ON THE VALUES AT NEGATIVE HALF-INTEGERS
OF THE DEDEKIND ZETA FUNCTION
OF A REAL QUADRATIC FIELD

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Abstract. The zeta function \( \zeta(A,s) \) associated with a narrow ideal class \( A \)
for a real quadratic field can be decomposed into \( \sum_{Q} Z_{Q}(s) \), where \( Z_{Q}(s) \) is
a Dirichlet series associated with a quadratic form \( Q(x, y) = ax^2 + bxy + cy^2 \),
and the summation is over finite reduced quadratic forms associated to the nar-
row ideal class \( A \). The values of \( Z_{Q}(s) \) at nonpositive integers were obtained
by Zagier [16] and Shintani [12] via different methods. In this paper, we shall
obtain the values of \( Z_{Q}(s) \) at negative half-integers \( s = -1/2, -3/2, \ldots, -m+1/2, \ldots \).
The values of \( Z_{Q}(s) \) at nonpositive integers were also obtained by
our method, and our results are consistent with those given in [16].

1. Introduction

Let \( Q(x, y) = ax^2 + bxy + cy^2 \) be a binary quadratic form with integral
coefficients and of discriminant \( D = b^2 - 4ac \). Also let \( T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \) be an
element of \( GL_{2}(Z) \) with \( \det T = \alpha\delta - \beta\gamma = \pm 1 \). Then \( T \) acts on the collection
of forms of discriminant \( D \) by the action:

\[ Q \rightarrow Q|T(x, y) = (\alpha\delta - \beta\gamma)Q(\alpha x + \beta y, \gamma x + \delta y). \]

Two forms \( Q_1 \) and \( Q_2 \) are said to be equivalent in the narrow sense (resp.
wide sense) if \( Q_1 = Q_2|T \) for some \( T \in SL_{2}(Z) \) (resp. \( T \in GL_{2}(Z) \) and
\( \det T = \pm 1 \)). A quadratic form \( Q(x, y) = ax^2 + bxy + cy^2 \) is called reduced
(in the narrow sense) if \( a > 0, c > 0, \) and \( b > a + c \). \( Q \) is primitive if the
g.c.d. of \( a, b, c \) is 1.

In real quadratic fields, there is a natural correspondence between classes of
modules and \( SL_{2}(Z) \)-equivalent classes of primitive quadratic forms. Let \( M \)
be a full module (module of rank 2) in a real quadratic field. The zeta function

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of \( M \) is defined by

\[
\zeta(M, s) = N(M)^s \sum_{\xi \in M/E} \frac{1}{N(\xi)^s}, \quad \text{Re} \, s > 1,
\]

where \( E \) is the group of totally positive units \( \varepsilon \) satisfying \( \varepsilon M = M \), and \( N \) is the norm on the real quadratic field. For any totally positive number \( \lambda \), we have \( \zeta(\lambda M, s) = \zeta(M, s) \). Hence \( \zeta(M, s) \) can be considered as a zeta function associated with the module class \( A \) to which \( M \) belongs. Consequently, we write \( \zeta(A, s) \) instead of \( \zeta(M, s) \).

For a reduced quadratic form \( Q(x, y) = ax^2 + bxy + cy^2 \), we define

\[
Z_Q(s) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{(ap^2 + bpq + cq^2)^s} + \frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{(ap^2)^s} + \frac{1}{2} \sum_{q=1}^{\infty} \frac{1}{(cq^2)^s}, \quad \text{Re} \, s > 1.
\]

In [16], Zagier proved that \( \zeta(A, s) \) can be decomposed into finite combinations of \( Z_Q(s) \), i.e.

\[
\zeta(A, s) = \sum_Q Z_Q(s),
\]

where the summation is over the reduced forms in the classes of forms associated to the module class of \( M \). Also Zagier gave the values of \( Z_Q(s) \) at nonpositive integers.

