PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 2, February 1989

ON THE VALUES AT NEGATIVE HALF-INTEGERS OF THE DEDEKIND ZETA FUNCTION OF A REAL QUADRATIC FIELD

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(Communicated by Larry J. Goldstein)

ABSTRACT. The zeta function $\zeta(A, s)$ associated with a narrow ideal class A for a real quadratic field can be decomposed into $\sum_Q Z_Q(s)$, where $Z_Q(s)$ is a Dirichlet series associated with a quadratic form $Q(x, y) = ax^2 + bxy + cy^2$, and the summation is over finite reduced quadratic forms associated to the narrow ideal class A. The values of $Z_Q(s)$ at nonpositive integers were obtained by Zagier [16] and Shintani [12] via different methods. In this paper, we shall obtain the values of $Z_Q(s)$ at negative half-integers $s = -1/2, -3/2, \ldots, -m+1/2, \ldots$. The values of $Z_Q(s)$ at nonpositive integers were also obtained by our method, and our results are consistent with those given in [16].

1. INTRODUCTION

Let $Q(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integral coefficients and of discriminant $D = b^2 - 4ac$. Also let $T = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$ be an element of $\mathrm{GL}_2(Z)$ with $\det T = \alpha\delta - \beta\gamma = \pm 1$. Then T acts on the collection of forms of discriminant D by the action:

$$Q \to Q|T(x, y) = (\alpha\delta - \beta\gamma)Q(\alpha x + \beta y, \gamma x + \delta y).$$

Two forms Q_1 and Q_2 are said to be equivalent in the narrow sense (resp. wide sense) if $Q_1 = Q_2|T$ for some $T \in \mathrm{SL}_2(Z)$ (resp. $T \in \mathrm{GL}_2(Z)$ and $\det T = \pm 1$). A quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ is called reduced (in the narrow sense) if $a > 0$, $c > 0$, and $b > a + c$. Q is primitive if the g.c.d. of a, b, c is 1.

In real quadratic fields, there is a natural correspondence between classes of modules and $\mathrm{SL}_2(Z)$-equivalent classes of primitive quadratic forms. Let M be a full module (module of rank 2) in a real quadratic field. The zeta function

Received by the editors December 16, 1987 and, in revised form, January 26, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11R42, 11M41; Secondary 11E32.
This work was supported by the Institute of Mathematics, Academia Sinica, and the National Science Foundation of Taiwan, Republic of China.

0002-9939/89 \$1.00 + \$.25 per page

of M is defined by

$$\zeta(M, s) = N(M)^s \sum_{\xi \in M/E} \frac{1}{N(\xi)^s}, \qquad \operatorname{Re} s > 1,$$

where E is the group of totally positive units ε satisfying $\varepsilon M = M$, and N is the norm on the real quadratic field. For any totally positive number λ, we have $\zeta(\lambda M, s) = \zeta(M, s)$. Hence $\zeta(M, s)$ can be considered as a zeta function associated with the module class A to which M belongs. Consequently, we write $\zeta(A, s)$ instead of $\zeta(M, s)$.

For a reduced quadratic form $Q(x, y) = ax^2 + bxy + cy^2$, we define

$$\begin{aligned} Z_Q(s) &= \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{1}{(ap^2 + bpq + cq^2)^s} + \frac{1}{2}\sum_{p=1}^{\infty} \frac{1}{(ap^2)^s} + \frac{1}{2}\sum_{q=1}^{\infty} \frac{1}{(cq^2)^s} \\ &= Z_Q^*(s) + \frac{1}{2}\sum_{p=1}^{\infty} \frac{1}{(ap^2)^s} + \frac{1}{2}\sum_{q=1}^{\infty} \frac{1}{(cq^2)^s}, \qquad \operatorname{Re} s > 1. \end{aligned}$$

In [16], Zagier proved that $\zeta(A, s)$ can be decomposed into finite combinations of $Z_Q(s)$, i.e.

$$\zeta(A, s) = \sum_Q Z_Q(s),$$

where the summation is over the reduced forms in the classes of forms associated to the module class of M. Also Zagier gave the values of $Z_Q(s)$ at nonpositive integers.

