

SMOOTH 2-KNOTS IN $S^2 \times S^2$ WITH SIMPLY CONNECTED COMPLEMENTS ARE TOPOLOGICALLY UNIQUE

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(Communicated by Haynes R. Miller)

ABSTRACT. For a given primitive homology class ξ of $H_2(S^2 \times S^2; \mathbf{Z})$, we show that there exists only one smoothly embedded 2-sphere in $S^2 \times S^2$, up to homeomorphism, which represents ξ and whose complement is simply connected.

1. INTRODUCTION

It is a consequence of Freedman's theorem [3] that (CP^2, S) , where S is a smoothly embedded 2-sphere in CP^2 representing a generator of $H_2(CP^2; \mathbf{Z})$, is pairwise homeomorphic to (CP^2, CP^1) . (See [6].) In this note, we will study the extent to which this kind of unknotting theorem holds in $S^2 \times S^2$.

Kuga [7] has characterized those homology classes in $S^2 \times S^2$ that can be represented by smoothly embedded 2-spheres. Let ζ and η be natural generators of $H_2(S^2 \times S^2; \mathbf{Z})$ representing the cross-section and fiber of the projection $S^2 \times S^2 \rightarrow S^2$ onto the first factor with $\zeta \cdot \zeta = \eta \cdot \eta = 0$ and $\zeta \cdot \eta = \eta \cdot \zeta = 1$. He has shown that $\xi = p\zeta + q\eta$, $p, q \in \mathbf{Z}$, can be represented by a smoothly embedded 2-sphere in $S^2 \times S^2$ if and only if $|p| \leq 1$ or $|q| \leq 1$. If we let S be a smoothly embedded 2-sphere in $S^2 \times S^2$ representing $\xi \in H_2(S^2 \times S^2; \mathbf{Z})$, we will call $S = (S^2 \times S^2, S)$ a 2-knot in $S^2 \times S^2$ representing ξ . It is easy to see that if the class ξ is not primitive, then $H_1(S^2 \times S^2 - S; \mathbf{Z})$ is nonzero. We are interested in 2-knots with simply connected complement, so we may assume, without loss of generality, that S represents the class $\xi = \zeta + p\eta$ for some $p \geq 0$. There is a standard 2-knot Σ_p in $S^2 \times S^2$ representing $\xi = \zeta + p\eta$, which is the image of $\phi_p: S^2 \rightarrow S^2 \times S^2$, defined by $\phi_p(x) = (x, \rho_p(x))$. Here $\rho_p: S^2 \rightarrow S^2$ is the canonical smooth map of degree p . We note that the exterior of Σ_p in $S^2 \times S^2$ is diffeomorphic to a D^2 -bundle $D(2p)$ over S^2 with

Received by the editors December 1, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 57N13, 57R40.

Key words and phrases. 2-knot, $S^2 \times S^2$.

the Euler number $2p$. Our main result is the following:

Theorem. *Let S be a 2-knot in $S^2 \times S^2$. Then $\pi_1(S^2 \times S^2 - S)$ is trivial if and only if there exists a homeomorphism of pairs*

$$\varphi: (S^2 \times S^2, S) \rightarrow (S^2 \times S^2, \Sigma_p)$$

for some nonnegative integer p .

Let K be a 2-knot in S^4 and S a 2-knot in $S^2 \times S^2$. Then we obtain another 2-knot S' in $S^2 \times S^2$ by forming the connected sum of pairs, $(S^2 \times S^2, S)$ and (S^4, K) ; i.e., $(S^2 \times S^2, S') = (S^2 \times S^2, S) \# (S^4, K)$. We shall say that $S' = (S^2 \times S^2, S')$ is obtained by the action of local 2-knot K on a 2-knot S in $S^2 \times S^2$. Thus the semigroup of local 2-knots acts on 2-knots in $S^2 \times S^2$. However, this action does not always give a new 2-knot in $S^2 \times S^2$. In fact

Corollary 1. *The action of local 2-knots on a 2-knot in $S^2 \times S^2$ with simply connected complement is trivial.*

In §3 we present proofs of the Theorem and Corollary 1, and in §4 we construct 2-knots in $S^2 \times S^2$ representing ζ whose complements are not simply connected. We note that the above results hold in the case of locally flat embeddings.

