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L^2-BOUNDEDNESS OF SPHERICAL MAXIMAL OPERATORS WITH MULTIDIMENSIONAL PARAMETER SETS

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ABSTRACT. For $s > 0$, let $M_s f(x) = \int_{|y|=1} f(x - sy)\, d\sigma(y)$ be the spherical mean operator on R^n. For a certain class of surfaces S in R_+^{n+1} with $\dim S = n - 2$ or $\dim S = n - 1$ with an additional condition, the maximal operator

$$\mathcal{M} f(x) = \sup_{(u,s)\in S} |M_s f(x - u)|$$

is shown to be bounded on $L^2(R^n)$. This extends (on $L^2(R^n)$) the theorem of Stein [7], where $S = \{(0, s) : s > 0\}$, and its generalizations to $\dim S = 1$ in Greenleaf [2] and Sogge and Stein [6].

1. INTRODUCTION

The spherical means $M_s f$, $s > 0$, of a function f defined on R^n are given by

$$M_s f(x) = \int_{|y|=1} f(x - sy)\, d\sigma(y), \tag{1}$$

where σ is the rotationally invariant measure on the unit sphere $\{ y \in R^n : |y| = 1 \}$ with total mass 1. It was shown by Stein [7] that if $n \geq 3$ and $p > n/(n-1)$, then the integral in (1) is, in fact, well defined for every function $f \in L^p_{\text{loc}}(R^n)$ and for almost every $x \in R^n$ and every $s > 0$, and the associated maximal operator defined by

$$f \to \sup_{s>0} |M_s f(x)| \tag{2}$$

is bounded on $L^p(R^n)$ (see also [9]). These results were extended to the case where the sphere was replaced by a hypersurface in R^n (see Greenleaf [2], and Sogge and Stein [6]). The following maximal operator, in which translations are involved, was considered by Sogge and Stein [6]:

$$f \to \sup_{s>0} \left| \int_\Omega f(x - \gamma(s) - sy) \psi(y)\, d\sigma(y) \right|, \tag{3}$$

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where Ω is a smooth hypersurface in R^n, $\psi \in C_c^\infty(\Omega)$, and γ is a curve in R^n. It was shown that if $n \geq 3$, the Gaussian curvature at each point of Ω does not vanish of infinite order, and γ is Lipschitzian, then the operator (3) is bounded on $L^p(R^n)$ whenever $p > p_0$ for some $p_0 = p_0(\Omega) < \infty$. One special case is the following : If γ is a Lipschitz curve in R^n and $n \geq 3$, then the maximal operator

$$f \to \sup_{s>0} |M_s f(x - \gamma(s))| \tag{4}$$

is bounded on $L^p(R^n)$ whenever $p > n/(n-1)$. This was proved also by Greenleaf [2] for the case when γ is a C^∞-curve.

We shall extend the maximal operator (4) by allowing the translations to be taken from a surface instead of a curve. Surfaces S which are suitable for this purpose are described as follows. Let T be a k-dimensional C^1-surface in R^{n+1} such that $T \cap R^n \times \{0\} \neq \varnothing$ and no tangent plane of T is parallel to $R^n \times \{0\}$, and let $\rho: R^1_+ \to R^1_+$ be a differentiable function such that for some constants c_1 and c_2 with $c_1 > c_2 > 0$, $\rho'(r) \geq c_2$ and $\rho(r) \leq c_1 r$ for $0 < r \leq 1$. We now set $S = \{(\rho(r)y, r) : (y, r) \in T, 0 < r \leq 1\}$, which we shall call a k-dimensional parameter set. Examples of k-dimensional parameter sets are k-dimensional cones which are not tangent to $R^n \times \{0\}$ and have vertices at the origin.