In this paper, we shall start with the zeta function

$$\tilde{\zeta}_2(s) = \sum_{s_1=1}^{\infty}\sum_{s_2=1}^{\infty}\sum_{s_{12}=0}^{\infty} \frac{1}{[s_1 s_2 + (s_1 + s_2)s_{12}]^s}, \qquad \operatorname{Re} s > 3/2,$$

associated with the principal Delaunay-Voronoi cone as considered in [3]. Letting $s_1 = p$, $s_2 = q$, and $s_{12} = (ap + cq)/(b - a - c)$, we get the zeta function $Z_Q^*(s)$ up to a constant multiple $(b - a - c)^s$. With the method introduced in [3, 6, 9], we get an integral expression for $Z_Q^*(s)\Gamma(s)(s - 1/2)\pi^{1/2}$ when $\operatorname{Re} s \geq 3/2$, and the values of $Z_Q^*(s)$ at nonpositive integers and negative half-integers can be written as a finite sum of integrals which are functions in s and have analytic continuations in the whole complex plane.

Theorem 1. *Let m be a nonnegative integer. Then*

$$\begin{aligned} Z_Q^*(-m) &= -\frac{(2m+1)!}{2^{2m}}(b - a - c)^m \frac{1}{2\pi} N_1(-m), \\ Z_Q^*\left(-m + \frac{1}{2}\right) &= -\frac{B_{2m}}{2^{2m}}(b - a - c)^{m-(1/2)} \frac{1}{2\pi} N_2\left(-m + \frac{1}{2}\right), \qquad m \geq 1, \end{aligned}$$

where

$$N_1(s) = \int_0^1 (1-r^2)^{s-3/2} r\,dr$$
$$\cdot \int_0^{2\pi} \sum_{p=0}^{m+1} \frac{B_{2p}B_{2m+2-2p}R(r,\theta)^{2p-1}T(r,\theta)^{2m+1-2p}}{2p!(2m+2-2p)!} \left(+\frac{1}{4} \text{ if } m=0\right) d\theta,$$
$$N_2(s) = \int_0^1 (1-r^2)^{s-3/2} r\,dr \int_0^{2\pi} [R(r,\theta)^{2m-1} + T(r,\theta)^{2m-1}]\,d\theta,$$

for $\operatorname{Re} s > 1$, *with*

$$R(r,\theta) = (1 + r\sin\theta) + \frac{2a}{b-a-c}(1 - r\cos\theta),$$
$$T(r,\theta) = (1 - r\sin\theta) + \frac{2c}{b-a-c}(1 - r\cos\theta).$$

Here B_m $(m = 0, 1, \ldots,)$ *are Bernoulli numbers defined by*

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} \frac{B_m t^m}{m!}, \qquad |t| < 2\pi.$$

Note that $N_1(s)$ has an analytic continuation which is holomorphic except for possible simple poles at $s = \frac{1}{2}, -\frac{1}{2}, \ldots, -m+\frac{1}{2}$. On the other hand, $N_2(s)$ has an analytic continuation which is holomorphic except for possible poles at $s = \frac{1}{2}, -\frac{1}{2}, \ldots, -m+\frac{3}{2}$ if $-m+\frac{3}{2} < \operatorname{Re} s \le -m+\frac{1}{2}$. Thus $N_1(-m)$ and $N_2(-m+\frac{1}{2})$ can be obtained by the analytic continuation of $N_1(s)$ and $N_2(s)$. In particular, we have

Theorem 2. *For positive integers* m, *we have*

$$\frac{1}{2\pi} N_2\left(-m+\frac{1}{2}\right) = \sum_{l=0}^{m-1} \binom{2m-1}{2l} \frac{[1\cdot 3\cdots(2l-1)]}{[m(m-1)\cdots(m-l)](-2)^{l+1}}$$
$$\cdot \{(1+\delta_1)^{2m-1-2l}(1+\delta_1^2)^l + (1+\delta_2)^{2m-1-2l}(1+\delta_2^2)^l\}$$

with $\delta_1 = 2a/(b-a-c)$ *and* $\delta_2 = 2c/(b-a-c)$.

Consequently, by an elementary computation, we have the following:

$$Z_Q\left(-\frac{1}{2}\right) = \frac{1}{24}\frac{b}{\sqrt{b-a-c}} - \frac{1}{24}(\sqrt{a}+\sqrt{c}),$$
$$Z_Q(-1) = \frac{1}{24}\left(\frac{b}{a}+\frac{b}{c}\right) + \frac{1}{4},$$
$$Z_Q\left(-\frac{3}{2}\right) = \frac{1}{1620} \cdot \frac{P(a,b,c)}{(b-a-c)^{3/2}} + \frac{1}{240}(a^{3/2}+c^{3/2}) \quad \text{with}$$
$$P(a,b,c) = 6a^3 - b^2 + 6c^3 - 3a^2b - 3bc^2$$
$$- 6ab^2 - 6b^2c - 6a^2c - 6ac^2 + 30abc,$$

$$Z_Q(-2) = \frac{1}{1440}\left(\frac{b^3 - 6abc}{a^2} + \frac{b^3 - 6abc}{c^2}\right) + \frac{b}{144}.$$