The author would like to express his gratitude to Professors M. Kato and T. Kanenobu. He wishes also to thank Professor O. Saeki for pointing out Corollary 2 to him.

2. PRELIMINARIES

The principal $SO(2)$ -bundles over S^2 are classified by the Euler number $\in \mathbf{Z} \cong \pi_1(SO(2))$. Let $D(m)$ be the total space of the associated D^2 -bundle over S^2 indexed by $m \in \mathbf{Z}$. If we let $\nu: S^2 \rightarrow D(m)$ be the zero-section, it represents a generator of $H_2(D(m); \mathbf{Z})$ and its selfintersection number is m . We note that the boundary of $D(m)$ is the lens space $L(m, m-1)$. Here $L(0, -1)$ is $S^2 \times S^1$ and $L(1, 0)$ is S^3 .

Let S be a 2-knot in $S^2 \times S^2$ whose selfintersection number is m . Then a tubular neighborhood of S in $S^2 \times S^2$ is diffeomorphic to $D(m)$. Let E be the exterior of S in $S^2 \times S^2$. By identifying the boundary $L(m, m-1)$ of $D(m)$ with the boundary $L(m, m-1)$ of E via some diffeomorphism $h: L(m, m-1) \rightarrow L(m, m-1)$, every 2-knot in $S^2 \times S^2$ with exterior E is obtained. By the isotopy extension theorem, it is easily seen that the homeomorphism type of 2-knot in $S^2 \times S^2$ obtained in this manner depends only on the isotopy class of the diffeomorphism used to identify the two boundaries. Thus we will need the following propositions.

Let $\mathcal{D}(L(m, m-1))$ be the group of diffeomorphisms of $L(m, m-1)$ onto itself, and let $\mathcal{D}_0(L(m, m-1))$ be the normal subgroup of $\mathcal{D}(L(m, m-1))$ consisting of those diffeomorphisms which are isotopic to the identity. Then

the quotient group

$$\mathcal{M}(L(m, m-1)) = \mathcal{D}(L(m, m-1)) / \mathcal{D}_0(L(m, m-1))$$

is the diffeotopy group of $L(m, m-1)$.

Let $r: S^2 \rightarrow S^2$ be the antipodal map, and $s: S^1 \rightarrow S^1$ the map induced on the unit circle in the complex plane by complex conjugation. Let \mathcal{M}' be the subgroup of $\mathcal{M}(S^2 \times S^1)$ generated by the isotopy classes of the maps (r, id) and (id, s) . Then $\mathcal{M}' \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Proposition 2.1 ([4]).

$$\mathcal{M}(S^2 \times S^1) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

The first two factors correspond to \mathcal{M}' and the third is generated by τ , defined by

$$\tau(x, \theta) = (\rho(\theta)(x), \theta),$$

where $\rho(\theta)$ is rotation of S^2 about its poles through the angle θ . We note that the maps (r, id) and (id, s) extend to diffeomorphisms of $(S^2 \times D^2, S^2 \times \{0\})$ but τ does not extend to a continuous map of $S^2 \times D^2$ onto itself.

Proposition 2.2 ([1, 5]). *Let $|m| \geq 2$. Then*

$$\mathcal{M}(L(m, m-1)) \cong \mathbf{Z}_2.$$

When we view $L(m, m-1)$ as the associated S^1 -bundle over S^2 , $\mathcal{M}(L(m, m-1))$ is generated by the diffeomorphisms of $L(m, m-1)$ onto itself whose restriction to each fiber corresponds to the map induced on S^1 by complex conjugation. Hence an element of $\mathcal{M}(L(m, m-1))$ extends to a diffeomorphism of $(D(m), \nu(S^2))$.

