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PREORDERS COMPATIBLE WITH PROBABILITY MEASURES DEFINED ON A BOOLEAN ALGEBRA

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ABSTRACT. We use nonstandard analysis techniques to find conditions for the existence of a strictly positive measure weakly compatible with a preorder defined in a Boolean algebra generated by denumerably many atoms.

0. INTRODUCTION

When assigning probabilities, events can be considered as elements of a σ-complete Boolean algebra B subject to the usual operation of union, intersection, complementation and σ-union.

Sometimes it is not possible to directly assign probabilities to events; however, it may be possible to compare events qualitatively with respect to a probability. Thus, we may be able to introduce a preorder $\lesssim$ on B where $a \lesssim b$ can be intuitively interpreted as "b is at least as probable as a".

It is natural then to ask what conditions on B and $\lesssim$ are necessary and sufficient for the existence of a probability measure μ on B such that for $p, q \in B$, $p \lesssim q$ iff $\mu(p) \leq \mu(q)$. If this is the case, we say that μ is compatible with $\lesssim$.

Villegas [4] found necessary conditions for the existence of a unique measure when B is atomless, namely

(1) $\lesssim$ is a preorder on B, i.e. $p \lesssim q$ and $q \lesssim r$ imply $p \lesssim r$, and $p \lesssim q$ or $q \lesssim p$ for $p, q, r \in B$;

(2) $\neg(1 \lesssim 0)$ and $0 \lesssim p$ for every $p \in B$;

(3) If $p, q, r \in B$ with $p \cap q \approx r \cap q \approx 0$ then $p \lesssim r$ iff $p \cup q \lesssim r \cup q$ ($p \approx q$ is defined as $p \lesssim q$ and $q \lesssim p$);

(4) If $p_i, q \in B$, and $p_i \subseteq p_{i+1} \lesssim q$ for every $i \in \mathbb{N}$, then $\bigcup p_i \lesssim q$.

He also proved that in any σ-complete Boolean algebra where (1)–(4) are satisfied there are at most denumerably many atoms and they can be ordered

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in a decreasing sequence $a_0 \gtrsim a_1 \gtrsim a_2 \gtrsim \cdots$, where $\gtrsim$ is defined by $p \gtrsim q$ iff $q \lesssim p$.

If I is the ideal of all events p with $p \approx 0$, it is easy to show, using the quotient algebra B/I, that it is enough to solve the problem for strictly positive compatible σ-measures, where a measure is strictly positive if $\mu(a) = 0$ implies $a = 0$.

In [3] Chuaqui and Schwarze claimed that conditions (1)–(4) were sufficient for the existence of a σ-measure compatible with a preorder on B without the assumption that B is atomless. This however is false.

In [2] Kraft, Pratt and Seidenberg gave an example of a Boolean algebra generated by 5 atoms where (1)–(3) and trivially (4) are satisfied but the algebra has no compatible measure. It is necessary to add one further condition:

(5) Every finite subalgebra B' of B has a compatible measure.

With the methods used in [3] it is possible to show that conditions (1)–(5) are necessary and sufficient for the existence of a unique σ-measure compatible with a preorder defined on a σ-complete Boolean algebra in which $\bigcup a_i \lesssim\sim (\bigcup a_i)$.

The counterexample given in [2] also refutes a conjecture of de Finetti. He conjectured that for any finite preorder satisfying (1), (2), and (3) there exists a measure compatible with it. He also suggested that, if the conjecture is false, this is due to an inappropriate definition of the algebra B.

According to [2] it is possible to replace condition (5) by the following algebraic condition:

(5′) Let B' be any finite subalgebra of B.

Let $b_0, b_1, \dots, b_{n-1}$ be the atoms of B'. Let $p_0, p_1, \dots, p_{m-1}, q_0, q_1, \dots, q_{m-1}$ be elements of B' such that every b_i is contained in the same number of p_j's and q_j's and $p_i \lesssim q_i$ for $i = 1, 2, \dots, m-1$. Then $q_0 \lesssim p_0$.

In [1] Chuaqui and Malitz studied the case where B is an atomistic Boolean algebra.