In particular, we prove the following result:

Theorem. *Let K be a real quadratic field of discriminant D and denote by G_K the (finite) set of positive divisors of integers of the form $(D - n^2)/4$ $(|n| < \sqrt{D}$, $n \equiv D \pmod 2)$. Then the value of the Dedekind zeta function of K, or of the zeta function of any ideal class of K, at a negative half-integral argument $s = 1/2 - m$ is a rational linear combination of the numbers $g^{1/2-m}$ $(g \in G_K)$, the denominators of the coefficients being bounded by an integer depending only on m (24 for $m = 1$, 1620 for $m = 2, \dots$).*

The above theorem is an easy consequence of Theorems 1 and 2 since the numbers a, c, and $b - a - c$ for any reduced form $ax^2 + bxy + cy^2$ belong to G_K.

2. The integral expression of Z_Q^* and the proof of Theorem 1

Fix a reduced quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ and let

$$Z_Q^*(s) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{1}{(ap^2 + bpq + cq^2)^s}, \qquad \operatorname{Re} s > 1.$$

In this section we shall obtain an integral expression for $Z_Q^*(s)\Gamma(s)\Gamma(s-1/2)\pi^{1/2}$ and the analytic continuation of this zeta function.

Lemma 1. *Let Y be the variable of a 2×2 real symmetric matrix and G be a fixed 2×2 positive definite symmetric matrix. Then we have, for $\operatorname{Re} s \geq 3/2$,*

$$\int_{Y>0} (\det Y)^{s-3/2} e^{-\operatorname{tr}(YG)}\, dY = (\det G)^{-s}\pi^{1/2}\Gamma(s)\Gamma\left(s - \frac{1}{2}\right).$$

Here $\operatorname{tr} X =$ *trace of X for any matrix X.*

Proof. See p. 225 of [1].

Proposition 1. *For $\operatorname{Re} s \geq 3/2$, we have*

$$\Gamma(s)\Gamma\left(s - \frac{1}{2}\right)\pi^{1/2}(b - a - c)^s Z_Q^*(s) = \int_{Y>0} (\det Y)^{s-3/2} \frac{dY}{(e^{A(Y)} - 1)(e^{B(Y)} - 1)},$$

where

$$Y = \begin{bmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{bmatrix} \quad \text{and} \quad \begin{cases} A(Y) = y_1 + \dfrac{a}{b - a - c}(y_1 + y_2 - 2y_{12}), \\ B(Y) = y_2 + \dfrac{c}{b - a - c}(y_1 + y_2 - 2y_{12}). \end{cases}$$

Proof. Apply Lemma 1 with

$$G = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} + \frac{ap + bq}{b - a - c}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

p, q being positive integers,

we get, for $\operatorname{Re} s \geq 3/2$,

$$\Gamma(s)\Gamma\left(s-\frac{1}{2}\right)\pi^{1/2}Z_Q^*(s)(b-a-c)^s = \sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\int_{Y>0}(\det Y)^{s-3/2}e^{-A(Y)p-B(Y)q}\,dY$$

$$= \int_{Y>0}(\det Y)^{s-3/2}\sum_{q=1}^{\infty}\sum_{p=1}^{\infty}e^{-A(Y)p-B(Y)q}\,dY$$

$$= \int_{Y>0}(\det Y)^{s-3/2}\frac{dY}{(e^{A(Y)}-1)(e^{B(Y)}-1)}.$$

Remark. Here the exchange of summation and integration is possible since the double series $\sum_{q=1}^{\infty}\sum_{p=1}^{\infty} e^{-A(Y)p-B(Y)q}$ is absolutely convergent and its partial sum is dominated by

$$\frac{1}{(e^{A(Y)}-1)(e^{B(Y)}-1)}.$$

Proposition 2. *$Z_Q^*(s)$ has an analytic continuation to the whole complex plane except a simple pole at $s = 1/2$. Furthermore, we have*

$$Z_Q^*(s) = 2\Gamma(1-s)\frac{e^{-\pi i s}}{e^{2\pi i s}+1}\cdot\frac{1}{2\pi i}\int_{L(\varepsilon)}u^{2s-3}I(s,u)\,du$$

where

$$I(s,u) = \frac{1}{\Gamma(s-1/2)\pi^{1/2}}\int_0^1(1-r^2)^{s-3/2}r\,dr\int_0^{2\pi}\frac{u^2\,d\theta}{(e^{R(r,\theta)u}-1)(e^{T(r,\theta)u}-1)},$$

