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CONDITIONAL YEH-WIENER INTEGRALS WITH VECTOR-VALUED CONDITIONING FUNCTIONS

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ABSTRACT. In this paper we establish various results involving conditional Yeh-Wiener integrals with vector-valued conditioning functions. We first develop a very simple formula for converting conditional Yeh-Wiener integrals into ordinary (i.e., nonconditional) Yeh-Wiener integrals. We then use this simple formula to evaluate the conditional Yeh-Wiener integral of various functionals, including functionals in which the sectional average of an L_2-function occurs naturally. Finally, we use these results to obtain a general Cameron-Martin type translation theorem for conditional Yeh-Wiener integrals.

1. INTRODUCTION

For $Q = [0, S] \times [0, T]$ let $C(Q)$ denote Yeh-Wiener space, i.e., the space of all real-valued continuous functions $x(s, t)$ on Q such that $x(0, t) = x(s, 0) = 0$ for every (s, t) in Q. Yeh [10] defined a Gaussian measure m_y on $C(Q)$ such that as a stochastic process $\{x(s, t), (s, t) \in Q\}$ has mean $E[x(s, t)] = \int_{C(Q)} x(s, t) m_y(dx) = 0$ for every (s, t) in Q and covariance $E[x(s, t)x(u, v)] = \min\{s, u\} \min\{t, v\}$. This process is called the standard Yeh-Wiener (or two-time parameter Wiener) process on Q.

For each partition $\tau = \tau_{m,n} = \{(s_i, t_j), i = 1, \ldots, m, u = 1, \ldots, n\}$ of Q with $0 = s_0 < s_1 < \cdots < s_m = S$ and $0 = t_0 < t_1 < \cdots < t_n = T$, define $X_\tau: C(Q) \to \mathbf{R}^{mn}$ by $X_\tau(x) = (x(s_1, t_1), \ldots, x(s_m, t_n))$. Let $\mathscr{B}^{mn}$ be the σ-algebra of Borel sets in $\mathbf{R}^{mn}$. A Yeh-Wiener interval is a set of the type

$$I = \{x \in C(Q): X_\tau(x) \in B\} = X_\tau^{-1}(B), \; B \in \mathscr{B}^{mn}.$$

The Yeh-Wiener measure m_y of such a set is given by

$$m_y(I) = \int_B K(\tau, \vec{\xi})\, d\vec{\xi}, \tag{1.1}$$

where

$$\vec{\xi} = (\xi_{1,1}, \ldots, \xi_{m,n}) \in \mathbf{R}^{mn}$$

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and

$$K(\tau, \vec{\xi}) = (2\pi)^{-mn/2}\left[\prod_{i=1}^{m}(s_i - s_{i-1})\right]^{-n/2}\left[\prod_{j=1}^{n}(t_j - t_{j-1})\right]^{-m/2}$$
$$\cdot \exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}(\xi_{i,j} - \xi_{i-1,j} - \xi_{i,j-1} + \xi_{i-1,j-1})^2/(s_i - s_{i-1})(t_j - t_{j-1})\right\},$$

with $\xi_{i,0} = \xi_{0,j} = 0$.

Let $\mathscr{F}_\tau$ be the σ-algebra generated by the sets $\left\{X_\tau^{-1}(B): B \in \mathscr{B}^{mn}\right\}$ with $\tau = \tau_{m,n}$ fixed. Then, by the definition of conditional expectation (see Tucker [9] and Yeh [13]), for each Yeh-Wiener integrable function $F(x)$

$$\mu_\tau(B) \equiv \int_{X_\tau^{-1}(B)} F(x)m_y(dx) = \int_{X_\tau^{-1}(B)} E(F|\mathscr{F}_\tau)m_y(dx) \tag{1.2}$$
$$= \int_B E(F(x)|X_\tau(x) = \vec{\xi})P_{X_\tau}(d\vec{\xi}), \qquad B \in \mathscr{B}^{mn},$$

where $P_{X_\tau}(B) = m_y(X_\tau^{-1}(B))$, and $E(F(x)|X_\tau(x) = \vec{\xi})$ is a Borel measurable function of $\vec{\xi}$ which is unique up to Borel null sets in $\mathbf{R}^{mn}$. Since $X_\tau^{-1}(B)$ is a Yeh-Wiener interval, it follows from (1.1) that

$$P_{X_\tau}(d\vec{\xi}) = K(\tau, \vec{\xi})\, d\vec{\xi}. \tag{1.3}$$

Conditional expectations of the type $E(F(x)|X_\tau(x) = \vec{\xi})$ are often called conditional Yeh-Wiener integrals, and have been studied extensively by many authors (see [2], [3], [4] and further references in these papers). Some results are obtained by Chang and Ahn [2] and Chung and Ahn [4] for conditional Yeh-Wiener integrals when $X_\tau(x) = x(S, T)$, i.e., when $m = n = 1$.