In this paper, we shall start with the zeta function

\[
\zeta_2(s) = \sum_{s_1=1}^{\infty} \sum_{s_2=1}^{\infty} \sum_{s_{12}=0}^{\infty} \frac{1}{(s_1s_2 + (s_1 + s_2)s_{12})^s}, \quad \text{Re} \, s > 3/2,
\]

associated with the principal Delaunay-Voronoi cone as considered in [3]. Letting \( s_1 = p, \ s_2 = q \), and \( s_{12} = (ap + cq)/(b - a - c) \), we get the zeta function \( Z_Q^*(s) \) up to a constant multiple \( (b - a - c)^s \). With the method introduced in [3, 6, 9], we get an integral expression for \( Z_Q^*(s)\Gamma(s)(s - 1/2)\pi^{1/2} \) when \( \text{Re} \, s \geq 3/2 \), and the values of \( Z_Q^*(s) \) at nonpositive integers and negative half-integers can be written as a finite sum of integrals which are functions in \( s \) and have analytic continuations in the whole complex plane.

**Theorem 1.** Let \( m \) be a nonnegative integer. Then

\[
Z_Q^*(-m) = -\frac{(2m + 1)!}{2^{2m}} (b - a - c)^m \frac{1}{2\pi} N_1(-m),
\]

\[
Z_Q^*(-m + \frac{1}{2}) = -\frac{B_{2m}}{2^{2m}} (b - a - c)^{m-(1/2)} \frac{1}{2\pi} N_2\left(-m + \frac{1}{2}\right), \quad m \geq 1,
\]
where

\[ N_1(s) = \int_0^1 (1 - r^2)^{s-3/2} r \, dr \]
\[ \cdot \int_0^{2\pi} \prod_{p=0}^{m+1} \frac{B_{2p} B_{2m+2-2p} R(r, \theta)^{2p-1} T(r, \theta)^{2m+1-2p}}{2^{2p}(2m + 2 - 2p)!} \left( + \frac{1}{4} \text{ if } m = 0 \right) d\theta, \]

\[ N_2(s) = \int_0^1 (1 - r^2)^{s-3/2} r \, dr \int_0^{2\pi} [R(r, \theta)^{2m-1} + T(r, \theta)^{2m-1}] d\theta, \]

for \( \text{Res} > 1 \), with

\[ R(r, \theta) = (1 + r \sin \theta) + \frac{2a}{b - a - c} (1 - r \cos \theta), \]
\[ T(r, \theta) = (1 - r \sin \theta) + \frac{2c}{b - a - c} (1 - r \cos \theta). \]

Here \( B_m \ (m = 0, 1, \ldots) \) are Bernoulli numbers defined by

\[ \frac{t}{e^t - 1} = \sum_{m=0}^{\infty} \frac{B_m t^m}{m!}, \quad |t| < 2\pi. \]

Note that \( N_1(s) \) has an analytic continuation which is holomorphic except for possible simple poles at \( s = \frac{1}{2}, -\frac{1}{2}, \ldots, -m + \frac{1}{2} \). On the other hand, \( N_2(s) \) has an analytic continuation which is holomorphic except for possible poles at \( s = \frac{1}{2}, -\frac{1}{2}, \ldots, -m + \frac{3}{2} \) if \(-m + \frac{3}{2} < \text{Res} \leq -m + \frac{1}{2} \). Thus \( N_1(-m) \) and \( N_2(-m + \frac{1}{2}) \) can be obtained by the analytic continuation of \( N_1(s) \) and \( N_2(s) \).

In particular, we have

**Theorem 2.** For positive integers \( m \), we have

\[ \frac{1}{2\pi} N_2 \left( -m + \frac{1}{2} \right) = \sum_{l=0}^{m-1} \binom{2m - 1}{2l} \frac{[1 \cdot 3 \cdot \cdots (2l-1)]}{[m(m-1) \cdots (m-l)](-2)^{l+1}} \cdot \{(1 + \delta_1)^{2m-1-2l}(1 + \delta_1^2)^l + (1 + \delta_2)^{2m-1-2l}(1 + \delta_2^2)^l\} \]

with \( \delta_1 = 2a/(b - a - c) \) and \( \delta_2 = 2c/(b - a - c) \).

