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LACK OF UNIFORM STABILIZATION FOR NONCONTRACTIVE SEMIGROUPS UNDER COMPACT PERTURBATION

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ABSTRACT. Let $G(t)$, $t \geq 0$, be a strongly continuous semigroup on a Hilbert space X (or, more generally, on a reflexive Banach space with the approximating property), with infinitesimal generator A. Let: (i) either $G(t)$ or $G^*(t)$ be strongly stable, yet (ii) not uniformly stable as $t \to +\infty$. Then, for any compact operator B on X, the semigroup $S_B(t)$ generated by $A+B$ cannot be uniformly stable as $t \to +\infty$. This result is 'optimal' within the class of compact perturbations B. It improves upon a prior result in [G.1] by removing the assumption that $G(t)$ be a contraction for positive times. Moreover, it complements a result in [R.1] where $G(t)$ was assumed to be a group, contractive for negative times. Our proof is different from both [R.1 and G.1]. Application include physically significant dynamical systems of hyperbolic type in feedback form, where the results of either [R.1 or G.1] are not applicable, as the free dyamics is not a contraction.

1. INTRODUCTION AND MAIN RESULTS

Let $G(t)$, $t \geq 0$, be a strongly continuous semigroup (possibly, a group, $t \in R$) of bounded linear operators on the Hilbert space X with norm $\| \ \|$, and let A be its (linear, closed) infinitesimal generator with domain $\mathscr{D}(A)$ dense in X. Let B be, for the time being, an element of $\mathscr{L}(X)$, the space of all bounded linear operator on X with the usual norm likewise indicated by $\| \ \|$. We then denote by $S_B(t)$, $t \geq 0$, the strongly continuous semigroup (group) generated by $A_B = A + B$, $\mathscr{D}(A_B) = \mathscr{D}(A)$, which corresponds to the abstract dynamics $\dot{x} = Ax + u$, $x(0) = x_0$, with u in "feedback" form $u = Bx$, whose solution is given by $x(t, x_0) = S_B(t)x_0$. Feedback problems of this type arise, and are of interest, in the context of control theoretic studies for dynamical systems, where an aim is to select the (feedback) operator B as to force the corresponding (feedback) dynamics $S_B(t)$ to possess stability properties not enjoyed by the original (free) dynamics $G(t)$; most notably, to force $S_B(t)$ to be uniformly (exponentially) stable in $\mathscr{L}(X)$. This means that

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there exist positive constants $M, \delta > 0$ such that

$$\|S_B(t)\| \le Me^{-\delta t}, \qquad t \ge 0. \tag{1.0}$$

We then say that the original dynamics $G(t)$ has been "uniformly stabilized by means of a feedback operator B" in the uniform norm of $\mathscr{L}(X)$. Having formulated the goal expected of the feedback operator B, it is likewise of interest to single out classes of operators $B \in \mathscr{L}(X)$ and of free dynamics $G(t)$, where the above objective cannot be achieved. An important class in physical applications is the class of *compact* (feedback) operators $B \in \mathscr{L}(X)$. Thus, henceforth, unless otherwise stated, we shall take B to be a compact operator on X. Then, the following two results of negative character on the lack of uniform stabilization by a *compact* feedback B are well known, and extensively invoked in the literature of feedback control systems.

(1) Russell [R.1] *The semigroup $S_B(t)$ with B compact cannot be uniformly (exponentially) stable* (i.e., (1.0) cannot be satisfied) *provided that:* (i) $G(t)$ *is a group, $t \in R$, which moreover is* (ii) *contraction for negative times* $\|G(-t)\| \le 1$, $t \ge 0$.

A different proof of the above result was later given in [G.1] where, in addition, a new result of negative character is presented, which does not require the hypothesis that $G(t)$ be a group.