The main purpose of this paper is to develop a useful, but rather simple formula to convert conditional Yeh-Wiener integrals into ordinary (i.e., non-conditional) Yeh-Wiener integrals, and then to use this formula to obtain a general Cameron-Martin Translation Theorem for conditional Yeh-Wiener integrals when the conditioning function is vector-valued. We also use this formula to evaluate the conditional Yeh-Wiener integral of various functions. The sectional average of an L_2-function proves useful when we consider functions of the form $F(\int_Q h_1\, dx, \dots, \int_Q h_N\, dx)$ where each $\int_Q h_i\, dx$ is a stochastic integral with mean zero and variance $\|h_i\|_2^2$.

2. A useful but simple formula for conditional Yeh-Wiener integrals

For each partition $\tau = \tau_{m,n}$ of Q and $x \in C(Q)$, define the quasi-polyhedric function $[x]$ on Q by

$$\begin{aligned}[x](s,t) = x(s_{i-1},t_{j-1}) &+ [(s-s_{i-1})(t-t_{j-1})/(\Delta_i s\Delta_j t)]\Delta_{ij}x(s,t) \\ &+ [(s-s_{i-1})/\Delta_i s](x(s_i,t_{j-1}) - x(s_{i-1},t_{j-1})) \\ &+ [(t-t_{j-1})/\Delta_j t](x(s_{i-1},t_j) - x(s_{i-1},t_{j-1}))\end{aligned} \tag{2.1}$$

on each $Q_{ij} = (s_{i-1},s_i] \times (t_{j-1},t_j]$, $i = 1,\dots,m$, $j = 1,\dots,n$ where $\Delta_i s = s_i - s_{i-1}$, $\Delta_j t = t_j - t_{j-1}$, $\Delta_{ij}x(s,t) = x(s_i,t_j) - x(s_{i-1},t_j) - x(s_i,t_{j-1}) + x(s_{i-1},t_{j-1})$ and $[x](s,t) = 0$ if $st = 0$.

Similarly, for each $\vec{\xi} = (\xi_{1,1},\dots,\xi_{m,n}) \in \mathbf{R}^{mn}$, define the quasi-polyhedric function $[\vec{\xi}]$ on Q by

$$\begin{aligned}[\vec{\xi}](s,t) = \xi_{i-1,j-1} &+ [(s-s_{i-1})(t-t_{j-1})/(\Delta_i s\Delta_j t)]\Delta_{ij}\vec{\xi} \\ &+ [(s-s_{i-1})/\Delta_i s](\xi_{i,j-1} - \xi_{i-1,j-1}) \\ &+ [(t-t_{j-1})/\Delta_j t](\xi_{i-1,j} - \xi_{i-1,j-1})\end{aligned} \tag{2.2}$$

on each Q_{ij} where $\Delta_{ij}\vec{\xi} = \xi_{i,j} - \xi_{i-1,j} - \xi_{i,j-1} + \xi_{i-1,j-1}$, $\xi_{0,j} = \xi_{i,0} = 0$ for all i and j, and $[\vec{\xi}](s,t) = 0$ if $st = 0$.

We note that both $[x]$ and $[\vec{\xi}]$ belong to $C(Q)$ for each x in $C(Q)$ and each $\vec{\xi}$ in $\mathbf{R}^{mn}$. In addition for all i and j, $[x](s_i,t_j) = x(s_i,t_j)$ and $[\vec{\xi}](s_i,t_j) = \xi_{i,j}$. Also on each Q_{ij}, both $[x](s,t)$ and $[\vec{\xi}](s,t)$ are quadratic functions of two variables and each is a linear function of one variable for each value of the other variable. The functions $[x]$ with $m = n$ and binary partition points were used as binary quadratic approximations in [6] and [7].

The following result plays a key role in this paper.

