PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 2, February 1989

ON TCHEBYSHEFF SYSTEMS

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(Communicated by Richard R. Goldberg)

ABSTRACT. Let $u_1, \ldots, u_n$ be linearly independent continuously differentiable functions on the unit interval. In this paper, we obtain the following two results. One is a necessary and sufficient condition for the span of $\{1, u_1, \ldots, u_n\}$ to have a Markoff basis containing 1. The other is that any Markoff system $\{u_i\}_{i=1}^n$ has a Tchebysheff extension u_{n+1} which is continuously differentiable.

INTRODUCTION

Let $u_1, \ldots, u_n$ denote linearly independent functions in $C[0, 1]$, the space of all real-valued continuous functions on the closed unit interval $[0, 1]$. Then $\{u_i\}_{i=1}^n$ is said to be a Tchebysheff system (respectively a weak Tchebysheff system) if every nonzero function in the linear subspace $L[u_1, \ldots, u_n]$ spanned by $\{u_i\}_{i=1}^n$ has no more than $n-1$ zeros (respectively changes of sign) in $[0, 1]$. For brevity, a Tchebysheff system (respectively a weak Tchebysheff system) is called a T-system (respectively a WT-system) and the linear subspace $L[u_1, \ldots, u_n]$ called a T-space (respectively a WT-space).

As is well known, T-systems and WT-systems are of great use in considering best approximation problems with the uniform norm or the L^1-norm, and many important properties of these systems have been obtained (see [1, 2 and 5–10]).

The purpose of this paper is to show the following two results of a system of continuously differentiable functions $\{u_i\}_{i=1}^n$. One is a necessary and sufficient condition for the span of $\{1, u_1, \ldots, u_n\}$ to have a Markoff basis containing 1. The other is that, for any Markoff system $\{u_i\}_{i=1}^n$ on a closed interval, there is a Tchebysheff extension u_{n+1} of $\{u_i\}_{i=1}^n$ which is continuously differentiable, i.e., there exists a u_{n+1} of $C^1[0, 1]$ such that $\{u_i\}_{i=1}^{n+1}$ is a T-system. To do them, we pay attention to a subclass of WT-systems which does not contain the spline spaces but contains the T-systems and study some properties of the systems of this subclass. For the sake of convenience, we name this system,

Received by the editors September 14, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 41A50.
Key words and phrases. Weak Tchebysheff systems, Tchebysheff systems.

which is defined in §1, an integral Tchebysheff system or an IT-system and call the linear subspace spanned by an IT-system an IT-space.

1. IT-SYSTEMS

First we make the following preparations: By a subinterval of $[0, 1]$ we mean a nondegenerate one and a set $\{I_\lambda\}_{\lambda\in\Lambda}$ of subintervals of $[0, 1]$ is called disjoint if every $I_\lambda \cap I_{\lambda'}$, $\lambda \neq \lambda'$, is a degenerate interval or empty. For a given positive integer n, we set $A_n = \{(t_1, \dots, t_n) |\ 0 < t_1 < \cdots < t_n < 1\}$, and for a function u of $C[0, 1]$, the set of zeros of u is denoted by $Z(u)$. Let $u_1, \dots, u_n$ be functions in $C[0, 1]$ and $t_1, \dots, t_n$ points in $[0, 1]$. Then we denote the nth order determinant by

$$D\begin{pmatrix} u_1, \dots, u_n \\ t_1, \dots, t_n \end{pmatrix} = \begin{vmatrix} u_1(t_1), \dots, u_n(t_n) \\ \vdots \qquad \vdots \\ u_1(t_1), \dots, u_n(t_n) \end{vmatrix}.$$

Now we give the definition of IT-systems.

