ON THE NONSIMPLICITY OF SOME CONVERGENCE CATEGORIES

E. LOWEN AND R. LOWEN

(Communicated by Dennis Burke)

Abstract. The category $\text{TOP}$ of topological spaces is known to be Conv-simple in the sense that there exists a single object $E$ in $\text{Top}$ such that $\text{Top}$ is the epireflective hull of $\{E\}$ in the category $\text{Conv}$ of convergence spaces. $\text{Ptop}$, the category of pretopological spaces is also Conv-simple. We show that on the contrary the category $\text{Pstop}$ of pseudo topological spaces and Conv itself are not Conv-simple. More specifically every epireflective subcategory of Conv which contains all Hausdorff $c$-embedded locally compact spaces is not Conv-simple.

For references on topological categories, reflective subcategories and reflective hulls we refer to [6, 8, 9, 10].

Let $\mathcal{A}$ be a topological category.

All subcategories $\mathcal{B}$ are assumed to be full and isomorphism-closed.

A subcategory $\mathcal{B}$ of $\mathcal{A}$ is epireflective in $\mathcal{A}$ if it is closed with respect to the formation of products and subobjects in $\mathcal{A}$. In this context "$Y$ is a subobject of $X$" means that there exists an embedding from $X$ to $Y$. This notion coincides with the categorical notion of extremal subobject.

Every subcategory $\mathcal{B}$ of $\mathcal{A}$ is contained in a smallest epireflective subcategory, its epireflective hull, which is denoted by $R\mathcal{B}$. An object $A$ of $\mathcal{A}$ belongs to $R\mathcal{B}$ if and only if $A$ is a subobject of a product of objects of $\mathcal{B}$. A subcategory $\mathcal{B}$ of $\mathcal{A}$ is called $\mathcal{A}$-simple if there exists a single object $E$ of $\mathcal{B}$ such that $\mathcal{B}$ is the epireflective hull in $\mathcal{A}$ of the class $\{E\}$, i.e., $\mathcal{B} = R\{E\}$.

Several examples of this situation are well known. If we take $\mathcal{A} = \text{Conv}$ then its bireflective subcategory $\text{Top}$ is Conv-simple. Several subcategories of $\text{TOP}$ are Conv-simple too, see for instance [4, 5, 6, 7, 9, 14]. Simplicity remains true if $\text{TOP}$ is enlarged to the bireflective subcategory $\text{Ptop}$. This can be derived from results in [1].

In this paper we show that simplicity however does not extend to the larger bireflective subcategories $\text{Pstop}$ or to $\text{Conv}$ itself.
For an arbitrary convergence space $Y$ we construct a Hausdorff, locally compact, $c$-embedded convergence space $X$ such that for any source $(f_i : X \to Y)_{i \in I}$ the space $X$ is not initial in Conv. It follows that every epireflective subcategory of Conv containing all Hausdorff, locally compact, $c$-embedded spaces, is not Conv-simple.

For notions about Conv we refer to the basic literature [2, 3, 11, 12, 13].

The following set theoretical property is fundamental for our results. We use the following notations. If $X$ is a set and $\mathcal{U}$ is an ultrafilter on $X$ then

$$
\|\mathcal{U}\| = \min\{\text{card } U | U \in \mathcal{U}\}.
$$

An ultrafilter $\mathcal{U}$ on $X$ is uniform if $\|\mathcal{U}\| = \text{card } X$. If $\mathcal{B}$ is a filterbase on $X$ then the filter generated on $X$ is denoted by $\text{stack}_X \mathcal{B}$. If $B$ is a subset of $X$ then the filter generated by $\{B\}$ is denoted by $\text{stack}_X B$. Such filters are called principal filters.

**Proposition.** If $X$ and $Y$ are sets, $X$ is infinite and $\text{card } X > \text{card } Y$, if $\mathcal{U}$ is a uniform ultrafilter on $X$ and $f$ is a function from $X$ to $Y$ then there exists a uniform ultrafilter $\mathcal{W}$ on $X$ different from $\mathcal{U}$ such that

$$
\text{stack}_Y f(\mathcal{W}) = \text{stack}_Y f(\mathcal{U}).
$$

**Proof.** Let $X, Y, \mathcal{U}$ and $f$ be as in the assertion above. We can choose a set $U \in \mathcal{U}$ such that $\text{card } U = \|\mathcal{U}\|$ and $\text{card } f(U) = \|\text{stack}_Y f(\mathcal{U})\|$. Let

$$
\mathcal{P} = \{f^{-1}(y) | y \in f(U)\}
$$

and consider the following subcollections

$$
\mathcal{P}_1 = \{P \in \mathcal{P} | \text{card } (P \cap U) < \omega\},
$$

$$
\mathcal{P}_2 = \{P \in \mathcal{P} | \text{card } (P \cap U) \geq \omega\}.
$$

Notice that $\mathcal{P}_2$ is not empty. Indeed suppose that $\mathcal{P} = \mathcal{P}_1$. If $\text{card } \mathcal{P}_1 < \omega$ it follows that $\text{card } U$ is finite. If $\text{card } \mathcal{P}_1 \geq \omega$ then

$$
\text{card } U = \max \left\{ \text{card } \mathcal{P}_1, \sup_{P \in \mathcal{P}_1} \text{card } (P \cap U) \right\}
$$

$$
= \text{card } \mathcal{P}_1 = \text{card } f(U).
$$

Thus in both cases we have a contradiction. Now for every $P \in \mathcal{P}_2$ choose disjoint sets $A_p$ and $B_p$ such that $P \cap U = A_p \cup B_p$, and $\text{card } A_p = \text{card } B_p = \text{card } (P \cap U)$.