Theorem 1. *If* $\{x(s,t), (s,t) \in Q\}$ *is the standard Yeh-Wiener process, then the process* $\{x(s,t) - [x](s,t), (s,t) \in Q\}$ *and* $X_\tau(x) = (x(s_1,t_1),\dots,x(s_m,t_n))$ *are (stochastically) independent.*

Proof. Using the expression (2.1), we may write for (s,t) in Q_{ij}

$$\begin{aligned}x(s,t) - [x](s,t) = x(s,t) &- x(s_{i-1},t_{j-1}) \\ &- [(s-s_i)(t-t_j)/\Delta_i s\Delta_j t)]\Delta_{ij}x(s,t) \\ &- [(s-s_i)/\Delta_i s](x(s_i,t_{j-1}) - x(s_{i-1},t_{j-1})) \\ &- [(t-t_j)/\Delta_j t](x(s_{i-1},t_j) - x(s_{i-1},t_{j-1})).\end{aligned} \tag{2.3}$$

Thus, it suffices to show that (2.3) is independent of $x(s_p,t_q)$ for $p = 1,\dots,m$ and $q = 1,\dots,n$. In view of equation (2.3) and repeated use of the formula

$$E[x(s,t)x(u,v)] = \min\{s,u\}\min\{t,v\},$$

a straghtforward computation yields

$$E[x(s_p, t_q)\{x(s,t) - [x](s,t)\}] = 0.$$

Because both $x(s_p, t_q)$ and $x(s,t) - [x](s,t)$ are Gaussian and uncorrelated, we may conclude that they are independent.

The following corollary is an immediate consequence of Theorem 1.

Corollary 1.1. *The two processes* $\{x(s,t) - [x](s,t), (s,t) \in Q\}$ *and* $\{[x](s,t), (s,t) \in Q\}$ *are independent.*

The following theorem plays an important role in the development of a simple formula for conditional Yeh-Wiener integrals.

Theorem 2. *Let* $F \in L_1(C(Q), m_y)$. *Then for every* $B \in \mathscr{B}^{mn}$,

$$\int_{X_\tau^{-1}(B)} F(x) m_y(dx) = \int_B E[F(x - [x] + [\vec{\xi}])] P_{X_\tau}(d\vec{\xi}). \tag{2.4}$$

Proof. We first consider the case when F is the indicator function of a Yeh-Wiener measurable set M, say $F(x) = I_M(x)$. Then

$$\begin{aligned}\int_{X_\tau^{-1}(B)} I_M(x) m_y(dx) &= m_y(M \cap X_\tau^{-1}(B)) \\ &= \int_B m_y(x \in M | X_\tau(x) = \vec{\xi}) P_{X_\tau}(d\vec{\xi}) \\ &= \int_B m_y(x - [x] + [\vec{\xi}] \in M | X_\tau(x) = \vec{\xi}) P_{X_\tau}(d\vec{\xi}).\end{aligned}$$

But, as was shown in Theorem 1, $x - [x]$ and $X_\tau(x)$ are independent, and thus so are $x - [x] + [\vec{\xi}]$ and $X_\tau(x)$. Therefore

$$\begin{aligned}\int_{X_\tau^{-1}(B)} I_M(x) m_y(dx) &= \int_B m_y(x - [x] + [\vec{\xi}] \in M) P_{X_\tau}(d\vec{\xi}) \\ &= \int_B E[I_M(x - [x] + [\vec{\xi}])] P_{X_\tau}(d\vec{\xi}).\end{aligned}$$

Thus, (2.4) holds for the indicator function of any Yeh-Wiener measurable set. The general case follows by the usual arguments in integration theory.

From (1.2) and (2.4) we may conclude that for F in $L_1(C(Q), m_y)$, $E(F(x)|X_\tau(x) = \vec{\xi})$ and $E[F(x - [x] + [\vec{\xi}])]$ are equal for a.e. $\vec{\xi}$ in $\mathbf{R}^{mn}$. But while the former is Borel measurable by definition, the latter may only be Lebesgue measurable as the following example shows.

Example. Let A be a Lebesgue measurable null set in $\mathbf{R}^{mn}$ that is not Borel measurable and let $M = \{x \in C(Q): X_\tau(x) \in A\}$. Then $m_y(M) = 0$. Let $F(x) = I_M(x)$. Then $F(x) = I_A(X_\tau(x))$ and

$$\begin{aligned}E[F(x - [x] + [\vec{\xi}])] &= E[I_M(x - [x] + [\vec{\xi}])] \\ &= E[I_A(X_\tau(x - [x] + [\vec{\xi}]))] = E[I_A(\vec{\xi})] = I_A(\vec{\xi})\end{aligned}$$

which is Lebesgue but not Borel measurable on $\mathbf{R}^{mn}$.