(2) Gibson [G.1] *The semigroup $S_B(t)$ with B compact cannot be uniformly exponentially stable, provided that*

(i) $G(t)$ *is a semigroup of contractions*: $\|G(t)\| \le 1$, $t \ge 0$, *which* (ii) *is strongly stable*: $G(t)x \to 0$ *as* $t \to +\infty$, *for all* $x \in X$, *but* (iii) *not uniformly (exponentially) stable on* X.

An equivalent statement of Gibson's result is that under assumptions (i) and (ii), if $S_B(t)$ is uniformly stable with B compact, then so is $G(t)$.

We note explicitly that the proof given in [G.1] makes crucial use of the assumption that $G(t)$ be a contraction for $t \ge 0$. As such, Gibson's result does not apply to some physically significant situations, see §2.

The purpose of this note is to provide the following extension of Gibson's result, which in particular removes the assumption that $G(t)$ be a contraction for $t \ge 0$. Our proof is simple and direct, while the one in [G.1] is by contradiction and in effect needs the details supplied by [B.1, Proposition 4.11.2, p. 247]. Our result is optimal within the class of compact feedback operators B in the sense specified in Remark 1.1 below.

Theorem. *Assume the following hypotheses.*

(i) *Either*[1] $G(t)$ *is strongly stable; i.e.*

$$G(t)x \to 0 \quad as\ t \to +\infty,\ \forall x \in X, \tag{1.1}$$

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[1] Conditions (1.1) and (1.2) are not equivalent, in general. They are, however, if A (equivalently, A^*) has compact resolvent on X [B.1, p. 245].

or else $G^*(t)$ *is strongly stable, i.e.*

$$G^*(t)x \to 0 \quad as\ t \to +\infty\,,\ \forall x \in X\,; \tag{1.2}$$

(ii) B *is a bounded compact operator on* X *;*

(iii) *let the semigroup* $S_B(t)$, $t \geq 0$, *generated by the operator* $A_B = A + B$, *be uniformly (exponentially) stable*

$$\|S_B(t)\| \leq Me^{-\delta t}\,, \qquad t \geq 0 \tag{1.3}$$

for some positive constants $M\,,\delta$. *Then,* $G(t)$ *is uniformly (exponentially) stable.* □

Proof. We first write the proof under the assumption (1.2) that $G^*(t)$ is strongly stable. Our objective is to show that

$$\|G(t)\| < 1\,, \quad \text{for all } t \text{ sufficiently large.} \tag{1.4}$$

This, as is well known, will then imply that $G(t)$ decays exponentially in the uniform norm $\mathscr{L}(X)$, as desired. To this end, writing $A = A_B - B$, the variation of constants formula gives

$$G(t)x = S_B(t)x - \int_0^t S_B(t-\tau)BG(\tau)x\,d\tau\,, \qquad t \geq 0\,,\ x \in X\,. \tag{1.5}$$

2. Since B is compact, it is the uniform limit of (bounded) operators B_n of finite rank. We may, if we wish, define B_n constructively as follows (but it is not necessary). Let Λ_n be the orthogonal projection of X onto $X_n = \operatorname{span}\{e_1\,, \dots\,, e_n\}$, where $\{e_k\}_{k=1}^{\infty}$ are an orthonormal basis on X. As it is well known [N-S.1, p. 386], since B is compact, the operator $\Lambda_n B \Lambda_n$ converges to B in the uniform norm of $\mathscr{L}(X)$. Thus, given $\varepsilon > 0$, there exists N_ε such that

$$\forall n > N_\varepsilon \Rightarrow \|\Lambda_n B \Lambda_n - B\| \leq \varepsilon\delta/MC \tag{1.6}$$

where, as a consequence of either condition under assumption (i), we have

$$\|G^*(t)\| = \|G(t)\| \leq C\,, \qquad \forall t \geq 0 \tag{1.7}$$

by the Principle of Uniform Boundedness.