Further, let

$$
A := \left( \bigcup_{P \in \mathcal{P}_1} P \cap U \right) \cup \left( \bigcup_{P \in \mathcal{P}_2} A_p \right)
$$

and

$$
B := \left( \bigcup_{P \in \mathcal{P}_1} P \cap U \right) \cup \left( \bigcup_{P \in \mathcal{P}_2} B_p \right).
$$
Since $A \cup B = U$ one of the sets $A$ or $B$ belongs to $\mathcal{U}$. Suppose $B \in \mathcal{U}$. Let $g : B \to A$ be any function such that $g(x) = x$ if $x \in \bigcup_{P \in \mathcal{P}_1} P \cap U$ and such that $g$ maps $B_p$ bijectively onto $A_p$ if $P \in \mathcal{P}_1$. Then clearly

$$g(B) \cap U = \bigcup_{P \in \mathcal{P}_1} P \cap U$$

and thus considering as before cases as to whether $\text{card } \mathcal{P}_1 < \omega$ or $\text{card } \mathcal{P}_1 \geq \omega$ one finds

$$\text{card}(g(B) \cap B) < \text{card } U$$

from which it follows that $g(B) \notin \mathcal{U}$. Put

$$\mathcal{W} = \text{stack}_X \{ g(V \cap B) | V \in \mathcal{U} \}$$

Then $\mathcal{W}$ is a uniform ultrafilter on $X$, different from $\mathcal{U}$. On the other hand, by construction we have

$$\text{stack}_Y f(\mathcal{W}) = \text{stack}_Y f(\mathcal{U})$$.

**Theorem.** For every convergence space $Y$ there exists a Hausdorff locally compact $c$-embedded convergence space $X$ such that for any source $(f_i : X \to Y)_{i \in I}$ in $\text{Conv}$ the space $X$ is not initial.

**Proof.** Let $Y$ be an arbitrary convergence space. Take an infinite set $X$ with cardinality strictly larger than the cardinality of the underlying set of $Y$. Further we fix a point $a \in X$ and a uniform ultrafilter $\mathcal{U}$ on $X$. We make $X$ a pseudotopological space by defining the following convergent ultrafilters:

1. if $x \neq a$ then an ultrafilter $\mathcal{W}$ converges to $x$ if and only if $\mathcal{W} = \text{stack}_x \{ x \}$,
2. an ultrafilter $\mathcal{W}$ converges to $a$ if and only if $\mathcal{W} = \text{stack}_x \{ a \}$ or $\mathcal{W}$ is nonprincipal and $\mathcal{W} \neq \mathcal{U}$.

$X$ clearly is Hausdorff and locally compact and since every convergent nonprincipal ultrafilter satisfies $\overline{\mathcal{W}} = \mathcal{W} \cap \text{stack}_x \{ a \}$ it follows that $X$ is regular. The pretopological reflection of $X$ is a compact Hausdorff topological space. It follows that the pretopological reflection of $X$ coincides with the completely regular reflection. Hence $X$ is also $\omega$-regular, and thus $c$-embedded. Now let $(f_i : X \to Y)_{i \in Y}$ be any source in $\text{Conv}$. From the proposition it follows that $\mathcal{U}$ converges to $a$ in the initial structure of the source. Hence $X$ is not initial. $\square$

**Corollary.** Every epireflective subcategory of $\text{Conv}$ containing all Hausdorff locally compact $c$-embedded spaces is not $\text{Conv}$-simple. $\square$

In particular the previous result can be applied to conclude that the following epireflective subcategories of $\text{Conv}$ are not $\text{Conv}$-simple: $\text{Conv}$ itself, $\text{Pstop}$, the categories $T_1 \text{Conv}$, $T_1 \text{Pstop}$, $\text{HConv}$, $\text{HPstop}$, the categories of all $c$-embedded spaces, of all $\omega$-regular spaces and the category of all regular spaces.
REFERENCES


DEPARTEMENT WISKUNDE, VRIJE UNIVERSITEIT BRUSSEL, PLEINLAAN 2, B-1050 BRUSSEL
WISKUNDIGE ANALYSE, UNIVERSITEIT ANTERPEN, R.U.C.A., GROENENBORGERLAAN 171, B-2020 ANTWERPEN

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use