THE NUMBER OF INDECOMPOSABLE SEQUENCES
OVER AN ARTIN ALGEBRA OF FINITE TYPE

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ABSTRACT. Let $\Lambda$ be an artin algebra of finite representation type. For a finitely generated $\Lambda$-module $C$, there are only finitely many f.g. modules $A$ such that $0 \to A \to B \to C \to 0$ is indecomposable as a short exact sequence.

Let $\Lambda$ be an artin algebra of finite representation type and $\text{mod}\,\Lambda$ the category of finitely generated (f.g.) left $\Lambda$ modules. If $X$ and $C$ are in $\text{mod}\,\Lambda$, we write $\Lambda(X,C)$ for $\text{hom}_\Lambda(X,C)$ and $P(X,C)$ for the submodule of $\Lambda(X,C)$ comprising those maps $f: X \to C$ for which there exists a factorization

$$X \xrightarrow{f} C \xrightarrow{g} P$$

with $P$ projective. Also, let $\text{Tr}$ and $D$ be the usual transpose and dual. In this setting, Theorem 5.7 in M. Auslander's paper [A] may be stated as follows.

**Theorem A.** Let $C$ be in $\text{mod}\,\Lambda$. Let $A_1, \ldots, A_m$ be a complete list of all non-injective indecomposable modules in $\text{mod}\,\Lambda$ and let $X_i = \text{Tr}DA_i$. For each $i$, $\Lambda(X_i,C)/P(X_i,C)$ is an $(\text{End} X_i)^{\text{op}}$-module of finite length. Let $S_{i_1}, \ldots, S_{i_d}$ be a complete set of nonisomorphic simple $(\text{End} X_i)^{\text{op}}$-modules, and for each $(\text{End} X_i)^{\text{op}}$-submodule $H$ of $\Lambda(X_i,C)$ containing $P(X_i,C)$ let $n_1(A_i,H), \ldots, n_{d_i}(A_i,H)$ be the uniquely determined nonnegative integers so that the $(\text{End} X_i)^{\text{op}}$-socle of $\Lambda(X_i,C)/H$ is isomorphic to $\prod_{j=1}^{d_i} S_{i_j}^{n_j(A_i,H)}$. Finally let $n(A_i) = \max\{n_j(A_i,H)\}$ as $j$ runs through $1, 2, \ldots, d_i$, and as $H$ runs through all $(\text{End} X_i)^{\text{op}}$-submodules of $\Lambda(X_i,C)$ containing $P(X_i,C)$. Then

1. $n(A_i)$ is finite;
2. if $k > n(A_i)$ and $0 \to A_i^k \xrightarrow{g} B \to C \to 0$ is exact, then $A_i^k$ contains a submodule $A'$ (isomorphic to $A_i^{k-n(A_i)}$) such that $g(A')$ is a summand of $B$.

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Keeping the notation of Theorem A we have

**Theorem 1.** Fix \( C \) in \( \text{mod} \Lambda \). Then there are only a finite number of modules \( A \) in \( \text{mod} \Lambda \) for which \( 0 \rightarrow A \overset{g}{\rightarrow} B \rightarrow C \rightarrow 0 \) is indecomposable as a short exact sequence. In fact, if \( A \) has an injective summand, or if \( A \simeq \bigsqcup_{i=1}^{m} A_i^{p_i} \) with \( p_i > n(A_i) \) for some \( i \), the sequence decomposes.

**Proof.** If \( A \) has an injective summand then clearly the sequence decomposes. Suppose \( p_i > n(A_i) \), and form the pushout diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & A & \overset{g}{\rightarrow} & B & \rightarrow \ & C & \rightarrow & 0 \\
\downarrow \text{surj} & & \downarrow & & \downarrow & & \\
0 & \rightarrow & A_i^{p_i} & \overset{h'}{\rightarrow} & D & \rightarrow \ & C & \rightarrow & 0.
\end{array}
\]

