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A CHARACTERIZATION OF COMPLEX HYPERSURFACES IN $\mathbf{C}^m$

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ABSTRACT. We show that an isometric immersion of a connected Kaehler manifold M^{2n} into the euclidean space with (real) codimension two is holomorphic with respect to some complex structure of $\mathbf{R}^{2n+2}$ provided that the index of nullity μ of the curvature tensor satisfies $\mu < 2n - 4$ everywhere.

1. INTRODUCTION

In this article we consider the problem of whether a codimension two isometric immersion $f: M^{2n} \to \mathbf{R}^{2n+2}$, of a Kaehler manifold of real dimension $2n$ into euclidean space is holomorphic, i.e., when f is congruent to a Kaehler immersion of M in $\mathbf{C}^{n+1} \simeq \mathbf{R}^{2n+2}$. We will prove

(1.1) **Theorem.** *Let $f: M^{2n} \to \mathbf{R}^{2n+2}$ be an isometric immersion of a connected Kaehler manifold. Assume that the index of nullity μ of the curvature tensor R of M satisfies $\mu < 2n - 4$ everywhere. Then f is holomorphic.*

In fact we will show that the theorem remains true under the weaker assumption that the index of relative nullity ν satisfies $\nu < 2n - 4$ everywhere. We refer to [K-N] for basic facts and definitions.

The proof consists in a linear algebra argument which allows us to construct pointwise an extension of the complex structure on each tangent space to M to a complex structure in $\mathbf{R}^{2n+2}$ so that the second fundamental form of f is complex linear with respect to it. Then it is easy to see that this pointwise constructed operator is parallel in the normal bundle and thus constant in $\mathbf{R}^{2n+2}$ over M.

(1.2) *Remark.* The isometric product immersion $f_1 \times f_2: M_1^n \times M_2^n \to \mathbf{R}^{2n+2}$, of two real Kaehler hypersurfaces $f_i: M_i^n \to \mathbf{R}^{n+1}$, provides examples, of any dimensions, of isometric immersions with $\mu = \nu = 2n - 4$, which are not holomorphic. See [D-G] for the classification of such submanifolds.

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2. The main lemma

Let V, W be finite dimensional real vector spaces. We say that a bilinear form $\beta: V \times V \to W$ is *flat* with respect to a nondegenerate real valued symmetric bilinear form (inner product) $\langle\ ,\ \rangle: W \times W \to \mathbf{R}$ iff

$$\langle \beta(x, y), \beta(w, z)\rangle - \langle \beta(x, z), \beta(w, y)\rangle = 0,$$

for all $x, y, z, w \in V$.

For $x \in V$, we define the linear transformation $\beta(x): V \to W$ by $\beta(x)(y) = \beta(x, y)$. We say that $x \in V$ is a (left) *regular element* if $\dim \beta(x)(V) = \max_{z\in V} \dim \beta(z)(V)$. It is easily checked that the subset of regular elements of β in V is open and dense. The following result follows from equation (8) and (9) of [M, p. 462].

(2.1) **Lemma.** *Suppose that $x \in V$ is a regular element. Then for $n \in \ker \beta(x)$, we have*

$$\beta(V, n) \subset \beta(x)(V) \cap (\beta(x)(V))^{\perp}.$$

We say that the symmetric bilinear form $\langle\ ,\ \rangle: W \times W \to \mathbf{R}$ has signature (p, q) if $\dim W = p+q$, and there exists a basis $\zeta_1, \dots, \zeta_n$ such that $\langle \zeta_i, \zeta_j\rangle = \varepsilon\delta ij$, with $\varepsilon = 1$ for $1 \le j \le p$, $\varepsilon = -1$ for $p+1 \le j \le p+q$. We define the nullity of β by $N(\beta) = \{m \in V: \beta(z, m) = 0, \text{ for all } z \in V\}$. We now state and prove our main lemma.

(2.2) **Lemma.** *Suppose that the bilinear form $\beta: V \times V \to W$, $\beta \neq 0$, is flat with respect to an inner-product $\langle\ ,\ \rangle$ of signature (p, p), $1 \le p \le 2$. Assume $\dim V > \dim W$ and $\dim N(\beta) < \dim V - \dim W$. Then W admits an orthogonal direct sum decomposition $W = W_1 \oplus W_2$ such that the restriction of $\langle\ ,\ \rangle$ to W_1 is nondegenerate of signature (q, q), $1 \le q \le 2$, and if β_1 and β_2 are the W_1 and W_2 components of β respectively, then*

(i) $\beta_1 \neq 0$ *and* $\langle \beta_1(x, y), \beta_1(w, z)\rangle = 0$ *for all* $x, y, w, z \in V$.