But if $f(\vec{\xi})$ is Lebesgue measurable on $\mathbf{R}^{mn}$, then there exists a Borel measurable function $\hat{f}(\vec{\xi})$, which is unique up to Borel null sets, such that $\hat{f}(\vec{\xi}) = f(\vec{\xi})$ a.e. on $\mathbf{R}^{mn}$. Thus we may define:

Definition 1. If F is Yeh-Wiener integrable on $C(Q)$, then by

$$\widehat{E}[F(x - [x] + [\vec{\xi}])]$$

we mean any Borel measurable function of $\vec{\xi}$ which is equal to

$$E[F(x - [x] + [\vec{\xi}])$$

for a.e. $\vec{\xi}$ in $\mathbf{R}^{mn}$.

Thus we have the following useful formula which is quite simple to apply in various applications.

Theorem 3. *Let* $F \in L_1(C(Q), m_y)$. *Then*

$$E(F(x)|X_\tau(x) = \vec{\xi}) = \widehat{E}[F(x - [x] + [\vec{\xi}])]. \tag{2.5}$$

In particular, if F *is Borel measurable, then*

$$E(F(x)|X_\tau(x) = \vec{\xi}) = E[F(x - [x] + [\vec{\xi}])]. \tag{2.6}$$

The equalities in (2.5) *and* (2.6) *mean that both sides are Borel measurable functions of* $\vec{\xi}$ *and they are equal except for Borel null sets.*

3. Three examples

The following examples demonstrate that Theorem 3 is very useful and greatly simplies the calculations of conditional Yeh-Wiener integrals as compared to previous methods.

Example 1. For x in $C(Q)$ let $F(x) = \int_Q x(s,t)dsdt = \int_0^T \int_0^S x(s,t)dsdt$. Then by (2.6) and (2.2)

$$\begin{aligned} E\left(\int_Q x(s,t)dsdt|X_\tau(x) = \vec{\xi}\right) &= E\left[\int_Q (x(s,t) - [x](s,t) + [\vec{\xi}](s,t))dsdt\right] \\ &= \int_Q E[x(s,t) - [x](s,t) + [\vec{\xi}](s,t)]dsdt = \int_Q [\vec{\xi}](s,t)dsdt \\ &= \sum_{i=1}^m \sum_{j=1}^n \frac{\Delta_i s \Delta_j t}{4} \left\{\xi_{i,j} + \xi_{i-1,j} + \xi_{i,j-1} + \xi_{i-1,j-1}\right\}. \end{aligned} \tag{3.1}$$

In particular, if $m = n = 1$, then

$$E\left(\int_0^T \int_0^S x(s,t)dsdt|X(S,T) = \xi\right) = \frac{1}{4}\xi ST. \tag{3.2}$$

The conditional Yeh-Wiener integral in (3.2) was computed by Chung and Ahn [4] by a different method, and the computation was rather lengthy.

Example 2. For $x \in C(Q)$ let $F(x) = \int_Q x^2(s,t)dsdt$. Then by (2.6)

$$E\left(\int_Q x^2(s,t)dsdt | X_\tau(x) = \vec{\xi}\right) = E\left[\int_Q (x(s,t) - [x](s,t) + [\vec{\xi}](s,t))^2 dsdt\right]$$

$$= \int_Q E[(x(s,t) - [x](s,t))^2 + ([\vec{\xi}](s,t))^2 - 2[\vec{\xi}](s,t)(x(s,t) - [x](s,t))]dsdt.$$

As $x - [x]$ and $[x]$ are independent on Q by Corollary 1.1,

$$E[(x(s,t) - [x](s,t))[x](s,t)] = 0,$$

and hence by the use of (2.1) with the formula

$$E[x(s,t)x(u,v)] = \min\{s,u\}\min\{t,v\},$$

$$\int_Q E[(x(s,t) - [x](x,t))]^2 dsdt = \int_Q E[x(s,t)(x(s,t) - [x](s,t))]dsdt$$

$$= \int_Q stdsdt - \sum_{i=1}^m \sum_{j=1}^n \int_{Q_{ij}} \left\{ s_{i-1}t_{j-1} + \frac{(s - s_{i-1})^2(t - t_{j-1})^2}{\Delta_i s \Delta_j t} + \frac{t_{j-1}(s - s_{i-1})^2}{\Delta_i s} + \frac{s_{i-1}(t - t_{j-1})^2}{\Delta_j t} \right\} dsdt$$