Fix a k-dimensional parameter set S of the above form and consider the maximal operator defined by

$$\mathscr{M} f(x) = \sup_{(u,s)\in S} \left| M_{(u,s)} f(x) \right|, \tag{5}$$

where $M_{(u,s)} f(x) = M_s f(x-u)$ and $M_s f(x)$ is the spherical mean of f on the sphere $\{y \in R^n : |y - x| = s\}$. The previous results correspond to the case that S is 1-dimensional.

In this paper we shall prove the following theorems about the L^2-boundedness of the maximal operator $\mathscr{M}$ with multidimensional parameter set.

Theorem 1. *If $n \geq 3$ and $\dim S \leq n-2$, then $\mathscr{M}$ is bounded on $L^2(R^n)$.*

Theorem 2. *Suppose $\dim S = n-1$ and there is an $(n-1)$-dimensional subspace V of R^n such that for each $(u, s) \in S$, $\{y - u : (y, s) \in S\} \subset V$. If $n \geq 3$, $\mathscr{M}$ is bounded on $L^2(R^n)$.*

Examples of S's satisfying the hypothesis of Theorem 2 are provided by intersecting an n-dimensional parameter set (for example, the light cone) with a hyperplane in R^{n+1} passing through the origin.

Our theorems do not seem to assert optimal results. The following examples, however, show that some restriction on the dimension of the parameter sets is in the nature of the problem.

(a) If $\dim S = n+1$, then $\mathscr{M}$ is not bounded on any $L^p(R^n)$, $1 \leq p < \infty$: Suppose $0 < s < 1$ and choose a ball $B \subset S_s$. For each r with $0 < r \leq s$ define a continuous function f_r supported on the set $E_r = \{x: s - r \le |x| \le s + r\}$ by $f_r(x) = 1 - ||x| - s|/r$. Then, for every $u \in B$, $\mathcal{M} f_r(u) \ge M_{(u,s)} f(u) = M_s f(0) = 1$, and so

$$\|\mathcal{M} f_r\|_p \ge (\text{volume of } B)^{1/p} = c_p > 0.$$

On the other hand, $\|f_r\|_p \le (\text{volume of } E_r)^{1/p} = c_p' r^{1/p}$, and the last term tends to 0 as $r \to 0$. Therefore, $\mathcal{M}$ is not bounded on any $L^p(R^n)$, $1 \le p < \infty$.

(b) If $\dim S = n$, then $\mathcal{M}$ is not bounded on any $L^p(R^n)$, $1 \le p \le \frac{n+1}{n-1}$. (This, in particular, shows that $\dim S \le n - 1$ is necessary for the L^2-boundedness of $\mathcal{M}$ when $n = 3$.): Fix ε with $0 < \varepsilon < 1$, and let $E = \{x: |x - \varepsilon e_n| > \varepsilon, x_n > 0, \text{ and } |x| < \varepsilon\}$, where e_n is the nth standard basis element of R^n. Define a function f on R^n by $f(x) = \varphi(|x|)\chi_E(x)$, where $\varphi(r) = r^{-n+1}|\log r|^{-1}$ for $r > 0$. An elementary computation shows that $f \in L^p(R^n)$ provided $1 \le p \le \frac{n+1}{n-1}$, and $M_s f(s e_n) = \infty$ for every $s > \varepsilon$. Let $H = \bigcup_{\varepsilon < s < 1}(se_n + S_s)$, where S_s is the level set of S at the height s. Then $\mathcal{M} f(x) = \infty$ for every $x \in H$. Note that H contains an open set since $\dim S = n$. We thus get $\|\mathcal{M} f\|_p = \infty$ for $1 \le p \le \infty$. Therefore, $\mathcal{M}$ is not bounded on $L^p(R^n)$ for $1 \le p \le \frac{n+1}{n-1}$.

The method we use for the proofs of the above theorems is a variant of a method of Paley used in [5] (see also [4] and [11]). It makes use of positivity and estimates for $M_{(u,s)} M^*_{(v,t)} f(x)$, which we give in the next section. However, we shall give the proofs for functions in $C_c(R^n)$. A well-known measure-theoretic argument then extends the results on $C_c(R^n)$ to those on $L^2(R^n)$ (see [9]). We use the convention: c (or $c_1, c_2, \ldots$) means a positive constant which is not necessarily the same at each occurrence.

