k \neq i, k \neq j\}$. By Theorem 3, for each k the A_k-equivalence classes consist of $\{A_k\}$ and $\{A_l \mid k \neq l\}$. Therefore, if $k \neq i$ and $k \neq j$, then A_i and A_j are in the same A_k-equivalence class and thus, $G_{ij} \cap \sum\{G_{lm} \mid \text{type}(G_{lm}) \nleq \text{type}(A_k)\} \neq 0$ (refer to the proof of Theorem 3). It follows that $G_{ij} \subset G[A_k]$ whenever $k \neq i$ and $k \neq j$, so that $G_{ij} \subset \bigcap\{G[A_k] \mid k \neq i, k \neq j\} = G[M_{ij}]$. If $M \subset M_{ij}$, then $G_{ij} \subset G[M_{ij}] \subset G[M]$. We conclude that $\sum\{G_{ij} \mid A_i, A_j \notin M\} = G \cap \bigoplus\{A_i \mid A_i \notin M\} \subset G[M]$. On the other hand, it is clear from the projection maps that any element of $G[M]$, written as an n-tuple $(a_1, \dots, a_n)$ in $A_1 \oplus \cdots \oplus A_n$, must have $a_k = 0$ for all k such that $A_k \in M$. Therefore, $G[M] \subset G \cap \bigoplus\{A_i \mid A_i \notin M\}$.

Lemma 5. *Let $G = G(A_1, \dots, A_n)$ and $H = G(B_1, \dots, B_n)$ be strongly indecomposable and assume that $r_G[M] = r_H[M]$ for each finite set of types M. If $M_{ij} = \{A_k \mid k \neq i, k \neq j\}$, then,*

(a) *$X_{ij} = H[M_{ij}]$ has rank one and type $X_{ij} = \text{type}\, G_{ij}$.*

(b) *If S is a subset of $\{kl \mid 1 \leq k \neq l \leq n\}$, then $G_{ij} \cap (\sum\{G_{kl} \mid kl \in S\} = 0$ if and only if $X_{ij} \cap (\sum\{X_{kl} \mid kl \in S\} = 0$.*

Proof. (a) For a fixed ij, denote $M = M_{ij}$. By Lemma 4, $1 = r_G[M] = r_H[M]$. Thus, there is a (unique) pure rank-1 subgroup $X_{ij} = H[M_{ij}]$ of H. Note that $X_{ij} \cap H[A_i] = 0$ since $\text{rank}(H[M_{ij}] \cap H[A_i]) = \text{rank}(H[M_{ij} \cup \{A_i\}]) = \text{rank}(G[M_{ij} \cup \{A_i\}]) = 0$. Similarly, $X_{ij} \cap H[A_j] = 0$. We conclude $\text{type}\, X_{ij} \leq \inf\{\text{type}\, A_i, \text{type}\, A_j\} = \text{type}\, G_{ij}$. Furthermore, if $\tau = \text{type}\, G_{ij}$, then $G(\tau) = G[M_\tau]$ where $M_\tau = \{\sigma \in \text{typeset}\, G \mid \tau \nleq \sigma\}$ (see [AV-1]). Thus, $1 = \text{rank}\, G_{ij} = \text{rank}(G(\tau) \cap G[M_{ij}]) = \text{rank}(G[M_{ij} \cup M_\tau]) = \text{rank}(H[M_{ij} \cup M_\tau]) = \text{rank}(H(\tau) \cap H[M_{ij}])$. This shows that $\text{type}\, X_{ij} \geq \tau$, and therefore that type $X_{ij} = \text{type}\, G_{ij}$.