3. PROOF OF THE MAIN THEOREM

Lemma. *Let S be a 2-knot in $S^2 \times S^2$ with simply connected complement. Then, the exterior E of S in $S^2 \times S^2$ is homeomorphic to $D(2p)$ for some nonnegative integer p .*

Proof. Without loss of generality we may assume that S represents $\xi = \zeta + p\eta$ for some nonnegative integer p . Let $N(S)$ be a tubular neighborhood of S in $S^2 \times S^2$. Then $N(S)$ is diffeomorphic to $D(2p)$, so the boundary ∂E of E is the lens space $L(2p, 2p-1)$. By the Poincaré duality and excision,

$$H^2(E) \cong H_2(E, \partial E) \cong H_2(S^2 \times S^2, N)$$

and

$$H_3(S^2 \times S^2, N) \cong H_3(E, \partial E) \cong H^1(E) \cong 0.$$

Thus the homology exact sequence of the pair $(S^2 \times S^2, N)$ gives us a short exact sequence

$$0 \rightarrow H_2(N) \rightarrow H_2(S^2 \times S^2) \rightarrow H_2(S^2 \times S^2, N) \rightarrow 0.$$

Since ξ is a generator of $H_2(S^2 \times S^2; \mathbf{Z})$, $H_2(S^2 \times S^2, N) \cong \mathbf{Z}$. Hence $H_2(E; \mathbf{Z}) \cong H_2(E, \partial E; \mathbf{Z}) \cong \mathbf{Z}$. First we show the case of $p = 0$. Consider the following exact sequence of the pair $(E, \partial E)$,

$$\begin{array}{ccccccc}
 0 & \rightarrow & H_2(\partial E) & \rightarrow & H_2(E) & \xrightarrow{j} & H_2(E, \partial E) & \rightarrow & H_1(\partial E) & \rightarrow & 0. \\
 & & \parallel & & \parallel & & \parallel & & \parallel & & \\
 & & \mathbf{Z} & & \mathbf{Z} & & \mathbf{Z} & & \mathbf{Z} & &
 \end{array}$$

Hence j is the zero-map, and the intersection form $(H_2(E; \mathbf{Z}), \cdot)$ is isomorphic to $(\mathbf{Z}, (0))$, where $(0): \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$ is the zero map. Therefore it follows from Remarks (5.3) of [2] that E is homeomorphic to $D(0) = S^2 \times D^2$. Next we show the case of $p \neq 0$. Then ξ and η generate $H_2(S^2 \times S^2; \mathbf{Z})$, and the intersection form $(H_2(E; \mathbf{Z}), \cdot)$ is isomorphic to $(\mathbf{Z}, (2p))$, where $(2p): \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$ is the bilinear form defined by $(2p)(1, 1) = 2p$. Moreover $H_2(E, \partial E; \mathbf{Z}) \cong \mathbf{Z}$ is generated by the class w represented by $(\{*\} \times S^2) \cap E$. Since $\xi \cdot \eta = 1$, $\partial w \in H_1(L(2p, 2p - 1); \mathbf{Z})$ is represented by the ∂D^2 -fiber of the D^2 -bundle $N(S)$ over the 2-sphere S . Hence, Example (5.4) and Remarks (5.6) of [2] imply that E is homeomorphic to $D(2p)$. This completes the proof. \square

Proof of Theorem. The “if” part of the theorem is trivial. Next we suppose that $\pi_1(S^2 \times S^2 - S)$ is trivial. Since a tubular neighborhood of S in $S^2 \times S^2$ is diffeomorphic to $D(2p)$,

$$(S^2 \times S^2, S) \cong (D(2p) \cup_\gamma E, \nu(S^2)),$$

where $\gamma: L(2p, 2p - 1) \rightarrow L(2p, 2p - 1)$ is some “gluing” diffeomorphism and $\nu: S^2 \rightarrow D(2p)$ is the zero-section. By the Lemma, there exists a homeomorphism $h: E \rightarrow D(2p)$. Let \tilde{h} be the restriction of h to ∂E . It is easily seen that $D(2p) \cup_\gamma E$ is homeomorphic to $D(2p) \cup_{\tilde{h} \circ \gamma} D(2p)$.

First we show the case of $p = 0$. Since the maps (r, id) and (id, s) extend to diffeomorphisms of $(S^2 \times D^2, S^2 \times \{0\})$, it is sufficient to consider the case of $[\tilde{h} \circ \gamma] \in \mathcal{M}'$ or $[\tilde{h} \circ \gamma] = [\tau] \in \mathcal{M}(S^2 \times S^1)$. We suppose $[\tilde{h} \circ \gamma] = [\tau]$, and set $M = S^2 \times D^2 \cup_{\tilde{h} \circ \gamma} S^2 \times D^2$. Then M is an S^2 -bundle over S^2 . Since $[\tau]$ corresponds to the nontrivial element of $\pi_1(\text{SO}(3)) \cong \mathbf{Z}_2$, the second Stiefel-Whitney class $w_2(M)$ of M is nontrivial. This contradicts the fact that M is homeomorphic to $S^2 \times S^2$. Thus we may assume $[\tilde{h} \circ \gamma] \in \mathcal{M}'$, and $\tilde{h} \circ \gamma$ extends to a diffeomorphism g of $(S^2 \times D^2, S^2 \times \{0\})$. We define a required homeomorphism