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2. Estimates for $M_{(u,s)} M^*_{(v,t)} f(x)$.

We define two additional operators, in terms of which we shall give estimates for
$M_{(u,s)} M^*_{(v,t)} f(x)$. For $(u, s) \in R^{n+1}_+$, we define

$$P_{(u,s)} f(x) = \frac{1}{s^n} \int_{|y| \le 2s} f(x - u - y)\, dy, \tag{6}$$

and

$$N_{(u,s)} f(x) = \frac{1}{s^{n-1}} \int_{|y| \le \alpha s} \frac{1}{\delta_s(y)} f(x - u - y)\, dy, \tag{7}$$

where α is chosen so that $\alpha \ge 5$ and $|u| \le \frac{\alpha}{5} s$ for each $(u, s) \in S$, and $\delta_s(y)$ is defined by

$$\delta_s(y) = \begin{cases} \operatorname{dist}(y, -S_s) & \text{if } S_s \ne \varnothing, \\ \infty & \text{otherwise.} \end{cases}$$

Lemma 1. *Suppose $n \geq 3$. Then for $(u, s), (v, t) \in S$ and for $f \in C_c(R^n)$ with $f \geq 0$, we have*

$$M_{(u,s)}M^*_{(v,t)}f(x) \leq c_1 P_{(u,s)}M^*_{(v,t)}f(x) + c_2 M_{(u,s)}P^*_{(v,t)}f(x) + c_3 N_{(u,s)}f(x) + c_4 N^*_{(v,t)}f(x),$$

where asterisks denote adjoints.

Remark. As will be shown later, the formula in the above lemma together with the L^2-boundedness of the maximal operators corresponding to the operators $P_{(u,s)}$ and $N_{(u,s)}$ implies the following inequality, which in turn yields our theorems:

$$\|M^* f\|_2^2 \leq c_1 \|M^* f\|_2 \|f\|_2 + c_2 \|f\|_2^2,$$

where M is a linearization of the maximal operator $\mathscr{M}$ corresponding to the function f.

To prove the above lemma we divide it into two parts, Lemma 2 and Lemma 3, whose proofs are based on the following formula concerning compositions of spherical mean operators (see [3]): if $f \in C(R^n)$ and $n \geq 3$,

$$M_s M_t f(x) = \frac{c(n)}{(st)^{n-2}} \int_{|t-s|}^{t+s} [(r^2 - (t-s)^2)((t+s)^2 - r^2)]^{\frac{n-3}{2}} r M_r f(x)\, dr \tag{8}$$

with $c(n) = 2^{4-n}\pi^{\frac{n-1}{2}} / \Gamma(\frac{n-1}{2})$.

Lemma 2. *Suppose $n \geq 3$, $f \in C_c(R^n)$, $f \geq 0$, and $(u, s), (v, t) \in R^{n+1}_+$. If $0 < 2s \leq t$,*

(a) $$M_{(u,s)}M^*_{(v,t)}f(x) \leq cP_{(u,s)}M^*_{(v,t)}f(x),$$

and if $0 < 2t \leq s$,

(b) $$M_{(u,s)}M^*_{(v,t)}f(x) \leq cM_{(u,s)}P^*_{(v,t)}f(x).$$

Proof. For $s, t > 0$, define $\psi_{s,t}$ by

$$\psi_{s,t}(x) = \begin{cases} s^{-n}t^{1-n}|x|^{1-n}[(|x|^2 - (t-s)^2)((t+s)^2 - |x|^2)]^{\frac{n-1}{2}} & \text{if } |t-s| \leq |x| \leq t+s, \\ 0 & \text{otherwise.} \end{cases}$$