3. We rewrite (1.5) more conveniently as

$$\begin{aligned} G(t)x = S_B(t)x &- \int_0^t S_B(t-\tau)(B - \Lambda_n B \Lambda_n)G(\tau)x\,d\tau \\ &- \int_0^t S_B(t-\tau)\Lambda_n B \Lambda_n G(\tau)x\,d\tau\,. \end{aligned} \tag{1.8}$$

4. Henceforth, unless otherwise stated, let n be fixed and $> N_\varepsilon$, so that (1.6) holds. Then, (1.3) and (1.6) plainly imply for such n

$$\begin{aligned} &\left\| \int_0^t S_B(t-\tau)(B - \Lambda_n B \Lambda_n)G(\tau)x\,d\tau \right\| \\ &\qquad \leq \varepsilon\delta \int_0^t e^{-\delta(t-\tau)}\,d\tau \|x\| \leq \varepsilon\|x\|\,, \quad \text{uniformly in } t \geq 0\,. \end{aligned} \tag{1.9}$$

5. We now note that, since $G^*(t)$ is strongly continuous and strongly stable, we have

(1.10) $\Lambda_n B \Lambda_n G(t)$ uniformly continuous and uniformly stable as $t \to +\infty$

for any n fixed. Indeed

$$\begin{aligned}(1.11)\qquad \Lambda_n B \Lambda_n G(t)x &= \sum_{k=1}^{n} (\Lambda_n B \Lambda_n G(t)x \,, e_k) \,, e_k)_X \\ &= \sum_{k=1}^{n} (x \,, G^*(t)\Lambda_n B^* \Lambda_n e_k)_X e_k\end{aligned}$$

so that

$$(1.12)\qquad \|\Lambda_n B \Lambda_n G(t)x\| \leq \left(\sum_{k=1}^{n} \|G^*(t)\Lambda_n B^* \Lambda_n e_k\|\right)\|x\|$$

where the term in parenthesis tend to zero as t tends to $+\infty$, and uniform stability is verified. One similarly checks uniform continuity. Thus, given $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$(1.13)\qquad \forall t > T_\varepsilon \Rightarrow \|\Lambda_n B \Lambda_n G(t)\| < \varepsilon\delta/M$$

6. We split the second integral in (1.8) for $t > T_\varepsilon$ as

$$\begin{aligned}(1.14)&\int_0^t S_B(t-\tau)\Lambda_n B \Lambda_n G(\tau)x \, d\tau \\ &= \int_0^{T_\varepsilon} S_B(t-\tau)\Lambda_n B \Lambda_n G(\tau)x \, d\tau + \int_{T_\varepsilon}^t S_B(t-\tau)\Lambda_n B \Lambda_n G(\tau)x \, d\tau\end{aligned}$$

where, by (1.3) and (1.13), one plainly obtains for all $t > T_\varepsilon$ and $x \in X$

$$\begin{aligned}(1.15)\quad \left\|\int_{T_\varepsilon}^t S_B(t-\tau)\Lambda_n B \Lambda_n G(\tau)x \, d\tau\right\| &\leq \varepsilon\delta \int_{T_\varepsilon}^t e^{-\delta(t-\tau)} \, d\tau \|x\| \\ &\leq \varepsilon[1 - \exp(-\delta(t - T_\varepsilon))]\|x\| < \varepsilon\|x\| \quad \text{uniformly in } t > T_\varepsilon .\end{aligned}$$

7. On the other hand by (1.3)

$$\begin{aligned}\left\|\int_0^{T_\varepsilon} S_B(t-\tau)\Lambda_n B \Lambda_n G(\tau)x \, d\tau\right\| &\leq \text{const}_n \int_0^{T_\varepsilon} e^{-\delta(t-\tau)} \, d\tau \|x\| \\ (1.16)\qquad &\leq \text{const}_n \, e^{-\delta t} \frac{(e^{\delta T_\varepsilon} - 1)}{\delta} \|x\| \quad \text{for all } t > T_\varepsilon\end{aligned}$$

where $\|\Lambda_n B \Lambda_n G(\tau)\| \leq \text{const}_n \leq \text{const}$, uniformly in $\tau \geq 0$.