Because \( p_i > n(A_i) \), \( A_i^{p_i} \) has a submodule \( A' \) for which \( h'(A') \) is a summand of \( D \) (so \( A' \) is actually a summand of \( A_i^{p_i} \)) by Theorem A. Let \( A'' \), \( A''' \) be such that \( A' \oplus A'' = A \) and \( A' \oplus A''' = A_i^{p_i} \). Then we have a commutative diagram

\[
\begin{array}{cccccccc}
0 & \rightarrow & A' \oplus A'' & \overset{g}{\rightarrow} & B & \rightarrow & C & \rightarrow & 0 \\
\downarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} & & \downarrow \begin{bmatrix} 2 \\ 0 \end{bmatrix} & & \downarrow & & \\
0 & \rightarrow & A' \oplus A''' & \overset{\alpha(A') \oplus B'}{\rightarrow} & C & \rightarrow & 0.
\end{array}
\]

in which \( \alpha = h'|_{A'} \) is an isomorphism and \( h = h'|_{A'''}. \) Then \( A' \overset{1}{\rightarrow} A' \overset{g}{\rightarrow} g(A') \) is a monomorphism which is split by \( B \overset{\mu}{\rightarrow} \alpha(A') \overset{\alpha^{-1}}{\rightarrow} A', \) so \( g(A') \) is a summand of \( B \). This split monomorphism and the split inclusion of \( A' \) into \( A \) are coherent, i.e., the diagram

\[
\begin{array}{cccc}
A' & \overset{g}{\rightarrow} & g(A') \\
\text{incl} \uparrow \updownarrow & & \downarrow \uparrow \alpha^{-1} \circ \mu_1 \\
A' \oplus A'' & \overset{\text{proj}}{\rightarrow} & B
\end{array}
\]

commutes both ways. Thus the exact sequence \( 0 \rightarrow A' \overset{g}{\rightarrow} g(A') \rightarrow 0 \rightarrow 0 \) is a summand of \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \).

If we let \( R \) be a local PID which is also a \( k \)-algebra, we get some interesting consequences. We let \( f \) and \( g \) be matrices over \( R \), and say that \( X = F_2 \overset{f}{\rightarrow} F_1 \overset{g}{\rightarrow} F_0 \) is a **representation** of the diagram \( A_2 = \cdot \rightarrow \cdot \rightarrow \cdot \) over \( R \), where \( F_2, F_1, \) and \( F_0 \) are free \( R \)-modules (see, e.g., [DR]). If \( f \) and \( g \) are both
t \times t \text{ matrices with nonzero determinant, then the sequence } \varepsilon = 0 \to A \to B \to C \to 0 \text{ is naturally associated with } X, \text{ where } A = \text{coker}(f), \ B = \text{coker}(gf), \text{ and } C = \text{coker}(g), \text{ by the following commutative diagram:}

$$
\begin{array}{ccc}
F_2 & \longrightarrow & F_2 \\
\uparrow f & & \uparrow gf \\
0 & \longrightarrow & F_1 \\
& \downarrow g & \downarrow \\
& F_0 & \longrightarrow & C & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0.
\end{array}
$$

Two representations \( X, X' \) are said to be isomorphic if there is a commutative diagram

$$
\begin{array}{ccc}
F_2 & \stackrel{f}{\longrightarrow} & F_1 & \stackrel{g}{\longrightarrow} & F_0 \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
F'_2 & \stackrel{f'}{\longrightarrow} & F'_1 & \stackrel{g'}{\longrightarrow} & F'_0
\end{array}
$$

with \( \alpha, \beta, \text{ and } \gamma \) isomorphisms. It is shown in [C] that representations are isomorphic if and only if the corresponding sequences are isomorphic.

If \( m \) is the maximal ideal of \( R \) and \( f \) is a \( t \times t \) matrix, we let \( \nu(f) \) be the least integer \( n \) such that \( \det(f) \in m^n \) (where \( m^0 \) is the set of units of \( R \)). In this situation Theorem 1 yields the following.

**Corollary.** Let \( g \) be a fixed \( t \times t \) matrix with nonzero determinant, and let \( \nu(g) = r \). Then for a fixed integer \( n \), there are only finitely many nonisomorphic indecomposable representations \( X = F_2 \stackrel{f}{\longrightarrow} F_1 \stackrel{g}{\longrightarrow} F_0 \) with \( \nu(f) \leq n \).

**Proof.** If \( 0 \to A \to B \to C \to 0 \) is the sequence associated with \( X \), then the length of an indecomposable summand of \( C \) (respectively \( A \)) is bounded by \( r \) (respectively \( n \)); so every such sequence may be considered to be a sequence of \( R/m^s \)-modules, where \( s = \max\{r, n\} \). But \( R/m^s \) is an artin algebra of finite type, so Theorem 1 may be applied.

An application of this corollary, proved in [C], is that if \( \nu(f) < t \), where \( t \) is as above, then \( X \) must decompose.

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REFERENCES


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