If $0 < 2s \leq t$, then (8) implies

$$M_s M_t f(x) \leq c(\psi_{2s,t} * f)(x). \tag{9}$$

Next let φ be the characteristic function of the unit ball $\{y \in R^n : |y| \leq 1\}$, and set $\varphi_s(x) = s^{-n}\varphi(s^{-1}x)$ for $s > 0$. To get an estimate for $M_t\varphi_s(x)$ we assume first that $|t-s| \leq |x| \leq t+s$. Let A denote the intersection of the sphere $\{y \in R^n : |y| = t\}$ and the ball $\{y \in R^n : |y-x| \leq s\}$, and let D denote the $(n-1)$-dimensional disc in R^n whose boundary is the intersection of the sphere $\{y \in R^n : |y| = t\}$ and the sphere $\{y \in R^n : |y-x| = s\}$. Then

$$M_t\varphi_s(x) = c\,\frac{\text{area}(A)}{t^{n-1}s^n} \geq c\,\frac{\text{area}(D)}{t^{n-1}s^n} = c\,\frac{d^{n-1}}{t^{n-1}s^n},$$

where d is the radius of D; and an elementary calculation shows that $d^2 = (|x|^2-(t-s)^2)((t+s)^2-|x|^2)/4|x|^2$. Observe that $M_t\varphi_s(x)=0$ if $|x|>t+s$ or $|x|<|t-s|$. Hence if $0<2s\le t$,

$$M_t\varphi_{2s}(x) \ge c\psi_{2s,t}(x). \tag{10}$$

Therefore, (a) follows from (9) and (10), since

$$P_{(u,s)}M^*_{(v,t)}f(x) = c\,(M_t\varphi_{2s}*f)(x-u+v)$$

and

$$M_{(u,s)}M^*_{(v,t)}f(x) = M_sM_tf(x-u+v).$$

Similarly, (b) is obtained.

Lemma 3. *Suppose* $n\ge 3, f\in C_c(R^n), f\ge 0$, *and* $(u,s),(v,t)\in S$. *If* $0<t/2<s\le t$,

$$M_{(u,s)}M^*_{(v,t)}f(x) \le c\,N_{(u,s)}f(x), \tag{a}$$

and if $0<s/2<t\le s$,

$$M_{(u,s)}M^*_{(v,t)}f(x) \le c\,N^*_{(v,t)}f(x). \tag{b}$$

Proof. To prove (a) suppose $0<t/2<s\le t$. Then (8) implies

$$M_{(u,s)}M^*_{(v,t)}f(x) \le c\frac{1}{s^{n-1}}\int_{t-s\le|y+v|\le t+s}\frac{1}{|y+v|}f(x-u-y)\,dy. \tag{11}$$

One can show that there is a $w\in S_s$ such that $|w-v|\le c(t-s)$ for some constant c independent of s,t,v, and w. Hence $\delta_s(y)\le|y+w|\le|y+v|+|w-v|$ for every y. If $t-s\le|y+v|\le t+s$, then $|w-v|\le c\,(t-s)\le c\,|y+v|$, and so $\delta_s(y)\le c\,|y+v|$. Furthermore, $|y|\le\alpha s$ is derived from $|y+v|\le t+s$ and $0<t/2<s\le t$.

These observations together with (11) imply (a). Similarly, (b) is obtained.

We now define maximal operators corresponding to the operators $P_{(u,s)}$ and $N_{(u,s)}$, whose L^2-boundedness is essential for the L^2-boundedness of the maximal operator $\mathscr{M}$ defined in (5):

$$\mathscr{P}f(x) = \sup_{(u,s)\in S}|P_{(u,s)}f(x)|, \quad\text{and}\quad \mathscr{N}f(x) = \sup_{(u,s)\in S}|N_{(u,s)}f(x)|.$$

From the well-known boundedness of the Hardy-Littlewood maximal operator one derives

Lemma 4. $\mathscr{P}$ *is bounded on* $L^p(R^n)$ *for* $1<p\le\infty$.