(b) Let S be a subset of $\{ij | 1 \leq i \neq j \leq n\}$. We first show that $\{G_{ij} \mid ij \in S\}$ independent in G implies $\{X_{ij} \mid ij \in S\}$ independent in H. Apply induction on $|S|$, the case $|S| = 1$ being trivial. In general, $\{G_{ij} \mid ij \in S\}$ independent implies that there is an index k which appears in exactly one element kl of S [AV-2, Lemma 1.4]. Let $S' = S \setminus \{kl\}$. By induction $\{X_{ij} \mid ij \in S'\}$ is independent in H. By the choice of k, $X_{ij} \subset H[A_k]$ for all $ij \in S'$. Since $X_{kl} \cap H[A_k] = 0$ as above, X_{kl} is independent of $\{X_{ij} \mid ij \in S'\}$. This proves the "only if" portion of (b).

For the converse, we show that if S is a subset of $\{ij \mid 1 \leq i \neq j \leq n\}$, then $(\sum_{ij \in S} G_{ij}) \cap G_{kl} \neq 0$ implies $(\sum_{ij \in S} X_{ij}) \cap X_{kl} \neq 0$. Without loss of generality we may assume that, given a double index kl, S is minimal with respect to $(\sum_{ij \in S} G_{ij}) \cap G_{kl} \neq 0$. In this case, either $S = \{kl\}$, or $kl \neq S$ and $S \cup \{kl\} = c$ forms a circuit in the complete graph with vertices $1, \dots, n$ and edges ij connecting i and j [AV-2, Lemma 1.4]. In the first instance the desired conclusion is immediate. In the second, $\sum_{ij \in S} G_{ij} = \bigoplus_{ij \in S} G_{ij}$ is a full subgroup of (has torsion index in) $G[M]$, where $M = \{A_m \mid m$ is not a vertex of $c\}$. To see this, note that for each $ij \in S$, $G_{ij} \subset G[M]$ by Lemma 4, that the sum $\sum G_{ij}$ is direct since S is minimal, and that $\bigoplus G_{ij}$ is full in $G[M]$ since $G[M] = G \cap \bigoplus\{A_i \mid A_i \notin M\}$ by Lemma 4. Now, by the first part of the proof and the definition of X_{ij}, $\sum_{ij \in S} X_{ij} = \bigoplus X_{ij}$ is a subgroup of $H[M]$. However, $r_G[M] = r_H[M]$, so that $\bigoplus_{ij \in S} X_{ij}$ is a full subgroup of $H[M]$. Again by definition, $X_{kl} \subset H[M]$, and the lemma follows.

Theorem 6. *Let $G = G(A_1, \dots, A_n)$ and $H = G(B_1, \dots, B_n)$ be strongly indecomposable. Then $r_G[M] = r_H[M]$ for each finite set of types M if and only if G and H are quasi-isomorphic.*

Proof. The "if" direction is clear. For the converse, up to quasi-isomorphism we may, as usual, assume the n-tuples $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ are adjusted and trimmed. Recall that $-g_{ji} = g_{ij} = (1, -1) \in G \cap (A_i \oplus A_j)$. We will define a quasi-monomorphism $\varphi\colon G \to H$ by defining $0 \neq \varphi(g_{ij}) = x_{ij} \in X_{ij}$ (type G_{ij} = type X_{ij} by Lemma 5) for all indices ij, and showing that for all subsets S of indices and all choices of coefficients $a_{ij} \in Q$,

$$(7) \qquad \sum_{ij \in S} a_{ij} g_{ij} = 0 \text{ in } G \text{ if and only if } \sum_{ij \in S} a_{ij} x_{ij} = 0 \text{ in } H.$$

The theorem will then follow by symmetry.