The purely atomistic case has mathematical interest in its own right since, as Villegas proved, if there exists a compatible measure, there are at most countably many atoms, so that B can be viewed as the power set of ω, the set of natural numbers.

Chuaqui and Malitz were able to show that conditions (1)–(5) imply the existence of a weakly compatible strictly positive measure μ (i.e., a probability measure such that $p \lesssim q$ implies $\mu(p) \le \mu(q)$), thus allowing the possibility that $p \prec q$ but $\mu(p) = \mu(q)$.

They also showed that when B has a nonempty atomless part, conditions (1)–(5) imply the existence of a compatible measure.

When comparing events, sometimes the order obtained (by subjective or objective methods) is not total, so that we just have a partial preorder, where $\lesssim$ defined in B is a partial preorder if $\lesssim$ is reflexive in B and transitive (i.e., $p \lesssim p$, $p \lesssim q$ and $q \lesssim r$ implies $p \lesssim r$ for $p, q, r \in B$).

The main purpose of this work is to find conditions for the existence of a strictly positive σ-measure weakly compatible with a partial preorder defined in the σ-complete Boolean algebra B of all subsets of ω, the set of natural numbers. This will be accomplished using nonstandard analysis.

1. Preliminaries

Throughout this work B will be the σ-complete Boolean algebra of all subsets of ω. Let $\{a_i : i \in \omega\}$ be the set of atoms of B. $\precsim$ will be a partial preorder defined on B which extends $\subseteq$.

If $p \cap q = \mathbf{0}$, $p + q$ is $p \cup q$. If for all $i, j \in I$ with $i \neq j$, $p_i \cap p_j = \mathbf{0}$, $\sum\{p_i : i \in I\} = \bigcup\{p_i : i \in I\}$. Note that these operations are defined only for disjoint elements.

$p \prec q$ is defined as $p \precsim q$ but $\neg(q \precsim p)$.

$p \approx q$ is defined as $p \precsim q$ and $q \precsim p$.

Let s_n be defined as $s_n = \sum\{a_k : k > n\}$. Such elements are called tails.

By a probability measure on B we understand a σ-additive strictly positive real valued measure μ on B, such that $0 < \mu(p) \leq \mu(\sum\{a_i : i \geq 0\}) < \infty$, for all $p \in B$ with $\mathbf{0} < p$.

μ is compatible with $\precsim$ iff for all $p, q \in B$, $p \precsim q$ iff $\mu(p) \leq \mu(q)$.

μ is weakly compatible with $\precsim$ if for all $p, q \in B$, $p \precsim q$ implies $\mu(p) \leq \mu(q)$.

2. Main theorem

Throughout the discussion we will assume that $\precsim$ has the following properties:

(1) Every finite subalgebra B' of B generated by atoms and tails has a strictly positive measure weakly compatible with $\precsim$.

(II) For all $k \in \omega$ there exists $k' \in \omega$ such that $s_{k'} \precsim a_k$.

We will prove that conditions (I) and (II) are necessary and sufficient for the existence of a σ-measure strictly positive weakly compatible with $\precsim$.

Condition (I) is obviously necessary, condition (II) is necessary to insure convergence.

Definition 1. For $p \in B$ let $p_{(n)}$ and $p^{(n)}$ be defined as follows:

$$p_{(n)} = p \cap \sum\{a_i : i \leq n\}, \qquad p^{(n)} = p \cup s_n.$$

Notice that for all n, $p_{(n)} \subseteq p \subseteq p^{(n)}$.

Definition 2. $B_1 = \{p^{(n)} : p \in B, n \in \omega\} \cup \{p_{(n)} : p \in B, n \in \omega\}$.

For all $p \in B_1$, let c_p be a constant. Then (I) implies that

$$T = \{c_p \leq c_q : p, q \in B_1 \text{ and } p \precsim q\} \cup \{c_p = c_q + c_r : p, q, r \in B_1 \text{ and } p \approx q + r\} \cup \mathrm{Th}\,\mathbf{R}$$

has a model $\mathfrak{M}$.