The boundedness of $\mathscr{N}$ depends on the dimension of the parameter set S as is shown in the following lemmas.

Lemma 5. *If* $k = \dim S \le n-2$, *then* $\mathcal{N}$ *is bounded on* $L^p(R^n)$ *for* $(n+1-k)/(n-k) < p \le \infty$.

Lemma 6. *Suppose* $\dim S = n-1$ *and that there is an* $(n-1)$*-dimensional subspace* V *of* R^n *so that* $S_s - u \subset V$ *for every* $(u, s) \in S$. *Then* $\mathcal{N}$ *is bounded on* $L^p(R^n)$ *for* $1 < p \le \infty$.

Proof of Lemma 5. Suppose $1 < p, q < \infty$, $1/p + 1/q = 1$, and $(u, s) \in S$. Then, by Hölder's inequality,

$$|N_{(u,s)}f(x)| \le s^{1-\frac{n}{q}}\left[\int_{|y|\le\alpha s}(\delta_s(y))^{-q}\,dy\right]^{1/q}\left[s^{-n}\int_{|y|\le\alpha s}|f(x-u-y)|^p\,dy\right]^{1/p}.$$

But, since $|u| \le \frac{\alpha}{5}s \le \alpha s$,

$$\begin{aligned}\left[s^{-n}\int_{|y|\le\alpha s}|f(x-u-y)|^p\,dy\right]^{1/p} &\le \left[s^{-n}\int_{|y|\le 2\alpha s}|f(x-y)|^p\,dy\right]^{1/p}\\ &\le c\sup_{r>0}\left[r^{-n}\int_{|y|\le r}|f(x-u)|^p\,dy\right]^{1/p},\end{aligned}$$

and the maximal operator given by the last expression, as is well known, is bounded on $L^r(R^n)$ for $r > p$. So it suffices to show that for each q with $1 < q < n+1-k$ (i.e., $p > (n+1-k)/(n-k)$),

$$I = I_q = \int_{|y|\le\alpha s}(\delta_s(y))^{-q}\,dy \le c\,s^{n-q}.$$

Recall that the parameter set S was defined in the first section via a k-dimensional C^1-surface T and a differentiable function ρ. Examing the definition, one derives

$$|\{y \in R^n : \delta_s(y) < r\}| \le \begin{cases} c\,s^{k-1}r^{n+1-k} & \text{if } 0 < r \le \rho(s),\\ c\,r^n & \text{if } r > \rho(s).\end{cases}$$

Putting $\lambda(r) = \lambda_s(r) = |\{y \in R^n : |y| \le \alpha s,\ 1/\delta_s(y) > r\}|$ for $r > 0$, we then get

$$\lambda(r) \le \begin{cases} cs^{k-1}r^{k-1-n} & \text{if } r \ge \frac{1}{\rho(r)},\\ cr^{-n} & \text{if } \frac{1}{\alpha s} \le r < \frac{1}{\rho(s)},\\ cs^n & \text{if } 0 < r < \frac{1}{\alpha s}.\end{cases}$$

Note that $\lim_{r\to 0} r^q\lambda(r) = \lim_{r\to\infty} r^q\lambda(r) = 0$ for $1 < q < n+1-k$. Hence

$$\begin{aligned} I &= q\int_0^\infty r^{q-1}\lambda(r)\,dr\\ &\le cqs^n\int_0^{\frac{1}{\alpha s}} r^{q-1}\,dr + cq\int_{\frac{1}{\alpha s}}^{\frac{1}{\rho(s)}} r^{q-n-1}\,dr\\ &\quad + cqs^{k-1}\int_{\frac{1}{\rho(s)}}^\infty r^{q+k-n-2}\,dr.\end{aligned}$$

Since $0 < \rho(s) \leq cs$, it is easy to see that each of the above integrals is bounded by cs^{n-q}. The proof is thus completed.