8. We return to identity (1.8), use (1.14), as well as the uniform bounds (1.3), (1.9), (1.15), (1.16) and thus obtain (1.4) as desired. The proof under assumption (1.2) is complete.

Let us now assume strong stability of $G(t)$ as in (1.1). We then apply the above proof to $A^* = A_B^* - B^*$, i.e. to the identity

$$G^*(t)x = S_B^*(t)x - \int_0^t S_B^*(t-\tau)B^* G^*(\tau)x \, d\tau \,, \qquad t > 0 \,,\ x \in X$$

instead of (1.5), where now assumption (i) is used to assert that $\Lambda_n B^* \Lambda_n G^*(t)$ is uniformly stable as $t \to +\infty$ for n fixed sufficiently large. We thus obtain that $G^*(t)$ is uniformly (exponentially) stable. But then so is $G(t)$, since $\|G(t)\| = \|G^*(t)\|$. The proof is complete □

Remark 1.1. An equivalent statement of our Theorem is that, with B compact, if $G(t)$ [or $G^*(t)$] is strongly stable but not uniformly (exponentially) stable on X, then $S_B(t)$ cannot be uniformly (exponentially) stable on X. This result, under the standing assumption that B is compact, is optimal in the sense that if assumption (i) regarding (1.1), (1.2) is removed from the Theorem, then its conclusion is false. In fact, to see this, it suffices to take the case where $G(t)$ is a compact semigroup for all $t >$ some $t_0 \geq 0$ (examples: parabolic equations on a smooth bounded domain of R^n, where in fact $t_0 = 0$; functional differential or delay differential equations of retarded type with finite delay, where $t_0 =$ largest delay). In this case, a suitable compact (indeed, even finite rank, bounded) operator B may well force $S_B(t)$ to be uniformly stable, even though $G(t)$ is not stable even in the weak topology [T.1] (In this present case of compact semigroup $G(t)$, stability of $G(t)$ in the weak topology is equivalent to stability of $G(t)$ in the uniform topology [B.1, p. 245]). The present case occurs e.g. when A has finitely many unstable eigenvalues with real part nonnegative.

In contrast, Russell's or Gibson's result, as well as our Theorem above, refer e.g. to hyperbolic dynamics or functional differential equations of neutral type. Assumption (ii) in Russell's result that $\|G(-t)\| \leq 1$ for $t \geq 0$ is meant to exclude the case e.g. of the wave equation with, say, viscous constant damping, where $G(t)$ itself is uniformly (exponentially) stable for $t > 0$ and thus so is $S_B(t)$ with the compact operator $B = 0$.

Remark 1.2. The proof of the Theorem, say under assumption (1.2), extends to Banach spaces having the "approximation property" that compact operators are uniform limits of finite rank operators. This property holds for the most commonly used Banach spaces, although it is not true for all separable reflexive Banach spaces [E.1]. The proof under assumption (1.1) requires further that X be a reflexive Banach space, so that $G^*(\cdot)$ and $S_B^*(\cdot)$ are likewise strongly continuous semigroups (groups) on X^* with generators A^* and A_B^*, respectively □

2. Applications

In this section we present examples of semigroups where our Theorem above is applicable, while Russell's and Gibson's results are not.