We can assume $\mathfrak{M}$ is an $\aleph_1$-saturated nonstandard model of T.

For $p \in B$, we define $\mathbf{p} \in |\mathfrak{M}|$ in the following way:

(a) If there is a $q \in B_1$ such that $p \approx q$, let $\mathbf{p} = c_q^{\mathfrak{M}}$.

(b) If not, as $\mathfrak{M}$ is $\aleph_1$-saturated there exists $\{p_n\}_{n\in {}^*\mathbb{N}}$ and $\{p^n\}_{n\in {}^*\mathbb{N}}$ internal sequences such that $p_n = \mathbf{p}_{(\mathbf{n})}$ and $p^n = \mathbf{p}^{(\mathbf{n})}$ for $n \in \mathbb{N}$. Let $M = \{n \in {}^*\mathbb{N}$: For $0 \le k \le n$, $\{p_k\}$ is increasing, $\{p^k\}$ is decreasing, and for all $r \le n$ and $k \le n$ $(p_r < p^k)\}$.

Then M is internal and $\mathbb{N} \subseteq M$, so there must exist $\lambda \in M$, such that $\lambda \in {}^*\mathbb{N} - \mathbb{N}$. We choose such a λ and define $\mathbf{p} = p_\lambda$. It is clear that for all $p \in B$, $n \in \mathbb{N}$, $\mathbf{p}_{(\mathbf{n})} \le \mathbf{p} \le \mathbf{p}^{(\mathbf{n})}$.

Definition 3. Let $<_1$ and $\approx_1$ be defined on B, as follows: $p <_1 q$ iff there exists a_k such that $\mathbf{p} + \mathbf{a_k} < \mathbf{q}$; $p \approx_1 q$ iff for all a_k, $|\mathbf{p} - \mathbf{q}| \le \mathbf{a_k}$. $p \precsim_1 q$ is defined as $p <_1 q$ or $p \approx_1 q$. It is clear that $p \precsim_1 q$ iff for all k, $\mathbf{p} < \mathbf{q} + \mathbf{a_k}$.

Lemma 1. $\precsim_1$ *is a simple preorder.*

Proof. We will prove first that $\precsim_1$ is simple. If $\neg(p \approx_1 q)$, then there exists a_k such that $|\mathbf{p} - \mathbf{q}| > a_k$. Then $\mathbf{p} - \mathbf{q} > \mathbf{a_k}$ and $q <_1 p$, or $\mathbf{p} - \mathbf{q} < -\mathbf{a_k}$ and $p <_1 q$.

To show that $\precsim_1$ is transitive, let $p, q, r \in B$ be such that $p \precsim_1 q$ and $q \precsim_1 r$. Given a_k, there exists $s_{k'}$ such that $s_{k'} \precsim a_k$ and since $p \precsim_1 q$ and $q \precsim_1 r$, $\mathbf{p} < \mathbf{q} + \mathbf{a_{k'+1}}$ and $\mathbf{q} < \mathbf{r} + \mathbf{a_{k'+2}}$. Then $\mathbf{p} < \mathbf{r} + \mathbf{a_{k'+1}} + \mathbf{a_{k'+2}} < \mathbf{r} + \mathbf{s_{k'}} < \mathbf{r} + \mathbf{a_k}$, and therefore $p \precsim_1 r$.

Lemma 2. *If* $p \precsim_1 q$, *then there exists* $n \in \mathbb{N}$, $m \in \mathbb{N}$ *such that* $\mathbf{p}^{(\mathbf{n})} < \mathbf{q}_{(\mathbf{m})}$.