$$= \frac{S^2T^2}{4} - \sum_{i=1}^m \sum_{j=1}^n \left\{ s_{i-1}t_{j-1}\Delta_i s \Delta_j t + \frac{1}{9}(\Delta_i s)^2(\Delta_j t)^2 + \frac{1}{3}t_{j-1}(\Delta_i s)^2 \Delta_j t + \frac{1}{3}s_{i-1}\Delta_i s(\Delta_j t)^2 \right\}$$

$$= \frac{S^2T^2}{4} - \frac{1}{9}\left\{\sum_{i=1}^m \Delta_i s(s_i + 2s_{i-1})\right\}\left\{\sum_{j=1}^n \Delta_j t(t_j + 2t_{j-1})\right\}.$$

Thus,

$$E\left(\int_Q x^2(s,t)dsdt | X_\tau(x) = \vec{\xi}\right) = \frac{S^2T^2}{4} + \int_Q ([\vec{\xi}](s,t))^2 dsdt - \frac{1}{9}\left\{\sum_{i=1}^m \Delta_i s(s_i + 2s_{i-1})\right\}\left\{\sum_{j=1}^n \Delta_j t(t_j + 2t_{j-1})\right\}. \tag{3.3}$$

In particular, if $m = n = 1$, then

$$E\left(\int_0^T \int_0^S x^2(s,t)dsdt | x(S,T) = \xi\right) = \frac{S^2T^2}{4} + \int_Q \left(\frac{st\xi}{ST}\right)^2 dsdt - \frac{1}{9}S^2T^2 = \frac{5}{36}(ST)^2 + \frac{1}{9}ST\xi^2 \tag{3.4}$$

which agrees with Chung and Ahn's computation in [4].

Example 3. For $x \in C(Q)$ let $F(x) = \exp\{\int_Q x(s,t)dsdt\}$. Then using (2.6) we see that

$$\begin{aligned}
&E\left(\exp\left\{\int_Q x(s,t)dsdt\right\}\Big| X_\tau(x) = \vec{\xi}\right) \\
&\quad = E\left[\exp\left\{\int_Q (x(s,t) - [x](s,t) + [\vec{\xi}](s,t))dsdt\right\}\right] \\
&\quad = \exp\left\{\int_Q [\vec{\xi}](s,t)dsdt\right\} E\left[\exp\left\{\int_Q (x(s,t) - [x](s,t)dsdt\right\}\right].
\end{aligned}$$

In particular, for each fixed $u \in C(Q)$,

$$\begin{aligned}
&E\left(\exp\left\{\int_Q x(s,t)dsdt\right\}\Big| X_\tau(x) = X_\tau(u)\right) \\
&\quad = \exp\left\{\int_Q [u](s,t)dsdt\right\} E\left[\exp\left\{\int_Q (x(s,t) - [x](s,t))dsdt\right\}\right].
\end{aligned}$$

Thus

$$\lim_{\|\tau\|\to 0} E\left(\exp\left\{\int_Q x(s,t)dsdt\right\}\Big| X_\tau(x) = X_\tau(u)\right) = \exp\left\{\int_Q u(s,t)dsdt\right\}$$

as expected.

4. Conditional expectation of functions involving stochastic integrals

In the next section we will be translating $x(s,t)$ by the function $x_0(s,t) = \int_0^t \int_0^s h(u,v)dudv$ for fixed $h \in L_2(Q)$. It turns out that using $\tilde{h}$, the sectional average of h, simplies both the statement and the proof of the conditional translation theorem.

Definition 2. Let $\tau = \{(s_i, t_j),\ i = 1, \dots, m \text{ and } j = 1, \dots, n\}$ be a partition of Q. Then for each function $h \in L_2(Q)$, we define the sectional average of h by

$$\tilde{h}(s,t) = \frac{1}{\Delta_i s \Delta_j t} \int_{Q_{ij}} h(u,v)dudv$$

on each $Q_{ij} = (s_{i-1}, s_i] \times (t_{j-1}, t_j]$ and $\tilde{h}(s,t) = 0$ if $st = 0$.

The following theorem gives an interesting relationship involving $\tilde{h}$ and $[x]$ that is very useful in computing conditional expectations.