Example 2.1. Let Ω be a smooth bounded domain in R^n, n typically ≥ 2, with boundary Γ. We consider the following feeback dynamics, involving the wave equation with feedback action exercised in the Dirichlet boundary conditions in the form of a nonlocal, finite dimensional range operator

(2.1a) $x_{tt}(t,\xi) = \Delta x(t,\xi)$ in $(0,T) \times \Omega$,

(2.1b) $x(0,\xi) = x_0(\xi); x_t(0,\xi) = x_1(\xi)$, $\quad \xi \in \Omega$,

(2.1c) $x(t,\sigma) = [\langle x(t,\cdot), w_1(\cdot)\rangle + \langle x_t(t,\cdot), w_2(\cdot)\rangle] g(\sigma)$ in $(0,T)x\Gamma$

where $\langle \cdot, \cdot \rangle$ denotes the $L_2(\Omega)$-inner product, and w_i, g are, for the time being, vectors in $L_2(\Omega)$ and $L_2(\Gamma)$, respectively. Throughout this example, we let $\mathscr{A}: L_2(\Omega) \supset \mathscr{D}(\mathscr{A}) \to L_2(\Omega)$ be the positive, selfadjoint operator $\mathscr{A} f = -\Delta f$ with homogeneous Dirichlet boundary conditions, which we then extend by isomorphism techniques from $L_2(\Omega) \to [\mathscr{D}(\mathscr{A})]'$, the dual of $\mathscr{D}(\mathscr{A})$ with respect to $L_2(\Omega)$, while preserving the same symbol $\mathscr{A}$. We shall distinguish two cases.

Case 1. Let $w_1 = 0$. It is proved in [L.T.1, T.2] that if the vectors w_2 and g satisfy some technical conditions (2.9), (2.10) below, then with reference to the dynamics (2.1), if we set

$$\begin{vmatrix} x(t) \\ x_t(t) \end{vmatrix} = G(t) \begin{vmatrix} x_0 \\ x_1 \end{vmatrix} \tag{2.2}$$

we have

(i) (well-posedness of (2.1)) $G(t)$ is a strongly continuous group of operators with infinitesimal generator

$$A = \begin{bmatrix} 0 & I \\ -\mathscr{A} & 0 \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & -Dg\langle \cdot, w_1 \rangle \\ 0 & 0 \end{bmatrix} \right) \tag{2.3}$$

on each of the following spaces

$$I_\theta = \mathscr{D}(\mathscr{A}^\alpha) \times \mathscr{D}(\mathscr{A}^\beta); \qquad \alpha = \tfrac{1}{4} - \rho - \tfrac{\theta}{2};\ \beta = -\tfrac{1}{4} - \rho - \tfrac{\theta}{2};\ 0 \le \theta \le 1 \tag{2.4}$$

where for $s > 0$ we use the conventional notation $\mathscr{D}(\mathscr{A}^{-s}) = [\mathscr{D}(\mathscr{A}^s)]'$, the dual space to $\mathscr{D}(\mathscr{A}^s)$ with respect to $L_2(\Omega)$. Moreover, $\rho > 0$ is an arbitrary small constant fixed once and for all by the relation

$$D\text{: continuous } L_2(\Gamma) \to H^{1/2}(\Omega) \subset H^{1/2-2\rho}(\Omega) = \mathscr{D}(\mathscr{A}^{1/4-\rho}). \tag{2.5}$$

Here, D is the Dirichlet map for the corresponding elliptic problem defined by

$$h = Dv, \quad \text{where } \Delta h = 0 \text{ on } \Omega; \qquad h = v \quad \text{on } \Gamma; \tag{2.6}$$

(ii) $G(t)$ is strongly stable on I_θ

$$||G(t)x||_{I_\theta} \to 0 \quad \text{as } t \to +\infty,\ \forall x \in I_\theta; \tag{2.7}$$

(ii) $G(t)$ is *not* uniformly stable here.