Proof. There exists a_k such that $\mathbf{p} + \mathbf{a_k} < \mathbf{q}$. Let m be such that $s_m \precsim a_k$. Then $\mathbf{s_m} \le \mathbf{a_k}$ and $\mathbf{p} + \mathbf{s_m} < \mathbf{q}$. We can choose s_n such that $s_n \precsim a_{m+1}$, so

$$\mathbf{p} + \mathbf{s_n} + \mathbf{s_{m+1}} < \mathbf{p} + \mathbf{a_{m+1}} + \mathbf{s_{m+1}} = \mathbf{p} + \mathbf{s_m} < \mathbf{p} + \mathbf{a_k} < \mathbf{q}.$$

But $\mathbf{q} \le \mathbf{q}^{(\mathbf{m+1})} = \mathbf{q}_{(\mathbf{m+1})} + \mathbf{s_{m+1}}$ therefore $\mathbf{p} + \mathbf{s_n} + \mathbf{s_{m+1}} < \mathbf{q}_{(\mathbf{m+1})} + \mathbf{s_{m+1}}$ and thus $\mathbf{p}^{(\mathbf{n})} < \mathbf{q}_{(\mathbf{m+1})}$.

Lemma 3. *For all* $p, q \in B_1$ *we have that* $p \precsim q$ *implies* $p \precsim_1 q$.

Proof. If not, there exist $p, q \in B$ such that $p \precsim q$ but $\neg(p \precsim_1 q)$. Then, Lemma 1 implies that $q \precsim_1 p$. Now Lemma 2 implies that there exists $n, m \in \mathbb{N}$ such that $\mathbf{q}^{(\mathbf{n})} < \mathbf{p}_{(\mathbf{m})}$. But $p_{(m)} \precsim p \precsim q \precsim q^{(n)}$ so $p_{(m)} \precsim q^{(n)}$, also $p_{(m)}$, $q^{(n)} \in B_1$, therefore $\mathbf{p_m} = (\mathbf{p_m})^{\mathfrak{M}} \le (\mathbf{q^n})^{\mathfrak{M}} = \mathbf{q^n}$: a contradiction.

Lemma 4. *For all* $q, p_i \in B$, $i = 1, 2, \ldots$ *with* $p_i \subseteq p_{i+1} \precsim_1 q$. *We have* $\bigcup p_i \precsim_1 q$.

Proof. If not, there exist $q, p_i \in B$ with $p_i \subseteq p_{i+1} \precsim_1 q$ but $q \precsim_1 \bigcup p_i$. But then, there exist k, r such that $\mathbf{q}^{(\mathbf{r})} < (\bigcup p_i)_{(\mathbf{k})}$. But $(\bigcup p_i)_{(k)} = (p_n)_{(k)}$ for some n, therefore $\mathbf{q} \le \mathbf{q}^{(\mathbf{r})} < (\mathbf{p_n})_{(\mathbf{k})}$.

If there exists a_t, with $a_t \subseteq \bigcup p_i \sim (p_n)_{(k)}$, then there exists m such that

$$\mathbf{q} + \mathbf{a_t} < (\mathbf{p_n})_{(\mathbf{k})} + \mathbf{a_t} \le (\mathbf{p_n})_{(\mathbf{k})} + \mathbf{a_t} \le (\mathbf{p_m})_{(\mathbf{t})} \le \mathbf{p_m},$$

which contradicts the fact that $p_m \precsim_1 q$.

If there is no such a_t, then $\bigcup p_i = p_n$. But $p_n \precsim_1 q$, therefore $\bigcup p_i \precsim_1 q$.

Lemma 5. *Every finite subalgebra B' of B has a strictly positive σ-measure μ compatible with $\precsim_1$.*

Proof. Let $\{b_0, b_1, \dots, b_{n-1}\}$ be the set of atoms of B'. By [2] it is enough to show that for all elements $p_0, p_1, \dots, p_{m-1}, q_0, \dots, q_{m-1}$ such that each b_j, $j < n$, appears in the same number of p_i's and q_i's with $p_i \precsim_1 q_i$ for $i = 1, 2, \dots, m-1$. Then $q_0 \precsim_1 p_0$. Let

$$p_i = \alpha_{i0}b_0 + \alpha_{i1}b_1 + \cdots + \alpha_{in-1}b_{n-1}; \quad q_i = \beta_{i0}b_0 + \beta_{i1}b_1 + \cdots + \beta_{in-1}b_{n-1},$$
$$\text{for } i < m \text{ and } \alpha_{ij}, \beta_{ij} \in \{0, 1\}.$$

By hypothesis we know $\sum_{\theta \le j \le m-1} \alpha_{ij} = \sum_{\theta \le j \le m-1} \beta_{ij}$, for $i < n$. Given a_k there exists $s_{k'}$ such that $s_{k'} \precsim a_k$.