$L(\varepsilon)$ is the contour in the complex plane consisting of the interval $[\varepsilon, +\infty)$ twice, in both directions (in and out) and the circle $|z| = \varepsilon$ in counterclockwise direction, and

$$\begin{cases} R(r,\theta) = (1+r\sin\theta) + \dfrac{2a}{b-a-c}(1-r\cos\theta), \\ T(r,\theta) = (1-r\sin\theta) + \dfrac{2c}{b-a-c}(1-r\cos\theta). \end{cases}$$

Proof. The first assertion was proved in [16]. Here we prove the integral expression from Proposition 1. By changing variables: $u = (y_1+y_2)/2$, $v = (y_1-y_2)/2$, $w = y_{12}$, the integral expression for $\Gamma(s)\Gamma(s-1/2)Z_Q^*(s)(b-a-c)^s$ is transformed into

$$2\int_{u^2-v^2-w^2>0,\,u>0}\frac{(u^2-v^2-w^2)^{s-3/2}\,du\,dv\,dw}{(e^{[u+v+\delta_1(u-w)]}-1)(e^{[u-v+\delta_2(u-w)]}-1)},$$

where $\delta_1 = 2a/(b-a-c)$ and $\delta_2 = 2c/(b-a-c)$.

Let $v = ux$, $v = uy$ and then let $x = r\cos\theta$, $y = r\sin\theta$. It follows that

$$\Gamma(s)\Gamma(s-1/2)\pi^{1/2s}Z_Q^*(s)(b-a-c)^s$$

$$= 2\int_0^{\infty}u^{2s-3}\,du\int_0^1(1-r^2)^{s-3/2}r\,dr\int_0^{2\pi}\frac{u^2\,d\theta}{(e^{R(r,\theta)u}-1)(e^{T(r,\theta)u}-1)}.$$

As shown in [6], $I(s, u)$ has an analytic continuation which is a meromorphic function in s. The integration with respect to u can be changed into a contour integral. Thus we have

$$\Gamma(s)Z_Q^*(s)(b-a-c)^s = 2(e^{4\pi is}-1)^{-1}\int_{L(\varepsilon)} u^{2s-3}I(s, u)\,du.$$

In light of the functional equation for the gamma function

$$\Gamma(s)\Gamma(1-s) = \frac{2\pi i e^{\pi is}}{e^{2\pi is}-1},$$

we then have

$$Z_Q^*(s)(b-a-c)^s = 2\Gamma(1-s)\frac{e^{-\pi is}}{e^{2\pi is}+1}\cdot\frac{1}{2\pi i}\int_{L(\varepsilon)} u^{2s-3}I(s, u)\,du.$$

The contour integral is convergent for all s. Thus it defines the analytic continuation of $Z_Q^*(s)$.

Proof of Theorem 1. When $s=-m$ or $s=-m+\frac{1}{2}$ $(m>0)$, then $2s-3$ is an integer. On the other hand, $I(s, u)$ is a holomorphic function in u. Consequently, the integrations along $[\varepsilon, \infty)$ twice in opposite directions will cancel and the evaluation of the contour integral is reduced to the calculation of residues of $u^{2s-3}I(s, u)$ at $u=0$ and $s=-m$ or $-m+\frac{1}{2}$.

Note that

$$\frac{u}{e^{R(r, \theta)u}-1} = \frac{1}{R(r, \theta)} - \frac{1}{2} + \sum_{m=1}^{\infty}\frac{B_{2m}u^{2m}R(r, \theta)^{2m-1}}{(2m)!}, \qquad |R(r, \theta)u| < 2\pi,$$

$$\frac{u}{e^{T(r, \theta)u}-1} = \frac{1}{T(r, \theta)} - \frac{1}{2} + \sum_{m=1}^{\infty}\frac{B_{2m}u^{2m}T(r, \theta)^{2m-1}}{(2m)!}, \qquad |T(r, \theta)u| < 2\pi.$$

By considering the coefficients of u^{2m+2} $(s=-m)$ and u^{2m+1} $(s=-m+\frac{1}{2})$ in the power expansion of

$$\frac{u^2}{(e^{R(r, \theta)u}-1)(e^{T(r, \theta)u}-1)}$$

at $u=0$, we get our assertion for $Z_Q^*(-m)$ and $Z_Q^*(-m+\frac{1}{2})$ as listed in Theorem 1 of the Introduction.