Definition 1. Let $\{u_i\}_{i=1}^n$ be linearly independent functions in $C[0, 1]$. Then $\{u_i\}_{i=1}^n$ is said to be an IT-system if for any disjoint n-subintervals I_i, $i = 1, \dots, n$, of $[0, 1]$, the nth order determinant

$$\begin{vmatrix} \int_{I_1} u_1\,dx, \dots, \int_{I_n} u_1\,dx \\ \vdots \qquad \vdots \\ \int_{I_1} u_n\,dx, \dots, \int_{I_n} u_n\,dx \end{vmatrix} \neq 0.$$

Then we have

Theorem 1. *For linearly independent functions $\{u_i\}_{i=1}^n$ in $C[0, 1]$, the following conditions are equivalent*:

(1) $\{u_i\}_{i=1}^n$ *is an IT-system.*

(2) $\{u_i\}_{i=1}^n$ *is a WT-system and, for any $f \in L[u_1, \dots, u_n] - \{0\}$, $Z(f)$ is nowhere dense in* $[0, 1]$.

(3) $D\binom{\sigma u_1, \dots, u_n}{t_1, \dots, t_n} \geq 0$ *for all* $(t_1, \dots, t_n) \in A_n$, *where* $\sigma = 1$ *or* -1, *and* $B_n(\sigma u_1, \dots, u_n) = \{(t_1, \dots, t_n) |\ (t_1, \dots, t_n) \in A_n,\ D\binom{\sigma u_1, \dots, u_n}{t_1, \dots, t_n} > 0\}$ *is dense in* A_n.

Proof. (1) ⇒ (2). Suppose that $\{u_i\}_{i=1}^n$ is not a WT-system. Then there is a function $f \in L[u_1, \dots, u_n] - \{0\}$ such that, for some $0 < t_1 < \cdots < t_{n+1} < 1$,

$$f(t_i) \cdot f(t_{i+1}) < 0, \qquad i = 1, \dots, n. \tag{1.1}$$

By (1.1), there exist disjoint subintervals $I_i \subset [t_i, t_{i+1}]$, $i = 1, 2, \dots, n$, such that

$$\int_{I_i} f\,dx = 0, \qquad i = 1, \dots, n. \tag{1.2}$$

But (1.2) contradicts the definition of IT-systems. Next suppose that, for some $g \in L[u_1, \dots, u_n] - \{0\}$, $Z(g)$ is not nowhere dense in $[0, 1]$. From this assumption, we can easily find disjoint subintervals I_i, $i = 1, \dots, n$, such that g vanishes identically on each I_i. This is also contradictory to the fact that $\{u_i\}_{i=1}^n$ is an IT-system.

(2) $\Rightarrow$ (1). If $\{u_i\}_{i=1}^n$ is not an IT-system, then, for some $f \in L[u_1, \dots, u_n] - \{0\}$ and some disjoint subintervals I_i, $i = 1, \dots, n$, of $[0, 1]$,

$$\int_{I_i} f\,dx = 0, \qquad i = 1, \dots, n. \tag{1.3}$$

Since f does not vanish identically on each I_i, from the continuity of f, and (1.3), we obtain

$$f(t_i) \cdot f(s_i) < 0 \quad \text{for some } t_i, s_i \in I_i, \ i = 1, \dots, n.$$

This implies that $\{u_i\}_{i=1}^n$ is not a WT-system.

(1) $\Rightarrow$ (3). For $I_1 \leq \cdots \leq I_n$ (i.e., $I_1 \times \cdots \times I_n \subset A_n$), applying problem 68, p. 61, in Pólya and Szegö [3], we see that

$$\begin{aligned}
\det\left(\int_{I_j} u_i(x)\,dx\right)_{i,j=1}^n &= \det\left(\int_{[0,1]} u_i(x)\chi_{I_j}(x)\,dx\right)_{i,j=1}^n \\
&= \frac{1}{n!}\int\cdots\int_{0<x_1<\cdots<x_n<1} \det(u_i(x_j))_{i,j=1}^n \det(\chi_{I_i}(x_j))_{i,j=1}^n\,dx_1\cdots dx_n \\
&= \frac{1}{n!}\int_{I_n}\cdots\int_{I_1} \det(u_i(x_j))_{i,j=1}^n\,dx_1\cdots dx_n, \qquad (1.4)
\end{aligned}$$

where each $\chi_{I_j}(x)$, $1 \leq j \leq n$, is the characteristic function of I_j. Then (3) $\Rightarrow$ (1) follows immediately from (1.4). By (1) $\Leftrightarrow$ (2) and (1.4), if

$$\det(u_i(x_j))_{i,j=1}^n = 0$$

on an open subset of A_n, we can easily choose $I_1 \leq \cdots \leq I_n$ contained in $[0, 1]$ whose product is in this open set and thus $(\int_{I_j} u_i(x)\,dx)_{i,j=1}^n = 0$. This implies that (1) $\Rightarrow$ (3) holds.