We will use induction on the set of possible indices ij which can appear in S, beginning with the first nontrivial case, $S = \{12, 23, 31\}$. Since $g_{12} + g_{23} + g_{31} = 0$, by Lemma 5 there is a choice $0 \neq \varphi(g_{ij}) = x_{ij} \in X_{ij}$ for $ij \in \{12, 23, 31\}$, such that $x_{23} + x_{23} + x_{31} = 0$. In this case the one nontrivial relationship between the g_{ij}'s is of the form $a(g_{12} + gj_{23} + g_{31}) = 0$ for $a \in Q$. Suppose $ax_{12} + bx_{23} + cx_{31} = 0$ and assume, for example, $a \neq b$. Subtracting $a(x_{12} + x_{23} + x_{31}) = 0$ yields $(b - a)x_{23} + (c - a)x_{31} = 0$, which contradicts Lemma 5. Thus, (7) is satisfied for the first step of the induction.

Assume φ has been defined on G_{ij} for $i, j < m (m \geq 4)$ so that (7) is satisfied. We will show how to define φ on $G_{mi} = G_{im}$, $i < m$. Since $(G_{1m} + G_{m2}) \cap G_{21} \neq 0$, by Lemma 5 we can define $0 \neq \varphi(g_{1m}) = x_{1m}$ and $0 \neq \varphi(g_{m2}) = x_{m2}$ so that $x_{1m} + x_{m2} + x_{21} = 0$. To show that (7) holds for the set of indices $S = \{ij \mid i, j < m \text{ or } ij = 1m \text{ or } m2\}$, suppose $\sum_{ij \in S} a_{ij} x_{ij} = 0$. Without loss of generality we may assume $a_{1m} \neq 0$. Let $x = \sum_{ij \in S} a_{ij} x_{ij} - a_{1m}(x_{1m} + x_{m2} + x_{21}) = 0$. Note that if $a_{1m} \neq a_{m2}$ the equation $x = 0$ implies $(\sum\{X_{ij} \mid i, j < m\}) \cap X_{m2} \neq 0$, an impossibility by Lemma 5. Therefore, $x = \sum_{i,j<m} b_{ij} x_{ij} = 0$, for some choice of b_{ij}, so that by induction, $\sum_{ij \in S} a_{ij} g_{ij} - a_{1m}(g_{1m} + g_{m2} + g_{21}) = \sum_{i,j<m} b_{ij} g_{ij} = 0$. It follows that $\sum_{ij \in S} a_{ij} g_{ij} = 0$. A symmetric argument shows that $\sum_{ij \in S} a_{ij} g_{ij} = 0$ implies $\sum_{ij \in S} a_{ij} x_{ij} = 0$.

Suppose φ has been defined on G_{ij} for $i, j < m$ and for $i = m$, $1 \leq j \leq k-1$, where $3 \leq k < m$. Since $(G_{(k-1)m} + G_{mk}) \cap G_{k(k-1)} \neq 0$, by Lemma 5 there exist $0 \neq \varphi(g_{mk}) = x_{mk} \in X_{mk}$ and $a \in Q$ such that $y = x_{k(k-1)} + ax_{(k-1)m} + x_{mk} = 0$. We first show $a = 1$. If not, $y - (x_{k(k-1)} + x_{(k-1)m} + x_{m1} + x_{1k}) = 0$ implies that $X_{(k-1)m} \cap (X_{mk} + X_{m1} + X_{1k}) \neq 0$, which is impossible by Lemma 5.

Therefore $a = 1$. To show (7) holds for the augmented set of indices including mk is a calculation similar to that performed in the previous paragraph. The proof is then complete.