Remark. Here we use the following identities, which can be verified in an elementary way:

$$\lim_{s\to -m} 2\Gamma(1-s)\frac{e^{-\pi is}}{e^{2\pi is}+1}\cdot\frac{1}{\Gamma\left(s-\frac{1}{2}\right)\pi^{1/2}} = -\frac{(2m+1)!}{2^{2m}}\cdot\frac{1}{2\pi},$$

$$\lim_{s\to -m+1/2} 2\Gamma(1-s)\frac{e^{-\pi is}}{e^{2\pi is}+1}\cdot\frac{1}{\Gamma\left(s-\frac{1}{2}\right)\pi^{1/2}} = -\frac{(2m)!}{2^{2m}\pi}.$$

3. The proof of Theorem 2

The evaluation of $N_1(-m)$ and $N_2(-m+\frac{1}{2})$ can be done by the same arguments as in [3]. However, as the values of $Z_Q(s)$ at nonpositive integers were given in [16], it is unnecessary to compute $N_1(-m)$ (though it is possible). Here we only compute the value of $N_2(-m+\frac{1}{2})$.

Proof of Theorem 2. Note that

$$1 + r\sin\theta + \delta_1(1 - r\cos\theta) = 1 + \delta_1 + \sqrt{1+\delta_1^2}\, r\sin(\theta - \phi)$$

with $\phi = \tan^{-1}(1/\delta_1)$. Hence

$$\begin{aligned}\int_0^{2\pi} R(r,\theta)^{2m-1}\, d\theta &= \int_0^{2\pi} [1 + \delta_1 + \sqrt{1+\delta_1^2}\, r\sin(\theta-\phi)]^{2m-1}\, d\theta \\ &= \int_0^{2\pi} [1 + \delta_1 + \sqrt{1+\delta_1^2}\, r\sin\theta]^{2m-1}\, d\theta \\ &= \sum_{l=0}^{m-1} \binom{2m-1}{2l} (1+\delta_1)^{2m-1-2l} (1+\delta_1^2)^l \int_0^{2\pi} r^{2l} \sin^{2l}\theta\, d\theta.\end{aligned}$$

For sufficiently large s, we have

$$\int_0^1 \int_0^{2\pi} (1-r^2)^{s-3/2} r^{2l+1} \sin^{2l}\theta\, d\theta\, dr = \frac{[1\cdot 3\cdots(2l-1)]2\pi}{(2s-1)(2s+1)\cdots(2s-1+2l)}.$$

Thus the contribution from $R(r,\theta)^{2m-1}$ to $(1/2\pi)N_2(-m)$ is given by

$$\begin{aligned}&\sum_{l=0}^{m-1} \binom{2m-1}{2l} (1+\delta_1)^{2m-1-2l} (1+\delta_1^2)^l \frac{[1\cdot 3\cdots(2l-1)]}{[(-2m)(-2m+2)\cdots(-2m+2l)]} \\ &\quad = \sum_{l=0}^{m-1} \binom{2m-1}{2l} (1+\delta_1)^{2m-1-2l} (1+\delta_1^2)^l \frac{[1\cdot 3\cdots(2l-1)]}{m(m-1)\cdots(m-l)](-2)^{l+1}}.\end{aligned}$$

In the same way, we get the contribution from $T(r,\theta)^{2m-1}$ to

$$(1/2\pi)N_2(-m).$$

Corollary. *Let m be a positive integer. Then*

$$\begin{aligned}Z_Q\left(-m+\frac{1}{2}\right) = -\frac{B_{2m}}{2^{2m}} \sum_{l=0}^{m-1} \binom{2m-1}{2l} &\frac{[1\cdot 3\cdots(2l-1)]}{[m(m-1)\cdots(m-l)](-2)^{l+1}} \\ &\cdot \{(1+\delta_1)^{2m-1-2l}(1+\delta_1^2)^l + (1+\delta_2)^{2m-1-2l}(1+\delta_2^2)^l\} \\ &\cdot (b-a-c)^{m-1/2} - \frac{1}{2}(a^{m-1/2} + c^{m-1/2})\frac{B_{2m}}{2m},\end{aligned}$$

where $\delta_1 = 2a/(b-a-c)$, $\delta_2 = 2c/(b-a-c)$.

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