The technical conditions on the vector w_2, g are as follows. Set

$$\gamma = -\mathscr{A}^{-1/2-2\rho} Dg \in L_2(\Omega) \tag{2.8}$$

and require the following two conditions:

(i)

$$\langle w_2, \Phi_i\rangle\langle Dg, \Phi_i\rangle < 0, \qquad i = 1, 2, \ldots \tag{2.9}$$

where $\{\Phi_i, i = 1, 2, \ldots\}$ is an orthonormal basis of eigenvectors of $\mathscr{A}$ on $L_2(\Omega)$, with eigenvalues $\mu_i > 0$, for simplicity supposed all simple throughout;

(ii) there are two positive constants $0 < c < C < \infty$ such that

$$0 < c \le \frac{|\langle w_2, \Phi_i\rangle|}{|\langle \gamma, \Phi_i\rangle|} \le C < \infty, \qquad i = 1, 2, \ldots \tag{2.10}$$

As the proof in [L-T.1] shows, the semigroup $G(t)$ in (2.2) is generally *not* a contraction on I_θ. (Contraction is achieved in the special case where

$$\mathscr{A}^{3/4+\rho} w_2 = -k^2 \mathscr{A}^{1/4-\rho} Dg, \quad \text{or} \quad w_2 = k^2\gamma \tag{2.11}$$

for a positive constant k, with ρ fixed by (2.5), and γ defined by (2.8). Condition (2.10) implies that these vectors w_2 and γ belong to the domain of the same fractional power of $\mathscr{A}$, i.e., $\mathscr{D}(\mathscr{A}^{3/4+\rho})$.

Thus, under assumptions (2.9), (2.10) and with reference to $G(t)$ in (2.2), *our Theorem in* § 1 *is applicable* while we are not authorized in general to invoke either Russell's or Gibson's result. By our Theorem, we then conclude that: addition on the right hand side of (2.1a) of an operator B, compact on the space I_θ and acting on the pair $[x(t), x_t(t)]$, cannot force the new resulting dynamics to become uniformly (exponentially) stable in the norm of $\mathscr{L}(I_\theta)$ as $t \to +\infty$. In operator terms, this means that the operator $A + B$, A as in (2.3), is the generator of a strongly continuous group of operators on I_θ, which however is not uniformly stable in $\mathscr{L}(I_\theta)$. In short, problem (2.1) cannot be uniformly stabilized by a compact operator.

Case 2. We now allow w_2 to be possibly *non*zero. Here, following [L.T.2, L-T.3], we consider the following problems, in a sense the converse of the one in Case 1. We let $g \in L_2(\Gamma)$ satisfy

$$\langle Dg, \Phi_m\rangle \neq 0, \text{ equivalently } (g, \partial\Phi_m/\partial\nu)_{L_2(\Gamma)} \neq 0, \qquad m = 1, 2, \ldots \tag{2.12}$$

D and Φ_m as in Case 1, and we select a scalar sequence $\{\varepsilon_m^{+,-}, m = 1, 2, \ldots\}$ of nonzero complex numbers with $\varepsilon_m^+ = \overline{\varepsilon_m^-}$, $\operatorname{Re}\varepsilon_m^+ < 0$ and otherwise satisfying the technical assumptions of Theorem 3.3 in [L-T.3] (these state, qualitatively, that $|\varepsilon_m^{+,-}| \to 0$ as $m \to \infty$ "sufficiently fast"). Then, see [L-T.3, Application 4.3], there exist suitable *real* vectors $w_1 \in L_2(\Omega)$ and $w_2 \in \mathscr{D}(\mathscr{A}^{1/2})$ such that

(i) the corresponding operator $G(t)$ defined by (2.2) is a strongly continuous group of operators on the space $Z = L_2(\Omega) \times [\mathscr{D}(\mathscr{A}^{1/2})]'$ which admits the following expansion for $[x_0, x_1] \in Z$,

$$\begin{bmatrix} x(t) \\ x_t(t) \end{bmatrix} = G(t)\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \sum_{m=1}^{\infty} k_m\left(\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}\right) e^{\alpha_m^{+,-}t}\psi_m^{+,-}. \tag{2.13}$$