Theorem 4. *Let* $h \in L_2(Q)$. *Then*

(4.1) $$\int_Q h(s,t)\tilde{h}(s,t)dsdt = \int_Q \tilde{h}^2(s,t)dsdt,$$

(4.2) $$\|h - \tilde{h}\|_2^2 = \|h\|_2^2 - \|\tilde{h}\|_2^2 \geq 0, \text{ and}$$

(4.3) $$\int_Q hd[x] = \int_Q \tilde{h}dx = \int_Q \tilde{h}d[x] \quad \text{for each } x \in C(Q).$$

Proof. (4.1) follows easily from the definition of $\tilde{h}$ while (4.2) follows from (4.1). We note that each of the integrals in (4.3) exists since $\tilde{h}$ is of bounded variation on Q and $[x]$ is absolutely continuous on Q with $\partial^2[x](s,t)/\partial s\partial t$ in $L_2(Q)$. Next, using (2.1) we see that for each Q_{ij},

$$\int_{Q_{ij}} hd[x] = \frac{\Delta_{ij}x(s,t)}{\Delta_i s\Delta_j t}\int_{Q_{ij}} h(s,t)dsdt = \Delta_{ij}x(s,t)\tilde{h}(s_i,t_j)$$

from which (4.3) follows immediately.

Theorem 5. *Let* $h_1, \dots, h_N \in L_2(Q)$ *and let* $g_j = h_j - \tilde{h}_j$ *for* $j = 1, \dots, N$. *If* $F(\int_Q h_1\,dx, \dots, \int_Q h_N\,dx)$ *is Yeh-Wiener integrable, then*

$$(4.4)\quad E\left(F\left(\int_Q g_1\,dx, \dots, \int_Q g_N\,dx\right)\middle| X_\tau(x) = \vec{\xi}\right) = E\left[F\left(\int_Q g_1\,dx, \dots, \int_Q g_N\,dx\right)\right].$$

Furthermore, if $\{\alpha_1, \dots, \alpha_k\}$ *is a set of orthogonal functions on* Q *with* $\text{span}\{\alpha_1, \dots, \alpha_k\} = \text{span}\{g_1, \dots, g_N\}$, *then there exists a function* G *such that*

$$(4.5)\quad F\left(\int_Q g_1\,dx, \dots, \int_Q g_N\,dx\right) = G\left(\int_Q \alpha_1\,dx, \dots, \int_Q \alpha_k\,dx\right),$$

and so

(4.6)

$$E\left(F\left(\int_Q g_1\,dx, \dots, \int_Q g_N\,dx\right)\middle| X_\tau(x) = \vec{\xi}\right) = \prod_{j=1}^{k}(2\pi\|\alpha_j\|^2)^{-1/2}\int_{\mathbf{R}^k} G(v_1, \dots, v_k)\exp\left\{-\frac{1}{2}\sum_{j=1}^{k}\frac{v_j^2}{\|\alpha_j\|^2}\right\} dv_1\dots dv_k.$$

Proof. By Theorem 3,

$$E\left(F\left(\int_Q g_1\,dx, \dots, \int_Q g_N\,dx\right)\middle| X_\tau(x) = \vec{\xi}\right) = \widehat{E}\left[F\left(\int_Q g_1\,d(x-[x]+[\vec{\xi}]), \dots, \int_Q g_N d(x-[x]+[\vec{\xi}])\right)\right].$$

But, by use of (4.3),

$$\int_Q g_j d(x-[x]+[\vec{\xi}]) = \int_Q g_j\,dx - \int_Q g_j d[x] + \int_Q g_j d[\vec{\xi}] = \int_Q g_j\,dx - \int_Q \tilde{g}_j\,dx + \int_Q \tilde{g}_j d[\vec{\xi}].$$

However $g_j = h_j - \tilde{h}_j$ and so $\tilde{g}_j = \tilde{h}_j - \tilde{h}_j = 0$. Thus

$$\int_Q g_j d(x-[x]+[\vec{\xi}]) = \int_Q g_j\,dx,$$

from which (4.4) readily follows. Equations (4.5) and (4.6) are obvious.

Corollary 5.1. *Let $h_1, \dots, h_N \in L_2(Q)$ satisfy the condition $\int_Q h_j(s, t)dsdt = 0$ for $j = 1, \dots, N$. Then*

$$E\left(F\left(\int_Q h_1\, dx, \dots, \int_Q h_N\, dx\right) \middle| x(S, T) = \xi \right) = E\left[F\left(\int_Q h_1\, dx, \dots, \int_Q h_N\, dx\right)\right]$$

which can be evaluated using (4.6).

Proof. Since τ consists of the single point (S, T), it follows that $\tilde{h}_j(s, t) = \frac{1}{ST}\int_Q h_j(u, v)dudv = 0$ for $j = 1, \dots, N$. Hence $g_j = h_j$ for all j, and so the result follows from Theorem 5.