(ii) β_2 *is flat and* $\dim N(\beta_2) \ge \dim V - 2$.

Proof. First we claim that if $x \in V$ is a regular element, then the restriction of $\langle\rangle$ to $\beta(x)(V)$ is degenerate. Otherwise $\beta(x)(V) \cap (\beta(x)(V))^{\perp} = \{0\}$ and it follows from (2.1) that $\ker \beta(x) \subset N(\beta)$. Since, by definition, $N(\beta) \subset \ker \beta(x)$, we conclude $\dim N(\beta) = \dim \ker \beta(x) \ge \dim V - \dim W$, which is a contradiction.

Now assume that for all regular elements $x \in V$, $\beta(x)(V)$ is a null subspace of W, i.e., $\langle\ ,\ \rangle \equiv 0$ restricted to $\beta(x)(V) \times \beta(x)(V)$. Thus $\langle \beta(x, z), \beta(x, w)\rangle = 0$, for all $z, w \in V$. Since the set of regular elements is dense, we have by continuity that $\langle \beta(x, z), \beta(x, w)\rangle = 0$ for all $x, z, w \in V$. By flatness

$$0 = \langle \beta(x+y, z), \beta(x+y, w)\rangle = 2\langle \beta(x, w), \beta(y, z)\rangle,$$

for all $x, y, z, w \in V$. Setting $W_1 = W$, $W_2 = 0$, we obtain the conclusions of the lemma in this case.

Notice that if $p = 1$, the only degenerate subspaces of W are null subspaces. Thus, the case $p = 1$ is proved by the above argument.

It remains to consider the case when $p = 2$ and there exists a regular element $x \in V$ such that $\beta(x)(V)$ is not a null subspace. In this situation the null subspace $U(x) = \beta(x)(V) \cap (\beta(x)(V))^\perp$ satisfies $\dim U(x) = 1$. To see this, observe that if $\dim U(x) = 2$, we would conclude from $4 = \dim W = \dim \beta(x)(V) + \dim(\beta(x)(V))^\perp$, that $\beta(x)(V) = (\beta(x)(V))^\perp = U(x)$, and this is a contradiction. It follows that $2 \leq \dim \beta(x)(V) \leq 3$, and hence $\dim \ker \beta(x) \geq \dim V - 3$. We claim that the subspace $S(\beta) = \operatorname{span}\{\beta(y, z): y, z \in V\}$ is orthogonal to $U(x)$. Otherwise, there exists $u, v \in V$ such that $\langle \beta(u, v), \xi_1 \rangle \neq 0$, where $\xi_1 \in W$ is a null vector spanning $U(x)$. For $n \in \ker \beta(x)$, we have from 2.1 and flatness that

$$\beta(y, n) = 0 \quad \text{iff } \langle \beta(y, n), \beta(u, v)\rangle = \langle \beta(u, n), \beta(y, v)\rangle = 0.$$

Consider the linear map $B: \ker \beta(x) \to U(x)$, given by $B(n) = \beta(u, n)$. By the above $\ker B \subset N(\beta)$, and therefore $\dim N(\beta) \geq \dim \ker B \geq \dim \ker \beta(x) - \dim U(x) \geq \dim V - 4$, which is a contradiction and proves the claim.

We complete ξ_1, to a pseudo-orthonormal basis $\xi_1, \xi_2, \xi_3, \xi_4$ of W such that $\langle \xi_1, \xi_2 \rangle = 1$, $\langle \xi_2, \xi_2 \rangle = 0$, and $\langle \xi_i, \xi_j \rangle = 0$ for $1 \leq i \leq 2$, $3 \leq j \leq 4$ or $i = 3$, $j = 4$. The existence of such basis follows from [A, Theorem 3.8, p. 120]. We write $\beta = \sum_{j=1}^4 \phi^j \xi_j$, where each ϕ^j is an ordinary real-valued bilinear form. From $\xi_1 \in S(\beta)$ we get $\phi^1 \neq 0$, and from the fact that ξ_1 is orthogonal to $S(\beta)$, we conclude $\phi^2 = 0$. Set $W_1 = \operatorname{span}\{\xi_1, \xi_2\}$, $W_2 = \operatorname{span}\{\xi_3, \xi_4\}$, $\beta_1 = \phi^1 \xi_1$ and $\beta_2 = \phi^3 \xi_3 + \phi^4 \xi_4$. Then β_1 verifies (i) of (2.2), and thus $\beta_2 = \beta - \beta_1$ if flat. It remains to show that $S(\beta_2)$ is nondegenerate and the second part of (ii) in (2.2) will follow from $\beta_2 = 0$ or the fact that W_2 has signature (1,1). To see this, observe that if $\langle \sum_{j=1}^4 \beta_2(x_j, y_j), \beta_2(w, z)\rangle = 0$ for all $w, z \in V$, then $\langle \sum_{j=1}^4 \beta(x_j, y_j), \beta(w, z)\rangle = 0$, and thus $\sum_j \beta(x_j, y_j) \in W_1$. This implies $\sum_{j=1}^4 \beta_2(x_j, y_j) = 0$. This concludes the proof of the lemma.