Remark 1. (1) As a typical example of an IT-system, let $\{u_i\}_{i=1}^n$ be a T-system and w a nonnegative continuous function such that $Z(w)$ is nowhere dense in $[0, 1]$, then the system $\{wu_i\}_{i=1}^n$ is an IT-system.

(2) In the rest of this paper, without loss of generality, we assume that, for a WT-system $\{u_i\}_{i=1}^n$,

$$D\begin{pmatrix} u_1, \dots, u_n \\ t_1, \dots, t_n \end{pmatrix} \geq 0, \qquad (t_1, \dots, t_n) \in A_n.$$

2. Basic properties of IT-systems

We begin by stating the following theorem regarding WT-spaces.

Theorem A (Sommer and Strauss [6] and Stockenberg [7]). *Let U be an n-dimensional WT-space; then there exists an $(n-1)$-dimensional WT-subspace of U.*

In case of IT-spaces, we prove

Theorem 2. *Let U be an n-dimensional IT-space; then there exists an $(n-1)$-dimensional IT-subspace of U.*

Proof. By Theorem 1, U is a WT-space, and using Theorem A, U contains an $(n-1)$-dimensional WT-subspace U_0 of U. Since, for any $f \in U_0 - \{0\}$, $Z(f)$ is nowhere dense in $[0, 1]$, U_0 is an IT-subspace by Theorem 1.

Remark 2. As a result of Theorem 2, we observe that every n-dimensional IT-space has a basis $\{u_i\}_{i=1}^n$ such that each system $\{u_i\}_{i=1}^j$, $1 \le j \le n$, is an IT-system.

Let us recall that, for a system $\{u_i\}_{i=1}^n$, the convexity cone $K[u_1, \dots, u_n]$ is the set of all real-valued functions f defined on $(0, 1)$ for which the determinant

$$D\begin{pmatrix} u_1, \dots, u_n, f \\ t_1, \dots, t_n, t_{n+1} \end{pmatrix} \ge 0 \quad \text{for all } (t_1, \dots, t_{n+1}) \in A_{n+1}.$$

Furthermore we denote by $K_c[u_1, \dots, u_n]$ the set of all functions in $K[u_1, \dots, u_n]$ which are continuous on $[0, 1]$.

Zielke [9] and Zalik [8] proved that, for a T-system $\{u_i\}_{i=1}^n$, there is a u_{n+1} in $K_c[u_1, \dots, u_n]$ such that $\{u_i\}_{i=1}^{n+1}$ is a T-system.

We shall prove the similar result for an IT-system. First, we need the following

Lemma. *Let $\{u_i\}_{i=1}^n$ be an IT-system. Suppose that there exist a countable dense subset $S = \{(s_1^{(i)}, \dots, s_n^{(i)}),\ i \in N\}$ of A_n and a sequence $\{f_i\}_{i \in N}$ of functions in $K_c[u_1, \dots, u_n]$ such that, for every $(s_1^{(i)}, \dots, s_n^{(i)})$ and f_i,*

$$\left\{ t \,\middle|\, D\begin{pmatrix} u_1, \dots, u_n, f_i \\ s_1^{(i)}, \dots, s_n^{(i)}, t \end{pmatrix} > 0,\ t \in (s_n^{(i)}, 1) \right\}$$

is dense in the open interval $(s_n^{(i)}, 1)$. Then there is a u_{n+1} in $K_c[u_1, \dots, u_n]$ such that $\{u_i\}_{i=1}^{n+1}$ is an IT-system.