$p_i \precsim_1 q_i$, therefore $\mathbf{p_i} < \mathbf{q_i} + \mathbf{a_{k'+i}}$, $i < m$. We have $\sum_{1 \le i \le m-1} (\mathbf{p_i} - \mathbf{q_i}) < \sum_{1 \le i \le m-1} \mathbf{a_{k'+i}} < \mathbf{s_{k'}} < \mathbf{a_k}$. But

$$\sum_{1 \le i \le m-1} (\mathbf{p_i} - \mathbf{q_i}) = \sum_{1 \le j \le m-1} (\alpha_{j0} - \beta_{j0})\mathbf{b_0} + \sum_{1 \le j \le m-1} (\alpha_{j1} - \beta_{j1})\mathbf{b_1} + \cdots + \sum_{1 \le j \le m-1} (\alpha_{jn-1} - \beta_{jn-1})\mathbf{b_{n-1}}$$

and

$$\sum_{1 \le j \le m-1} (\alpha_{ji} - \beta_{ji}) = \sum_{\theta \le j \le m-1} \alpha_{ji} - \alpha_{0i} - \sum_{\theta \le j \le m-1} \beta_{ji} + \beta_{0i} = \beta_{0i} - \alpha_{0i}.$$

Therefore

$$(\beta_{00} - \alpha_{00})\mathbf{b_0} + (\beta_{01} - \alpha_{01})\mathbf{b_1} + \cdots + (\beta_{0n-1} - \alpha_{0n-1})\mathbf{b_{n-1}} < \mathbf{a_k},$$

so $\mathbf{q_0} - \mathbf{p_0} < \mathbf{a_k}$. But a_k was arbitrary, so $q_0 \precsim_1 p_0$.

Lemmas 4 and 5 together with [1, Theorem 5.3] imply the existence of a strictly positive σ-measure μ weakly compatible with $\precsim_1$; this together with Lemma 3 implies the following theorem.

Theorem. *Let B be a σ-complete Boolean algebra generated by countably many atoms $\{a_i : i \in \omega\}$ and $\precsim$ a partial preorder on B which extends $\subseteq$, such that*

(I) *Every finite subalgebra B' of B generated by atoms and tails has a strictly positive measure weakly compatible with $\precsim$.*

(II) *For any atom a_k there exists a tail $s_{k'}$ such that $s_{k'} \precsim a_k$.*

Then there exists a strictly positive σ-measure μ weakly compatible with $\precsim$.

3. Counterexamples

Example 1. Let B be a σ-complete Boolean algebra generated by $\{a_i : i \in \omega\}$. For $p \in B$ we define $\mathbf{p}$ in the following way:

If $p = \sum\{a_i : i \in J\}$ with $J \subseteq \omega$, then $\mathbf{p} = \lim_{i\to\infty} \sum_{0\le k\le i}[\chi_J(k)] \cdot (1/2)^{k+1}$, where χ_J is the characteristic function of J.

Now we define $\precsim$ on B. Let p and q be any elements of B.

(i) If $a_0 \subsetneq p \cup q$, then $p \precsim q$ iff $\mathbf{p} \le \mathbf{q}$.

(ii) If $p \subsetneq \sum\{a_i : i > 0\}$ and $a_0 \subseteq q$, then $p \prec q$.

(iii) $a_0 \prec \sum\{a_i : i > 0\}$.

(iv) If $a_0 \subseteq p \cap q$, then $p \precsim q$ iff $(p - a_0) \precsim (q - a_0)$.

Then $\precsim$ is a total preorder; furthermore for all n, $s_{n+2} \precsim a_n$ and every subalgebra B' of B has a measure compatible with $\precsim$. Therefore B has a σ-measure weakly compatible with $\precsim$. However, for any measure weakly compatible with $\precsim$, $\mu(a_0) = \mu(\sum\{a_i : i > 0\})$ holds, so B admits no compatible measure.