Proof of Lemma 6. Choose a unit vector $w \in R^n$ normal to V, and represent points in R^n in the form $y = y' + y''w$ with $y' \in V$ and $y'' \in R^1$. By the hypothesis about the level sets of S, there is a function $h: (0, 1] \to R^1$ such that for every $(u, s) \in S$, $u'' = h(s)$. We define distance functions δ'_s on V by $\delta'_s(y') = \delta_s(y' - h(s)w)$.

Since $\delta_s(y) = \delta_s(y' + y''w) = \left[(y'' + h(s))^2 + (\delta'_s(y'))^2\right]^{1/2}$, there exists a constant $c_\theta > 0$ for each θ with $0 < \theta < 1$ such that

$$\delta_s(y) \geq c_\theta\, |y'' + h(s)|^{1-\theta} (\delta'_s(y'))^\theta .$$

Hence for $(u, s) \in S$ and for $0 < \theta < 1$,

$$(12)\quad |N_{(u,s)} f(x)| \leq c\, s^{n-1} \int_{|y'| \leq \alpha s,\, y' \in V} (\delta'_s(y'))^{-\theta} \times \left[\int_{|y''| \leq \alpha s} |y'' + h(s)|^{\theta - 1} |f(x - u - y)|\, dy''\right] dy'.$$

Putting

$$\mathscr{H}^w f(x) = \sup_{t>0} \left(\frac{1}{t} \int_{-t}^{t} |f(x - rw)|\, dr\right),$$

one can show

$$(13)\qquad \int_{|y''| \leq \alpha s} |y'' + h(s)|^{\theta - 1} |f(x - u - y)|\, dy'' \leq c\, s^\theta \mathscr{H}^w f(x - u' - y').$$

For each $p > 1$, we now put

$$\mathscr{H}_p^V f(x) = \sup_{t>0} \left(\frac{1}{t^{n-1}} \int_{|y'| \leq t,\, y' \in V} |f(x - y')|^p\, dy'\right)^{1/p}.$$

If $1 < p, q < \infty$ and $1/p + 1/q = 1$, then (12) and (13) together with Hölder's inequality imply

$$|N_{(u,s)} f(x)| \leq c\, s^{\theta - \frac{n-1}{q}} \left[\int_{|y'| \leq \alpha s,\, y' \in V} (\delta'_s(y'))^{-\theta q}\, dy'\right]^{1/q} \mathscr{H}_p^V \mathscr{H}^w f(x).$$

Note that $\mathscr{H}_p^V$ and $\mathscr{H}^w$ are variants of the Hardy-Littlewood maximal operators, and so one can easily show that the composition $\mathscr{H}_p^V \mathscr{H}^w$ is bounded on $L^r(R^n)$ for $r > p$. It now suffices to show that for each q with $1 < q < \infty$ there is a θ with $0 < \theta < 1$ such that

$$I = I_q = \int_{|y'| \leq \alpha s,\, y' \in V} (\delta'_s(y'))^{-\theta q}\, dy' \leq c\, s^{n-1-\theta q}.$$

Let $\lambda(r) = |\{y' \in V : |y'| \leq \alpha s, 1/\delta_s'(y') > r\}|_{n-1}$ for $r > 0$, where $|\cdot|_{n-1}$ denotes the Lebesgue measure on V. As in the proof of Lemma 5, we then get

$$\lambda(r) \leq \begin{cases} cs^{n-2}r^{-1} & \text{if } r \geq \frac{1}{\rho(s)}, \\ cr^{-n+1} & \text{if } \frac{1}{\alpha s} \leq r < \frac{1}{\rho(s)}, \\ cs^{n-1} & \text{if } 0 < r < \frac{1}{\alpha s}. \end{cases} \tag{14}$$

Suppose $1 < q < \infty$ and choose θ such that $0 < \theta < 1$ and $0 < \theta q < 1$. Then $\lim_{r\to 0} r^{\theta q}\lambda(r) = \lim_{r\to\infty} r^{\theta q}\lambda(r) = 0$, and so

$$I = \theta q \int_0^\infty r^{\theta q - 1}\lambda(r)\,dr.$$

As in the proof of Lemma 5, (14) now implies $I \leq cs^{n-1-\theta q}$. Thus the proof is completed.