Here k_m are continuous linear functional on Z and

$$\alpha_m^{+,-} = \pm i\sqrt{\mu_m} + \varepsilon_m^{+,-}, \qquad 0 \neq |\varepsilon_m^{+,-}| \to 0 \tag{2.14}$$

are the new eigenvalues of the corresponding problem (2.1) with corresponding normalized eigenvectors $\{\psi_m^{+,-}\}$ which form a Riesz basis on Z; this means that there are two constants $0 < c < C < \infty$ such that for $z_0 = [x_0, x_1] \in Z$,

$$c \sum_{m=1}^{\infty} |k_m(z_0)|^2 e^{(\mathrm{Re}\,\alpha_m^{+,-})t} \leq \|G(t)z_0\|_Z^2 \leq C \sum_{m=1}^{\infty} |k_m(z_0)|^2 e^{(\mathrm{Re}\,\alpha_m^{+,-})t}, \tag{2.15}$$

(ii) $G(t)$ is strongly stable on Z

$$\|G(t)z\|_Z \to 0 \quad \text{as } t \to +\infty, \ \forall z \in Z, \tag{2.16}$$

(iii) $G(t)$ is *not* uniformly stable here.

Moreover, $G(t)$ is *not* contraction on Z in general, for either positive or negative times. Thus, our *Theorem in* §1 is *applicable* to $G(t)$ (equivalently, to problem (2.1)), while either Russell's result or Gibson's result are not in general.

Example 2.2. Let $X = l_p$, $1 < p < \infty$, $\Phi_i = [0, \ldots, 0, 1, 0, \ldots,]$ 1 in the ith position and let

$$A = \mathrm{diag}[-\lambda_1, -\lambda_2, -\lambda_3, \ldots] \tag{2.17}$$

where $\lambda_n = r_n + iv_n$, r_n, v_n real numbers, r_n positive and $\downarrow 0$, while $v_n \uparrow +\infty$. Then A generates a strongly continuous group $G_1(t)$ in X

$$G_1(t) = \mathrm{diag}[e^{-\lambda_1 t}, e^{-\lambda_2 t}, e^{-\lambda_3 t}, \ldots]. \tag{2.18}$$

We have [T.3]

(i)

$$G_1(t)x \to 0 \quad \text{as } t \to +\infty, \ \forall x \in X, \tag{2.19}$$

(ii)

$$\|G_1(t)\|_{\mathscr{L}(x)} \equiv 1, \qquad t \geq 0, \tag{2.20}$$

whereby Gibson's result would apply to $G_1(t)$, but Russell's result would not ($G_1(-t)\Phi_i = e^{(\mathrm{Re}\,\lambda_i)t}\Phi_i$, $t \geq 0$). Apply now a similarity transformation to $G_1(t)$

$$G(t) = QG_1(t)Q^{-1}, \qquad t \in R; \ Q, Q^{-1} \in \mathscr{L}(X)$$

where Q is not unitary, $Q^* \neq Q^{-1}$. Then, $G(t)$ is a strongly continuous group on X which is strongly stable here. Moreover, $G(t)$ is not contraction in general. Thus, *our Theorem in* §1 *is applicable to* $G(t)$, but Russell's and Gibson's results are not in general. [Note that if we take instead $v_n \equiv 0$, then the point 0 belongs to the essential spectrum of A, and hence it cannot be removed by a compact (or even relatively compact) perturbation B [K.1, p. 244]. Thus, the group generated by $A + B$ cannot be uniformly (exponentially) stable in this case. This example was given in [T.3].]

Example 2.3. In conclusion, we remark that if $G(t)$ is a strongly continuous, contraction semigroup on X which is strongly stable but not uniformly stable (so that $\|G(t)\| \equiv 1$, $t \geq 0$), the introduction of an equivalent norm on X would in general destroy contraction, but preserve the properties of strong stability and the lack of uniform stability. Thus, our Theorem is applicable in the equivalent norm, while Russell's and Gibson's result would not be applicable in general.

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