To conclude, we return to the invariants of Baer, Richman and [AV-3]. Note that if G is completely decomposable, then the number of summands of type τ is $\operatorname{rank}(G^*[\tau]/G[\tau]) = \operatorname{rank} G^*[\tau] - \operatorname{rank} G[\tau]$, where $G^*[\tau] = \bigcap\{G[\sigma] \mid \sigma < \tau\} = G[M]$ for $M = \{\sigma \mid \sigma < \tau\}$. Thus, the $r_G[M]$'s provide the invariants of Baer. On the other hand, if $G = G(A_1, \dots, A_n)$ is doubly incomparable, one can check that the Richman invariants, $\{\operatorname{type}(A_i) \mid 1 \leq i \leq n\}$ are the meet irreducible elements of cotypset $G = \{\sigma \mid \operatorname{type}(f(G)) = \sigma$ for some $f: G \to Q\} = \{\sigma \mid \operatorname{rank}(G^*[\sigma]/G[\sigma]) \neq 0\}$. Therefore these invariants also may be calculated from the $r_G[M]$'s. Finally, to calculate the invariants $r_G(\sigma, \tau) = \operatorname{rank}(G(\tau) + G[\sigma])/G[\sigma] = \operatorname{rank} G(\tau)/(G(\tau) \cap G[\sigma])$ of [AV-3], we use the fact noted previously that $G(\tau) = G[M_\tau]$, where $M_\tau = \{\sigma \mid \tau \nleq \sigma\}$. Indeed, it is then routine to check that $r_G(\sigma, \tau) = r_G[M_\tau] - r_G[M_\tau \cup \{\sigma\}]$.

References

[A-1] D. Arnold, *Finite rank torsion free abelian groups and rings*, Lecture Notes in Math., vol 931, Springer-Verlag, Berlin and New York, 1982.

[A-2] ——, *Representations of partially ordered sets and abelian groups*, Proceedings of the 1987 Perth Conference on Abelian Groups (to appear).

[A-3] ——, *Pure subgroups of finite rank completely decomposable groups*, Abelian Group Theory, Lecture Notes in Math., vol. 874, Springer-Verlag, Berlin and New York, 1981, pp. 1-31.

[AV-1] D. Arnold and C. Vinsonhaler, *Pure subgroups of finite rank completely decomposable groups*. II, Abelian Group Theory, Lecture Notes in Math., vol. 1006, Springer-Verlag, Berlin and New York, 1983, pp. 97–143.

[AV-2] ——, *Representing graphs for a class of torsion-free abelian groups*, Abelian Group Theory, Gordon and Breach, London, 1987, pp. 309–332.

[AV-3] ——, *Quasi-isomorphism invariants for a class of torsion-free abelian groups*, Houston J. Math. (to appear).

[AV-4] ——, *Endomorphism rings of Butler groups*, J. Austral. Math. Soc. **42** (1987), 322–329.

[AV-5] ——, *Coxeter functors and duality for abelian groups*, preprint.

[Ba] R. Baer, *Abelian groups without elements of finite order*, Duke Math. J. **3** (1937), 68–122.

[BB] S. Brenner and M. C. R. Butler, *Endomorphism rings of vector spaces and torsion free abelian groups*, J. London Math. Soc. **40** (1965), 183–187.

[BP] R. Beaumont and R. S. Pierce, *Torsion-free groups of rank two*, Mem. Amer. Math. Soc., No. 38 (1961).

[Bu-1] M. C. R. Butler, *A class of torsion-free abelian groups of finite rank*, Proc. London Math. Soc. **15** (1965), 680–698.

[Bu-2] ——, *Torsion-free modules and diagrams of vector spaces*, Proc. London Math. Soc. **18** (1968), 635–652.

[Bu-3] ——, *Some almost split sequences in torsion-free abelian group theory*, Abelian Group Theory, Gordon and Breach, New York, 1987, pp. 291–302.

[Re] J. D. Reid, *On the ring of quasi-endomorphisms of a torsion-free group*, Topics in Abelian Groups, Chicago, Ill, 1963, pp. 51–68.

[R-1] F. Richman, *Butler groups, valuated vector spaces, and duality*, Rend. Sem. Mat. Univ. Padova **72** (1984), 13–19.

[R-2] ——, *An extension of the theory of completely decomposable torsion-free abelian groups*, Trans. Amer. Math. Soc. **279** (1983), 175–185.

DEPARTMENT OF MATHEMATICS, NEW MEXICO STATE UNIVERSITY, LAS CRUCES, NEW MEXICO 88003

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06268