3. Proofs of theorems

We shall derive a lemma from the previous lemmas, from which the proofs of our theorems follow immediately.

Lemma 7. *Suppose that S is a parameter set satisfying the hypothesis of either Theorem 1 or Theorem 2. If $A_1, \dots, A_m$ are disjoint Borel cubes in R^n, and ξ is a function from $\{1, \dots, m\}$ into S, and if $Mf(x) = \sum_{i=1}^m \chi_{A_i}(x)M_{\xi(i)}f(x)$, then there exists a positive constant c independent of the cubes A_i, the integer m, and the function ξ such that $\|Mf\|_2 \leq c\|f\|_2$ for every $f \in C_c(R^n)$.*

Proof. Following Paley's method, define operators P and N by

$$Pf(x) = \sum_{i=1}^m \chi_{A_i}(x)P_{\xi(i)}f(x)$$

and

$$Nf(x) = \sum_{i=1}^m \chi_{A_i}(x)N_{\xi(i)}f(x).$$

It follows from Lemma 1 that for every $f \in C_c(R^n)$ with $f \geq 0$,

$$\begin{aligned} MM^*_{(v,t)}f(x) \leq c_1 PM^*_{(v,t)}f(x) + c_2 MP^*_{(v,t)}f(x) \\ + c_3 Nf(x) + c_4 N^*_{(v,t)}f(x). \end{aligned} \tag{15}$$

Integrating each term in (15) against an arbitary nonnegative function in $C_c(R^n)$, we get the following inequality for $f \in C_c(R^n)$ with $f \geq 0$:

$$\begin{aligned} M_{(v,t)}M^*f(x) \leq c_1 M_{(v,t)}P^*f(x) + c_2 P_{(v,t)}M^*f(x) \\ + c_3 N^*f(x) + c_4 N_{(v,t)}f(x), \end{aligned}$$

which, in turn, implies

$$MM^*f(x) \leq c_1 MP^*f(x) + c_2 PM^*f(x) + c_3N^*f(x) + c_4Nf(x).$$

Hence,

$$\begin{aligned}
\|M^* f\|_2^2 &= \int M^* f(x)\overline{M^* f(x)}\, dx = \int MM^* f(x) f(x)\, dx \\
&\le c_1 \int MP^* f(x) f(x)\, dx + c_2 \int PM^* f(x) f(x)\, dx \\
&\quad + c_3 \int N^* f(x) f(x)\, dx + c_4 \int Nf(x) f(x)\, dx \\
&= c_5 \int PM^* f(x) f(x)\, dx + c_6 \int Nf(x) f(x)\, dx \\
&\le c_5 \int \mathscr{P} M^* f(x) f(x)\, dx + c_6 \int \mathscr{N} f(x) f(x)\, dx \\
&\le c_5 \|\mathscr{P} M^* f\|_2 \|f\|_2 + c_6 \|\mathscr{N} f\|_2 \|f\|_2.
\end{aligned}$$

The boundedness of $\mathscr{P}$ and $\mathscr{N}$, established in Lemma 4, 5, and 6, now imply

$$\|M^* f\|_2^2 \le c_7 \|M^* f\|_2 \|f\|_2 + c_8 \|f\|_2^2,$$

and so $\|M^* f\|_2 \le c\, \|f\|_2$ with $c = (c_7 + \sqrt{c_7^2 - 4c_8})/2$.

Examining the process, one sees that the coefficient c does not depend on the cubes A_i, the integer m, and the function ξ. The proof is now completed by the fact that M and M^* have the same norm.