Theorem 6. *Let $h \in L_2(Q)$ and assume that $F(\int_Q h\, dx)$ is Yeh-Wiener integrable. Then*

$$E\left(F\left(\int_Q h\, dx\right) \middle| X_\tau(x) = \vec{\xi} \right) = \frac{1}{\sqrt{2\pi}\|h - \tilde{h}\|}\int_{-\infty}^{\infty} F(v)\exp\left\{\frac{-(v - \int_Q h\, d[\vec{\xi}])^2}{2\|h - \tilde{h}\|^2}\right\} dv.$$

Proof. By Theorem 3 and (4.3) we have

$$\begin{aligned} E\left(F\left(\int_Q h\, dx\right) \middle| X_\tau(x) = \vec{\xi} \right) &= \widehat{E}\left[F\left(\int_Q h\, d\left(x - [x] + [\vec{\xi}]\right)\right)\right] \\ &= \widehat{E}\left[F\left(\int_Q (h - \tilde{h})\, dx + \int_Q h\, d[\vec{\xi}]\right)\right] \\ &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} F(\|h - \tilde{h}\|u + \int_Q h\, d[\vec{\xi}])\exp\left\{-\frac{u^2}{2}\right\} du. \end{aligned}$$

5. Translation of conditional Yeh-Wiener integrals

The Cameron-Martin translation theorem in Yeh-Wiener space $C(Q)$ (see [11]) states that if $x_0(s, t) = \int_0^t \int_0^s h(u, v)dudv$ for some $h \in L_2(Q)$, and if T_1 is the translation of $C(Q)$ into itself given by $z = T_1(x) = x + x_0$, then for any Yeh-Wiener integrable functional F on $C(Q)$ and any Yeh-Wiener measurable set Γ

$$\int_\Gamma F(z)m_y(dz) = \int_{T_1^{-1}(\Gamma)} F(x + x_0)J(x_0, x)m_y(dx), \tag{5.1}$$

where

$$J(x_0, x) = \exp\left\{-\frac{1}{2}\int_Q h^2(s, t)dsdt\right\}\exp\left\{-\int_Q h(s, t)\, dx(s, t)\right\} \tag{5.2}$$

and $\int_Q h(s,t)\,dx(s,t)$ is a stochastic integral with mean zero and variance $\int_Q h^2(s,t)dsdt$. In particular, if $\Gamma = C(Q)$, then (5.1) becomes

$$E[F(z)] = E[F(x+x_0)J(x_0,x)]. \tag{5.3}$$

The following is the vector-valued conditional version of (5.3).

Theorem 7. *Let $x_0(s,t) = \int_0^t \int_0^s h(u,v)dudv$ on Q for some $h \in L_2(Q)$. Let $F \in L_1(C(Q), m_y)$. Then for each partition $\tau = \{(s_i, t_j),\ i = 1, \ldots, m,\ j = 1, \ldots, n\}$ of Q,*

$$\begin{aligned} &E(F(z)|X_\tau(z) = \vec{\xi}) \\ &\quad = E(F(x+x_0)J(x_0,x)|X_\tau(x+x_0) = \vec{\xi}) \\ &\quad\quad \cdot \exp\left\{-\frac{1}{2}\int_Q \tilde{h}^2(s,t)dsdt + \int_Q h(s,t)d[\vec{\xi}](s,t)\right\}. \end{aligned}$$

Proof. By Theorem 3, we have

$$E(F(z)|X_\tau(z) = \vec{\xi}) = \widehat{E}[F(z-[z]+[\vec{\xi}])]. \tag{5.4}$$

Noting that $[x+x_0](s,t) = [x](s,t) + [x_0](s,t)$, and using (5.3) we see that

$$E[F(z-[z]+[\vec{\xi}])] = E[F(x+x_0-[x]-[x_0]+[\vec{\xi}])J(x_0,x)]. \tag{5.5}$$

Next we rewrite $J(x_0,x)$ in the form

(5.6)
$$\begin{aligned} J(x_0,x) = {} & \exp\left\{-\frac{1}{2}\int_Q h^2(s,t)dsdt\right\}\exp\left\{-\int_Q h\,d(x-[x]+[\vec{\xi}]-[x_0])\right\} \\ & \exp\left\{-\int_Q h\,d[x]\right\}\exp\left\{\int_Q h\,d([\vec{\xi}]-[x_0])\right\}. \end{aligned}$$