$$\varphi: (S^2 \times D^2 \cup_\gamma E, S^2 \times \{0\}) \rightarrow (S^2 \times D^2 \cup_{\text{id}} S^2 \times D^2, S^2 \times \{0\})$$

by setting

$$\varphi = \begin{cases} g & \text{on } S^2 \times D^2, \\ h & \text{on } E. \end{cases}$$

Hence $(S^2 \times S^2, S)$ is pairwise homeomorphic to $(S^2 \times S^2, \Sigma_0)$.

Using the remarks after Proposition 2.2, we can show the case of $p \neq 0$ in the same manner as above. This completes the proof of the theorem. \square

Proof of Corollary 1. Let S be a 2-knot in $S^2 \times S^2$ with exterior $E(S)$, and let K be a 2-knot in S^4 with exterior $E(K)$. Set $(S^2 \times S^2, S') = (S^2 \times S^2, S) \# (S^4, K)$ and let $E(S')$ be the exterior of S' in $S^2 \times S^2$. Then $E(S') = E(S) \cup_T E(K)$, where $T = \partial E(S) \cap \partial E(K)$ is a meridional solid torus. Since $\pi_1(E(S))$ is trivial and $\pi_1(E(K))$ is normally generated by the class of the meridian, it follows from van Kampen's theorem that $\pi_1(E(S'))$ is trivial. Thus the result follows from our theorem. \square

Let $\rho_p: S^2 \rightarrow S^2$ be the canonical smooth map of degree p ; we define $\phi_{p,q}: S^2 \rightarrow S^2 \times S^2$ by $\phi_{p,q}(x) = (\rho_p(x), \rho_q(x))$. If $|p| = 1$ or $|q| = 1$, then the image $\Sigma_{p,q}$ of $\phi_{p,q}$ is a 2-knot in $S^2 \times S^2$ representing $p\zeta + q\eta$, whose complement is simply connected. We note that $\Sigma_p = \Sigma_{1,p} = \Sigma_{-1,-p} \subset S^2 \times S^2$ and $\Sigma_{p,1} = \Sigma_{-p,-1} \subset S^2 \times S^2$ as sets.

Corollary 2. *Let S be a 2-knot in $S^2 \times S^2$. If $\pi_1(S^2 \times S^2 - S)$ is trivial, then S is topologically ambient isotopic to $\Sigma_{1,p}$ or $\Sigma_{p,1}$ for some integer p .*

Proof. It follows from Theorem 1.1 of [10] that the homeotopy group of $S^2 \times S^2$ is isomorphic to

$$\begin{aligned} \text{Aut} \left(H_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) &= \left\{ A \in \text{GL}(2, \mathbf{Z}); {}^t A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \\ &= \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}. \end{aligned}$$

Since $\pi_1(S^2 \times S^2 - S)$ is trivial, we let S represent $\zeta + p\eta$ for some integer p . In fact we can show other cases without any essential change. It is easy to see from the Theorem that there is a homeomorphism of pairs, $\varphi: (S^2 \times S^2, S) \rightarrow (S^2 \times S^2, \Sigma_{1,p})$, such that $\varphi_*[S] = [\Sigma_{1,p}] = \zeta + p\eta$. Hence, in $\text{Aut}(H_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$

$$\varphi_* = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } p = 1, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } p \neq 1. \end{cases}$$

Therefore if $p \neq 1$, then φ is isotopic to the identity. If $p = 1$, then there is a homeomorphism of pairs, $\psi: (S^2 \times S^2, \Sigma_{1,p}) \rightarrow (S^2 \times S^2, \Sigma_{1,p})$, such that $\psi_* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence, by composing φ (if necessary) with ψ , $\varphi_* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and so φ is isotopic to the identity. \square

4. EXAMPLES OF 2-KNOTS IN $S^2 \times S^2$

We give examples of 2-knots in $S^2 \times S^2$ whose complements are not simply connected. We construct a 2-knot in $S^2 \times S^2$ in the same manner as in [8].

