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THE NUMBER OF INDECOMPOSABLE SEQUENCES OVER AN ARTIN ALGEBRA OF FINITE TYPE

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ABSTRACT. Let Λ be an artin algebra of finite representation type. For a finitely generated Λ-module C, there are only finitely many f.g. modules A such that $0 \to A \to B \to C \to 0$ is indecomposable as a short exact sequence.

Let Λ be an artin algebra of finite representation type and $\operatorname{mod}\Lambda$ the category of finitely generated (f.g.) left Λ modules. If X and C are in $\operatorname{mod}\Lambda$, we write ${}_\Lambda(X, C)$ for $\hom_\Lambda(X, C)$ and $P(X, C)$ for the submodule of ${}_\Lambda(X, C)$ comprising those maps $f\colon X \to C$ for which there exists a factorization

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & C \\ & \searrow \quad \nearrow & \\ & P & \end{array}$$

with P projective. Also, let Tr and D be the usual transpose and dual. In this setting, Theorem 5.7 in M. Auslander's paper [A] may be stated as follows.

Theorem A. *Let C be in $\operatorname{mod}\Lambda$. Let $A_1, \dots, A_m$ be a complete list of all noninjective indecomposable modules in $\operatorname{mod}\Lambda$ and let $X_i = \operatorname{Tr}\mathrm{D}A_i$. For each i, ${}_\Lambda(X_i, C)/P(X_i, C)$ is an $(\operatorname{End}X_i)^{\mathrm{op}}$-module of finite length. Let $S_{i_1}, \dots, S_{i_{d_i}}$ be a complete set of nonisomorphic simple $(\operatorname{End}X_i)^{\mathrm{op}}$-modules, and for each $(\operatorname{End}X_i)^{\mathrm{op}}$-submodule H of ${}_\Lambda(X_i, C)$ containing $P(X_i, C)$ let $n_1(A_i, H), \dots, n_{d_i}(A_i, H)$ be the uniquely determined nonnegative integers so that the $(\operatorname{End}X_i)^{\mathrm{op}}$-socle of ${}_\Lambda(X_i, C)/H$ is isomorphic to $\coprod_{j=1}^{d_i} S_{i_j}^{n_j(A_i, H)}$. Finally let $n(A_i) = \max\{n_j(A_i, H)\}$ as j runs through $1, 2, \dots, d_i$ and as H runs through all $(\operatorname{End}X_i)^{\mathrm{op}}$-submodules of ${}_\Lambda(X_i, C)$ containing $P(X_i, C)$. Then*

(1) *$n(A_i)$ is finite;*

(2) *if $k > n(A_i)$ and $0 \to A_i^k \xrightarrow{g} B \to C \to 0$ is exact, then A_i^k contains a submodule A' (isomorphic to $A_i^{k-n(A_i)}$) such that $g(A')$ is a summand of B.*

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Keeping the notation of Theorem A we have

Theorem 1. *Fix C in $\operatorname{mod}\Lambda$. Then there are only a finite number of modules A in $\operatorname{mod}\Lambda$ for which $0 \to A \xrightarrow{g} B \to C \to 0$ is indecomposable as a short exact sequence. In fact, if A has an injective summand, or if $A \simeq \coprod_{i=1}^{m} A_i^{p_i}$ with $p_i > n(A_i)$ for some i, the sequence decomposes.*

Proof. If A has an injective summand then clearly the sequence decomposes. Suppose $p_i > n(A_i)$, and form the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{g} & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \text{surj}} & & \downarrow & & \| & & \\ 0 & \longrightarrow & A_i^{p_i} & \xrightarrow[h']{} & D & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