Since $x-[x]$ and $[x]$ are two independent processes on Q by Corollary 1.1, it follows from (5.5) and (5.6) that

(5.7)
$$\begin{aligned} & E[F(z-[z]+[\vec{\xi}])] \\ &\quad = E\left[F(x+x_0-[x]-[x_0]+[\vec{\xi}])\exp\left\{-\int_Q h\,d(x-[x]+[\vec{\xi}]-[x_0])\right\}\right] \\ &\quad\quad \cdot E\left[\exp\left\{-\int_Q h\,d[x]\right\}\right]\exp\left\{-\frac{1}{2}\int_Q h^2(s,t)dsdt + \int_Q h\,d([\vec{\xi}]-[x_0])\right\}. \end{aligned}$$

Since $\int_Q \tilde{h}\,dx$ is a Gaussian random variable with mean zero and variance $\|\tilde{h}\|_2^2$ it follows using (4.3) that

(5.8)
$$E\left[\exp\left\{-\int_Q h\,d[x]\right\}\right] = E\left[\exp\left\{-\int_Q \tilde{h}\,dx\right\}\right] = \exp\left\{\frac{1}{2}\int_Q \tilde{h}^2(s,t)dsdt\right\}.$$

Next using Theorem 4 we have

$$(5.9)\qquad \int_Q h\, d[x_0] = \int_Q \tilde{h}\, dx_0 = \int_Q \tilde{h}(s,t)\frac{\partial^2 x_0(s,t)}{\partial s\partial t}dsdt = \int_Q \tilde{h}(s,t)h(s,t)dsdt = \int_Q \tilde{h}^2(s,t)dsdt.$$

Upon applying (2.5), (5.8), and (5.9) to (5.7), we obtain

$$\begin{aligned}&\widehat{E}[F(z-[z]+[\vec{\xi}])]\\ &= E(F(x+x_0)J(x_0,x)|X_\tau(x+x_0)=\vec{\xi})\\ &\cdot \exp\left\{\frac{1}{2}\int_Q \tilde{h}^2(s,t)dsdt\right\}\exp\left\{\int_Q h\, d[\vec{\xi}] - \int_Q \tilde{h}^2(s,t)dsdt\right\}.\end{aligned}$$

This together with (5.4) gives the desired result.

Note that if the partition τ consists of the single point (S,T), then Theorem 7 reduces to the following corollary:

Corollary 7.1. *Let F, h and x_0 be as in Theorem 7. Then*

$$E(F(z)|z(S,T)=\xi) = E(F(x+x_0)J(x_0,x)|x(S,T)=\xi - x_0(S,T)) \exp\left\{-\frac{x_0^2(S,T)}{2ST} + \frac{\xi x_0(S,T)}{S,T}\right\}.$$

This corollary was the main result in Chang and Ahn [2]; however their method was very different than ours.

By choosing $F(x)\equiv 1$ in Theorem 7 we obtain the following corollary:

Corollary 7.2. *Let h and x_0 be as in Theorem 7. Then*

$$E\left(\exp\left\{\int_Q h(s,t)dx(s,t)\right\}\middle| X_\tau(x)=\vec{\eta}\right) = \exp\left\{\frac{1}{2}\int_Q (h^2(s,t)-\tilde{h}^2(s,t))dsdt + \int_Q h\, d[\vec{\eta}]\right\}.$$

Proof. By setting $F\equiv 1$ and $\xi_{i,j}=\eta_{i,j}+x_0(s_i,t_j)\forall(s_i,t_j)\in\tau$ in Theorem 7, we obtain

$$(5.10)\qquad 1 = E(J(x_0,x)|X_\tau(x)=\vec{\eta})\exp\left\{-\frac{1}{2}\int_Q \tilde{h}^2(s,t)dsdt + \int_Q h\, d[\vec{\eta}] + \int_Q h\, d[x_0]\right\}.$$

Using (5.9) in (5.10) and using (5.2), we get that

$$E\left(\exp\left\{-\int_Q h(s,t)dx(s,t)\right\}\middle| X_\tau(x)=\vec{\eta}\right) = \exp\left\{\frac{1}{2}\int_Q (h^2(s,t)-\tilde{h}^2(s,t))dsdt - \int_Q h\, d[\vec{\eta}]\right\}.$$

Thus, the result readily follows by replacing $h(s,t)$ by $-h(s,t)$.

Remark. Corollary 7.2 is also a special case of Theorem 6.

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