We now prove Theorem 1 and Theorem 2 in one stroke. Let $\mathscr{C}$ be the family of cubes in R^n of the form $[a_1 2^{p_1}, b_1 2^{q_1}) \times \cdots \times [a_n 2^{p_n}, b_n 2^{q_n})$, where a_i, b_i, p_i, and q_i are all integers.

Fix a function $f \in C_c(R^n)$ with $f \ge 0$. If $\mathscr{M} f(x) \ne 0$, one can find a point $(u, s) \in S$ such that $\mathscr{M} f(x) < 2M_{(u,s)} f(x)$, and so can find a cube $Q \in \mathscr{C}$ such that $x \in Q$ and $\mathscr{M} f(y) \le 2M_{(u,s)} f(y)$ for every $y \in Q$ since $M_{(u,s)} f$ is continuous. Hence, one obtains a disjoint countable subfamily $\{Q_i\}_{i=1}^{\infty}$ of $\mathscr{C}$ and a countable subset $\{(u_i, s_i)\}_{i=1}^{\infty}$ of S such that

$$\mathscr{M} f(x) \le 2 \sum_{i=1}^{\infty} \chi_{Q_i}(x) M_{(u_i, s_i)} f(x)$$

for every $x \in R^n$. Put

$$Mf(x) = \sum_{i=1}^{\infty} \chi_{Q_i}(x) M_{(u_i, s_i)} f(x),$$

and

$$M^{(m)} f(x) = \sum_{i=1}^{m} \chi_{Q_i}(x) M_{(u_i, s_i)} f(x)$$

for $m = 1, 2, \ldots$. Since $Mf, M^{(m)}f \geq 0$ and $M^{(m)}f(x)$ increases to $Mf(x)$ as $m \to \infty$, by Fatou's lemma we have

$$\|Mf\|_2 = \sup_{g\geq 0, \|g\|_2=1} \int Mf(x)g(x)\,dx$$
$$\leq \sup_{g\geq 0, \|g\|_2=1} \left(\liminf_{m\to\infty} \int M^{(m)}f(x)g(x)dx\right).$$

Since $|\int M^{(m)}f(x)g(x)\,dx| \leq \|M^{(m)}f\|_2\|g\|_2$, Lemma 7 now implies that $\|Mf\|_2$
$\leq c\,\|f\|_2$ for some constant c depending only on the parameter set S, which together with the fact $\mathscr{M}f(x) \leq 2Mf(x)$, in turn, imply $\|\mathscr{M}f\|_2 \leq c\,\|f\|_2$. For general $f \in C_c(R^n)$,

$$\|\mathscr{M}f\|_2 \leq \|\mathscr{M}|f|\,\|_2 \leq c\,\|\,|f|\,\|_2 = c\|f\|_2.$$

A measure-theoretic argument similar to that in [9] extends this result to $L^2(R^n)$, and thus completes the proof.

From the theorems the following are immediately derived.

Corollary. *Let S be a parameter set in R^{n+1}. If it satisfies the hypothesis of either Theorem 1 or Theorem 2, and if $f \in L^2_{\mathrm{loc}}(R^n)$ and $n \geq 3$, then*

$$\lim_{\substack{(v,t)\to(0,0)\\(v,t)\in S}} M_{(v,t)}f(x) = f(x)$$

for almost every $x \in R^n$.

Corollary. *Let S be a parameter set in R^4_+ satisfying the hypothesis of either Theorem 1 or Theorem 2 with $n = 3$. If u is the solution of the wave equation $\Delta u = u_{tt}$ in R^3 with the initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = f(x)$ for some $f \in L^2_{\mathrm{loc}}(R^3)$, then*

$$\lim_{\substack{(v,t)\to(0,0)\\(v,t)\in S}} \frac{u(x-v, t) - u(x, 0)}{t} = f(x)$$

for almost every $x \in R^3$.

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