3. Proof of the theorem

Let $\alpha: T_pM \times T_pM \to N_pM$ be the vector valued second fundamental form of the immersion f at $p \in M$, where N_pM is the orthogonal complement of the tangent space T_pM in $\mathbf{R}^{2n+2}$. Set $W = N_pM \oplus N_pM$, and define an inner-product $\langle\langle\ ,\ \rangle\rangle$ of signature (2,2) in W by requiring that $\langle\langle \xi \oplus \eta, \gamma \oplus \delta \rangle\rangle = \langle \xi, \gamma \rangle - \langle \eta, \delta \rangle$, where $\langle\ ,\ \rangle$ denotes both the riemannian metrics on M and $\mathbf{R}^{2n+2}$.

Consider the bilinear form $\beta: T_pM \times T_pM \to W$ defined by $\beta(x, y) = \alpha(x, y) \oplus \alpha(x, Jy)$, where J is the complex structure in TM. It follows easily from the Gauss equations and the relation $\langle R(u, v)Jw, Jz\rangle = \langle R(u, v)w, z\rangle$, that β is flat. Clearly, $\dim(N(\beta) < 2n - 4$, and thus $\beta = \beta_1 \oplus \beta_2$ as in (2.2). We claim that $\beta_2 = 0$. Assume otherwise. We choose orthonormal bases

$\{\xi, \eta\}$, $\{\tilde{\xi}, \tilde{\eta}\}$ of N_pM, such that $S(\beta_1) = \text{span}\{\xi \oplus \tilde{\xi}\}$. Thus,

$$\langle \alpha(x, y), \xi \rangle = \langle \alpha(x, Jy), \tilde{\xi} \rangle \quad \text{for all } xT_pM, y \in V, \tag{3.1}$$

where $V = \ker(\beta_2) \subset T_pM$. Then $\dim V = 2n-2$, and $\beta(x, v) = \beta_1(x, v)$ for all $x \in T_pM$, $v \in V$. In particular $\langle \beta(x, v), \eta \oplus \{0\} \rangle = \langle \beta(x, v), \{0\} \oplus \tilde{\eta} \rangle = 0$, since $\langle \xi \oplus \tilde{\xi}, \eta \oplus \{0\} \rangle = \langle \xi \oplus \tilde{\xi}, \{0\} \oplus \tilde{\eta} \rangle = 0$. We obtain

$$\langle \alpha(x, v), \eta \rangle = 0 = \langle \alpha(x, Jv), \tilde{\eta} \rangle \quad \text{for all } x \in T_pM, \ v \in V. \tag{3.2}$$

We conclude from (3.2) that either $\tilde{\eta} = \pm\eta$ or $JV \cap V \subset N(\alpha)$, where the second possibility is in contradiction with the assumption of the theorem, since $\dim JV \cap V \geq 2n - 4$. In particular, it follows that $\tilde{\xi} = \pm\xi$, and from (3.1) we have $\langle \alpha(x, y), \xi \rangle = \pm\langle \alpha(x, Jy), \xi \rangle = \langle \alpha(x, J^2y), \xi \rangle = -\langle \alpha(x, y), \xi \rangle = 0$. Thus $V \subset N(\alpha)$ which is not possible. This proves our claim.

We have from $\beta = \beta_1$ that

$$\langle \alpha(x, y), \alpha(w, z) \rangle = \langle \alpha(x, Jy), \alpha(w, Jz) \rangle, \quad \text{for all } x, y, w, z \in T_pM.$$

In particular, $\|\alpha(x, y)\| = \|\alpha(x, Jy)\|$ and $\langle \alpha(x, y), \alpha(x, Jy) \rangle = 0$. This means that the complex structure J of TM extends to an almost complex structure J on the tangent bundle of $\mathbf{R}^{2n+2}$ restricted to f, such that the second fundamental form α is complex linear, i.e.,

$$\alpha(x, Jy) = J\alpha(x, y) = \alpha(Jx, y). \tag{3.3}$$

For dimension reasons, the orthogonal transformation J restricted to the normal bundle NM is parallel in the normal connection. Now, it follows easily using (3.3) that J is constant in $\mathbf{R}^{2n+2}$ along M. This completes the proof of the theorem.

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