This example shows that strengthening condition (5) does not yield a stronger result. Namely, even if every finite subalgebra B' of B has a measure compatible with $\precsim$, B admits only weakly compatible σ-measures.

The following two examples will show that condition (II) is necessary when the σ-algebra is infinite. In Example 2 the σ-algebra admits a finitely additive measure μ compatible with $\precsim$, but no σ-additive measure weakly compatible with it. In Example 3 the σ-algebra does not admit even a finitely additive measure compatible with $\precsim$.

Example 2. Let B be a Boolean algebra generated by denumerably many atoms $\{a_i : i \in \omega\}$, and Cof the filter over ω such that $a \subset \omega$ belongs to Cof if and only if $\omega \sim a$ is finite.

Let U be an ultrafilter over ω such that $\mathrm{Cof} \subset U$. Let $\mu : B \to [0, 1]$ be defined as:

$$\mu(p) = \begin{cases} \frac{1}{2} + \sum\{(\frac{1}{2})^{i+2} : a_i \subseteq p\}, & \text{if } \{i : a_i \subseteq p\} \in U, \\ \sum\{(\frac{1}{2})^{i+2} : a_i \subseteq p\}, & \text{otherwise}. \end{cases}$$

To show μ is finitely additive, let $p \cap q = 0$. If $\{i : a_i \subseteq p\} \in U$, then $\{i : a_i \subseteq q\} \notin U$ and $\{i : a_i \subseteq p \cup q\} \in U$. So:

$$\mu(p) + \mu(q) = \frac{1}{2} + \sum\left\{\left(\frac{1}{2}\right)^{i+2} : a_i \subseteq p\right\} + \sum\left\{\left(\frac{1}{2}\right)^{i+2} : a_i \subseteq q\right\}$$
$$= \frac{1}{2} + \sum\left\{\left(\frac{1}{2}\right)^{i+2} : a_i \subseteq p \cup q\right\} = \mu(p \cup q).$$

Now if $\{i : a_i \subseteq p\} \notin U$ and $\{i : a_i \subseteq q\} \notin U$, then $\{i : a_i \subseteq p \cup q\} = \{i : a_i \subseteq p\} + \{i : a_i \subseteq q\} \notin U$. So

$$\mu(p) + \mu(q) = \sum\left\{\left(\frac{1}{2}\right)^{i+2} : a_i \subseteq p\right\} + \sum\left\{\left(\frac{1}{2}\right)^{i+2} : a_i \subseteq q\right\}$$
$$= \sum\left\{\left(\frac{1}{2}\right)^{i+2} : a_i \subseteq p \cup q\right\} = \mu(p \cup q).$$

Then the order $\precsim$ induced by μ has a compatible measure, but no σ-additive compatible measure, since for all $n \in \omega$, $\sum\{a_i : 0 \leq 1 \leq n\} < \sum\{a_i : 1 > n\}$.

Example 3. Let B be generated by denumerably many atoms $\{a_i : i \in \omega\}$. For all $p \in B$ let $\mathbf{p}$ be a constant. Let

$$T = \{\mathbf{a_n} = \mathbf{a_m} : n, m \in \omega\} \cup \{\mathbf{p} < \mathbf{q} : p \subset q\}$$
$$\cup \{\mathbf{p} = \mathbf{q} : p = q\} \cup \{\mathbf{p} = \mathbf{q} + \mathbf{r} : p = q + r\}.$$

Then every finite subset T' of $T \cup \mathrm{Th}\,\mathbb{R}$, has a model, then $T \cup \mathrm{Th}\,\mathbb{R}$ has a model $\mathfrak{M}$. Let $\precsim$ be the preorder on B defined by: $p \precsim q$ iff $\mathbf{p}^{\mathfrak{M}} \leq \mathbf{q}^{\mathfrak{M}}$. As $\mathfrak{M}$ is a model of $\mathrm{Th}\,\mathbb{R}$, condition (I) is satisfied, but there exists no strictly positive measure weakly compatible with $\precsim$ since for all n, m, $a_n \sim a_m$.

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