PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 2, February 1989

SOME FOURIER-STIELTJES COEFFICIENTS REVISITED

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(Communicated by J. Marshall Ash)

ABSTRACT. We give a new proof of a result of R. Salem: The Fourier-Stieltjes coefficients of certain strictly increasing singular functions do not vanish at infinity.

Let p, q be two fixed positive numbers such that $p+q=1$ and $p \neq q$. Let the number x in the left-closed unit interval be represented by its dyadic expansion: $x = \sum_{l}^{\infty} d_n 2^{-n}$, where $d_n = 0$ or 1; in the ambiguous case of the dyadic rationals, choose the finite expansion. Let the function $H = H_p$ be defined in the closed unit interval l by

$$H(1) = 1 \, ; H(x) = \sum_{l}^{\infty} d_n p^{n-(s(n)-1)} q^{s(n)-1},$$

x in $[0,1)$, and $s(n) = \sum_{l}^{n} d_k$. Then Salem [14] proved that H is a strictly increasing singular continuous function on l and that its Fourier-Stieltjes coefficients do not vanish at infinity; he demonstrated this latter property of H by calculating the coefficients via approximating Riemann-Stieltjes sums. In this note we give a proof of this property by exploiting the self-similarity of the function H.

It is easy to see that H satisfies the self-similar functional equations:

$$(1) \qquad \begin{aligned} H(x/2) &= pH(x) \\ H((1+x)/2) &= p + qH(x) \end{aligned} \qquad 0 \leq x \leq 1.$$

Consider now the Fourier-Stieltjes coefficients of H defined by

$$c_n = c_n(dH) = \int_0^l \exp(-i2\pi nx)\, dH(x),$$

where $n = 0, \pm 1, \pm 2, \ldots$; the integrals are Riemann-Stieltjes integrals.

Received by the editors May 3, 1988. Presented at the Centennial Celebration of the AMS in Providence, RI, on August 9, 1988 (844-42-81).

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42A16, 42A38, 26A30; Secondary 39B20, 60G40.

Key words and phrases. Fourier-Stieltjes coefficient, singular function.

Theorem. *The Fourier-Stieltjes coefficients of H do not vanish at infinity.*

Proof. Consider $c_n(dH)$ and set $n = 2^m$, where $m > 0$. Then

$$c_{2^m} = \int_0^l \exp(-i\pi 2^{m+1} x)\, dH(x).$$

Splitting the integral into two integrals—from 0 to $\frac{1}{2}$ and $\frac{1}{2}$ to 1—and substituting suitably, we have c_{2^m} equals

$$\int_0^l e^{-i\pi 2^m x} dH(x/2) + \int_0^l e^{-i\pi 2^m (1+x)} dH((1+x)/2).$$

Using now the self-similar equations (1), c_{2^m} becomes

$$\int_0^l \exp(-i\pi 2^m x) p dH(x) + \int_0^l \exp(-i\pi 2^m x) q dH(x).$$

Combining these integrals, we have $c_{2^m} = c_{2^{m-1}}$. Thus $c_{2^m} = c_1$ for all nonnegative integers m. If we can show that $c_1 \neq 0$, then we will be done.

Consider now the imaginary part of c_1:

$$\operatorname{Im}(c_1) = -\int_0^l \sin(2\pi x) dH(x).$$

Using the same steps as before—splitting the integral into two parts (from 0 to $\frac{1}{2}$ and $\frac{1}{2}$ to 1), substituting suitably, and using the self-similar equations (1)—we have

$$\operatorname{Im}(c_1) = (q-p) \int_0^l \sin(\pi x) dH(x) \neq 0.$$

The last inequality follows from: $p \neq q$; $\sin(\pi x)$ is positive and H is strictly increasing for $0 < x < 1$. Hence, $c_1 \neq 0$.

Remarks 1. This proof was suggested by Hardy and Rogosinski's use [6, Theorem 41, p. 27] of the self-similarity of Cantor's singular function to show that its Fourier-Stieltjes coefficients do not vanish at infinity. Note, however, that there are uncountably many "self-similar Cantor-like" singular continuous functions on l whose Fourier-Stieltjes coefficients do vanish at infinity. See Salem [15] or Zygmund [17, Volume 2, Theorem 11.16]. Note also that there are uncountably many H_p's on the interval l.

2. An equivalent form of the functional equations (1) play a central role in Dubins and Savage's [4, p. 85] gambling strategy called BOLD play.

3. In [12], de Rham used the functional equations (1) to define the functions H_p.

4. Lastly, we note that the functions H_p have been appearing in the literature in various forms since at least 1906. They gained a wide exposure in the well-known book of Riesz and Sz.-Nagy [13, p. 48]. Besides those authors already cited, see Cesàro [3], Hellinger [7, p. 27], Faber [5, p. 395], Lomnicki and Ulam [11, p. 268], Kakutani [9], Hewitt and Stromberg [8, p. 278], Billingsley [1, p. 36; 2, pp. 85, 361], Takacs [16], and Laczkovich [10]. The books of Billingsley have some graphs of H_p for some specific p's.

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