Because $p_i > n(A_i)$, $A_i^{p_i}$ has a submodule A' for which $h'(A')$ is a summand of D (so A' is actually a summand of $A_i^{p_i}$) by Theorem A. Let A'', A''' be such that $A' \oplus A'' = A$ and $A' \oplus A''' = A_i^{p_i}$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' \oplus A'' & \xrightarrow{g} & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \left[\begin{smallmatrix} 1 & 0 \\ 0 & \rho \end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} \mu_1 \\ \mu_2 \end{smallmatrix}\right]} & & \| & & \\ 0 & \longrightarrow & A' \oplus A''' & \xrightarrow[\left[\begin{smallmatrix} \alpha & 0 \\ 0 & h \end{smallmatrix}\right]]{} & \alpha(A') \oplus B' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

in which $\alpha = h'|_{A'}$ is an isomorphism and $h = h'|_{A'''}$. Then $A' \xrightarrow{1} A' \xrightarrow{g} g(A')$ is a monomorphism which is split by $B \xrightarrow{\mu_1} \alpha(A') \xrightarrow{\alpha^{-1}} A'$, so $g(A')$ is a summand of B. This split monomorphism and the split inclusion of A' into A are *coherent*, i.e., the diagram

$$\begin{array}{ccc} A' & \overset{g|_{A'}}{\longrightarrow} & g(A') \\ \text{incl} \uparrow\downarrow \ \text{proj} & & \downarrow\uparrow \alpha^{-1} \circ \mu_1 \\ A' \oplus A'' & \underset{g}{\longrightarrow} & B \end{array}$$

commutes both ways. Thus the exact sequence $0 \to A' \xrightarrow{g|_{A'}} g(A') \to 0 \to 0$ is a summand of $0 \to A \to B \to C \to 0$.

If we let R be a local PID which is also a k-algebra, we get some interesting consequences. We let f and g be matrices over R, and say that $X = F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0$ is a *representation* of the diagram $A_2 = \cdot \to \cdot \to \cdot$ over R, where F_2, F_1, and F_0 are free R-modules (see, e.g., [DR]). If f and g are both

$t \times t$ matrices with nonzero determinant, then the sequence $\varepsilon = 0 \to A \to B \to C \to 0$ is naturally associated with X, where $A = \operatorname{coker}(f)$, $B = \operatorname{coker}(gf)$, and $C = \operatorname{coker}(g)$, by the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & F_2 & = & F_2 & & & & \\
 & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle gf} & & & & \\
0 & \longrightarrow & F_1 & \xrightarrow[g]{} & F_0 & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

Two representations X, X' are said to be isomorphic if there is a commutative diagram

$$\begin{array}{ccccc}
F_2 & \xrightarrow{f} & F_1 & \xrightarrow{g} & F_0 \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
F_2' & \xrightarrow[f']{} & F_1' & \xrightarrow[g']{} & F_0'
\end{array}$$

with α, β, and γ isomorphisms. It is shown in [C] that representations are isomorphic if and only if the corresponding sequences are isomorphic.

If $\underline{m}$ is the maximal ideal of R and f is a $t \times t$ matrix, we let $\nu(f)$ be the least integer n such that $\det(f) \in \underline{m}^n$ (where $\underline{m}^0$ is the set of units of R). In this situation Theorem 1 yields the following.

Corollary. *Let g be a fixed $t \times t$ matrix with nonzero determinant, and let $\nu(g) = r$. Then for a fixed integer n, there are only finitely many nonisomorphic indecomposable representations $X = F_2 \xrightarrow{f} F_2 \xrightarrow{g} F_0$ with $\nu(f) \le n$.*

Proof. If $0 \to A \to B \to C \to 0$ is the sequence associated with X, then the length of an indecomposable summand of C (respectively A) is bounded by r (respectively n); so every such sequence may be considered to be a sequence of $R/\underline{m}^s$-modules, where $s = \max\{r, n\}$. But $R/\underline{m}^s$ is an artin algebra of finite type, so Theorem 1 may be applied.

An application of this corollary, proved in [C], is that if $\nu(f) < t$, where t is as above, then X must decompose.

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