Proof. Setting $u_{n+1}(t) = \sum_{i=1}^{\infty} 2^{-i} \|f_i\|^{-1} \cdot f_i(t)$, where $\|\cdot\|$ denotes the uniform norm on $[0, 1]$, u_{n+1} is clearly contained in $K_c[u_1, \dots, u_n]$ and the subset $\{(t_1, \dots, t_n) |\ D\binom{u_1, \dots, u_{n+1}}{t_1, \dots, t_{n+1}} > 0,\ (t_1, \dots, t_{n+1}) \in A_{n+1}\}$ is dense in A_{n+1}. Hence, from Theorem 1, the conclusion follows immediately.

Now we show

Theorem 3. *If $\{u_i\}_{i=1}^n$ is an IT-system, there is a u_{n+1} in $K_c[u_1, \dots, u_n]$ such that $\{u_i\}_{i=1}^{n+1}$ is an IT-system.*

Proof. In case $n = 1$, by setting $u_2(t) = t \cdot u_1(t)$, $\{u_1, u_2\}$ is an IT-system.

In case $n \ge 2$, by Remark 2, we assume that each system $\{u_i\}_{i=1}^k$, $1 \le k \le n$, is an IT-system. Since, by Theorem 1, $B_n(u_1, \dots, u_n)$ and $\{(t, s) | (t, s) \in A_n,\ s \in B_{n-1}(u_1, \dots, u_{n-1})\}$ are open and dense in A_n, we can take a countable dense subset $S = \{(s_1^{(i)}, \dots, s_n^{(i)}),\ i \in N\}$ in A_n such that

$$D\begin{pmatrix} u_1, \dots, u_n \\ s_1^{(i)}, \dots, s_n^{(i)} \end{pmatrix} > 0 \quad \text{for } i \in N \tag{2.1}$$

and

$$D\begin{pmatrix} u_1, \dots, u_{n-1} \\ s_2^{(i)}, \dots, s_n^{(i)} \end{pmatrix} > 0 \quad \text{for } i \in N.$$

For every $(s_1^{(i)}, \dots, s_n^{(i)}) \in S$, using the same method as the proof of Theorem 1 in Zalik [8], we find a $V(t) \in K_c[u_1, \dots, u_n]$ such that

$$V(t) = \begin{cases} 0, & t \in [0, s_n^{(i)}], \\ \sum_{i=1}^n a_i u_i(t), & t \in [s_n^{(i)}, 1], \text{ where } a_n = 1. \end{cases} \tag{2.2}$$

Hence putting

$$f(t) = D\begin{pmatrix} u_1, \dots, u_n, V \\ s_1^{(i)}, \dots, s_n^{(i)}, t \end{pmatrix} = V(t) \cdot D\begin{pmatrix} u_1, \dots, u_n \\ s_1^{(i)}, \dots, s_n^{(i)} \end{pmatrix} \quad \text{for } t \in [s_n^{(i)}, 1], \tag{2.3}$$

by (2.1), (2.2) and (2.3), f is nonnegative and contained in $L[u_1, \dots, u_n]|_{[s_n^{(i)}, 1]} - \{0\}$, where $L[u_1, \dots, u_n]|_{[s_n^{(i)}, 1]}$ denotes the linear space obtained by restricting $L[u_1, \dots, u_n]$ to $[s_n^{(i)}, 1]$. Since $\{u_i\}_{i=1}^n$ is an IT-system, $\{t | f(t) > 0, t \in (s_n^{(i)}, 1)\}$ is dense in $(s_n^{(i)}, 1)$ by Theorem 1. Thus the condition of Lemma holds.

3. Main theorems

In the first place, we need the following

Definition 2. (1) Let $\{u_i\}_{i=1}^n$ be a T-system (respectively a WT-system). If each system $\{u_i\}_{i=1}^j$, $1 \le j \le n$, is a T-system (respectively a WT-system), then $\{u_i\}_{i=1}^n$ is said to be a Markoff system (respectively a weak Markoff system).