Let K be a 2-knot in S^4 and C a smoothly embedded circle in S^4 disjoint from K . Then we may assume that C is standardly embedded in S^4 . The exterior of C in S^4 is diffeomorphic to $S^2 \times D^2$. Let ζ denote the generator of $H_2(S^4 - C)$. Since K is contained in $S^2 \times D^2$, this gives us a 2-knot S in $S^2 \times S^2 = S^2 \times D^2 \cup S^2 \times D^2$. It follows from van Kampen's theorem that $\pi_1(S^2 \times S^2 - S)$ is isomorphic to $\pi_1(S^4 - K)/H$, where H is the normal closure generated by the element represented by C in $\pi_1(S^4 - K)$.

Proposition 4.1. *If C is homologous in $S^4 - K$ to a meridian of K , then the 2-knot S in $S^2 \times S^2$ constructed from K and C represents ζ .*

Proof. We consider the following exact sequences,

$$\begin{aligned} \rightarrow H_2(K) \xrightarrow{i} H_2(S^4 - C) &\rightarrow H_2(S^4 - C, K) \rightarrow 0, \\ \rightarrow H^1(S^4 - K) \xrightarrow{j} H^1(C) &\rightarrow H^2(S^4 - K, C) \rightarrow 0. \end{aligned}$$

The 2-knot S in $S^2 \times S^2$ represents ζ if and only if i is an isomorphism. By the duality, $H_2(S^4 - C, K) \cong H^2(S^4 - K, C)$. Since

$$H_2(K) \cong H_2(S^4 - C) \cong H^1(S^4 - K) \cong H^1(C) \cong \mathbf{Z},$$

$\mathbf{Z}/i(\mathbf{Z}) \cong \mathbf{Z}/j(\mathbf{Z})$. Hence i is an isomorphism if and only if j is also. Meanwhile if C is homologous in $S^4 - K$ to a meridian of K , j is an isomorphism. This completes the proof. \square

Remark. In general, if C is homologous in $S^4 - K$ to the p th power of a meridian of K , then S represents $p\zeta$.

Example 4.1. Let K be the 5-twist-spun 2-knot of the trefoil [11]. Then $\pi_1(S^4 - K) \cong \mathbf{Z} \times \mathcal{D}$, where \mathcal{D} is the binary dodecahedral group $\langle a, b; a^3 = b^5 = (ab)^2 \rangle$, and \mathbf{Z} is generated by μ , which is homologous to a meridian of K . Then the subgroup generated by $c = a^3$ in \mathcal{D} is the center of \mathcal{D} . Let C be an embedded circle representing μc^{-1} in $\pi_1(S^4 - K)$ and H the normal closure generated by μc^{-1} in $\pi_1(S^4 - K)$. The 2-knot S in $S^2 \times S^2$ constructed from $K \subset S^4$ and $C \subset S^4$ represents ζ , and $\pi_1(S^2 \times S^2 - S) \cong \pi_1(S^4 - K)/H \cong \mathcal{D}$. Therefore, $(S^2 \times S^2, S)$ is not pairwise homeomorphic to $(S^2 \times S^2, \Sigma_0)$.

Example 4.2. Let K be a classical knot and let m and l be the meridian-longitude pair for K . Then the resultant manifold $M(K; 1/n)$ obtained by $1/n$ -Dehn surgery on K becomes a homology 3-sphere, and $\pi_1(M(K; 1/n)) \cong \pi_1(S^3 - K)/H$, where H is the normal closure generated by ml^n in $\pi_1(S^3 - K)$. Now let L be the spun 2-knot of K . $\pi_1(S^4 - L) \cong \pi_1(S^3 - K)$. Let C be an embedded circle in $S^4 - L$ representing ml^n in $\pi_1(S^4 - L)$. Then the 2-knot S constructed from L and C represents ζ , and $\pi_1(S^2 \times S^2 - S) \cong \pi_1(M(K; 1/n))$. Thus we have

Proposition 4.2. *For every homology 3-sphere M obtained by Dehn surgery, there exists a 2-knot in $S^2 \times S^2$ which represents ζ and whose group is isomorphic to $\pi_1(M)$. \square*

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