(2) For an n-dimensional linear subspace U of $C[0, 1]$, a basis $\{u_i\}_{i=1}^n$ of U is called a Markoff basis (respectively a weak Markoff basis) if it is a Markoff system (respectively a weak Markoff system).

Providing that U is an $n(\ge 2)$-dimensional linear subspace of continuously differentiable functions, containing constants, Zwick [10] has given the following characterization of U having a weak Markoff basis.

Theorem B. *Let U be an n-dimensional subspace of $C^1[0, 1]$ which contains constants. Then U has a weak Markoff basis $\{u_i\}_{i=1}^n$ with $u_1 = 1$ if and only if the space of derivatives U' is a WT-space.*

Replacing a weak Markoff basis with a Markoff basis, we obtain

Theorem 4. *Let U be an n-dimensional subspace of $C^1[0, 1]$ which contains constants. Then U has a Markoff basis $\{u_i\}_{i=1}^n$ with $u_1 = 1$ if and only if the space of derivatives U' is an IT-space.*

Proof. Suppose that U has a Markoff basis $\{u_i\}_{i=1}^n$ with $u_1 = 1$. Since, by Theorem B, U' is a WT-space, applying Theorem 1, it is sufficient to show that

$Z(f)$ is nowhere dense in $[0,1]$ for every $f \in U' = \{0\}$. To do this, assume that a nonzero function $f = \sum_{i=1}^{n} a_i u_i'$ vanishes identically on a subinterval $[a,b]$ of $[0,1]$. Then the function $\sum_{i=1}^{n} a_i u_i$ is equal to a constant on $[a,b]$. But this contradicts the fact that U has a Markoff basis.

Conversely suppose that U' is an IT-space. From Theorem 1 and Theorem B, we have a weak Markoff basis $\{u_i\}_{i=1}^{n}$ with $u_1 = 1$ of U. Furthermore we shall show that each system $\{u_i\}_{i=1}^{j}$, $2 \le j \le n$, is a T-system. As in the first half of this proof, suppose that a function $g = \sum_{i=1}^{j} b_i u_i \in L[u_1, \dots, u_n] - \{0\}$ vanishes identically on a subinterval $[c,d]$ of $[0,1]$. Since $b_2^2 + \cdots + b_j^2 \neq 0$, $g' = \sum_{i=2}^{j} b_i u_i'$ is contained in $U' - \{0\}$ and vanishes identically on $[c,d]$. But this is contradictory to the assumption on U'. Hence applying Theorem 2.45 in Schumaker [4], we can conclude tht each $\{u_i\}_{i=1}^{j}$ is a T-system on the open interval $(0,1)$. By Theorem B, $\{u_i'\}_{i=2}^{j}$ is a WT-system on $[0,1]$. If u in $L[u_1, \dots, u_j] - \{0\}$ has j zeros in $[0,1]$, then between each pair of zeros u' attains positive and negative values and hence has j points of sign change. This is a contradiction. Thus each system $\{u_i\}_{i=1}^{j}$, $2 \le j \le n$, is a T-system on $[0,1]$.

Theorem 5. *If $\{u_i\}_{i=1}^{n}$ is a Markoff system consisting of continuously differential functions, then there exists a continuously differentiable function u_{n+1} such that $\{u_i\}_{i=1}^{n+1}$ is a Markoff system.*

Proof. Since $\{u_i\}_{i=1}^{n}$ is a Markoff system on $[0,1]$, u_1 does not vanish on $[0,1]$. By setting $v_i = u_i/u_1$, $1 \le i \le n$, we can reduce to a Markoff system $\{v_i\}_{i=1}^{n}$ with $v_1 = 1$. By Theorem 4, $v_2', \dots, v_n'$ is an IT-system. Then, using Theorem 3, we have such an f of $C[0,1]$ that $(v_2', \dots, v_n', f)$ is an IT-system. Hence, setting $v_{n+1}(t) = \int_0^t f(x)\,dx$, we easily observe that $\{v_i\}_{i=1}^{n+1}$ is a Markoff system by Theorem 4.

Acknowledgment

The author wishes to express his gratitude to the referee who gave him many valuable comments on this article.

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