Consequently, by an elementary computation, we have the following:

\[ Z_Q \left( -\frac{1}{2} \right) = \frac{1}{24} \sqrt{b - a - c} - \frac{1}{24} (\sqrt{a} + \sqrt{c}), \]
\[ Z_Q(-1) = \frac{1}{24} \left( \frac{b}{a} + \frac{b}{c} \right) + \frac{1}{4}, \]
\[ Z_Q \left( -\frac{3}{2} \right) = \frac{1}{1620} \cdot \frac{P(a, b, c)}{(b - a - c)^{3/2}} + \frac{1}{240} (a^{3/2} + c^{3/2}) \] with
\[ P(a, b, c) = 6a^3 - b^2 + 6c^3 - 3a^2b - 3bc^2 \]
\[ - 6ab^2 - 6b^2c - 6a^2c - 6ac^2 + 30abc, \]
\[ Z_Q(-2) = \frac{1}{1440} \left( \frac{b^3 - 6abc}{a^2} + \frac{b^3 - 6abc}{c^2} \right) + \frac{b}{144}. \]

In particular, we prove the following result:

**Theorem.** Let \( K \) be a real quadratic field of discriminant \( D \) and denote by \( G_K \) the (finite) set of positive divisors of integers of the form \( (D - n^2)/4 \ (|n| < \sqrt{D}, \ n \equiv D \pmod{2}) \). Then the value of the Dedekind zeta function of \( K \), or of the zeta function of any ideal class of \( K \), at a negative half-integral argument \( s = 1/2 - m \) is a rational linear combination of the numbers \( g^{1/2-m} \ (g \in G_K) \), the denominators of the coefficients being bounded by an integer depending only on \( m \) (24 for \( m = 1 \), 1620 for \( m = 2, \ldots \)).

The above theorem is an easy consequence of Theorems 1 and 2 since the numbers \( a, c \), and \( b - a - c \) for any reduced form \( ax^2 + bxy + cy^2 \) belong to \( G_K \).

2. THE INTEGRAL EXPRESSION OF \( Z^*_Q \) AND THE PROOF OF THEOREM 1

Fix a reduced quadratic form \( Q(x, y) = ax^2 + bxy + cy^2 \) and let

\[ Z^*_Q(s) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{(ap^2 + bpq + cq^2)^s}, \quad \text{Re} \ s > 1. \]

In this section we shall obtain an integral expression for \( Z^*_Q(s)\Gamma(s)\Gamma(s-1/2)\pi^{1/2} \) and the analytic continuation of this zeta function.

**Lemma 1.** Let \( Y \) be the variable of a \( 2 \times 2 \) real symmetric matrix and \( G \) be a fixed \( 2 \times 2 \) positive definite symmetric matrix. Then we have, for \( \text{Re} \ s \geq 3/2 \),

\[ \int_{Y>0} (\det Y)^{s-3/2} e^{-\text{tr}(YG)} dY = (\det G)^{-s} \pi^{1/2} \Gamma(s) \Gamma \left( s - \frac{1}{2} \right). \]

Here \( \text{tr} X = \text{trace of } X \) for any matrix \( X \).

**Proof.** See p. 225 of [1].

**Proposition 1.** For \( \text{Re} \ s \geq 3/2 \), we have

\[ \Gamma(s)\Gamma \left( s - \frac{1}{2} \right) \pi^{1/2} (b - a - c)^s Z^*_Q(s) = \int_{Y>0} (\det Y)^{s-3/2} \frac{dY}{(e^{A(Y)} - 1)(e^{B(Y)} - 1)}, \]

where

\[ Y = \begin{bmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{bmatrix} \quad \text{and} \quad \begin{align*} A(Y) &= y_1 + \frac{a}{b - a - c} (y_1 + y_2 - 2y_{12}), \\ B(Y) &= y_2 + \frac{c}{b - a - c} (y_1 + y_2 - 2y_{12}). \end{align*} \]

**Proof.** Apply Lemma 1 with

\[ G = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} + \frac{ap + bq}{b - a - c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \]

\( p, q \) being positive integers.
we get, for \( \Re s \geq 3/2 \),
\[
\Gamma(s) \Gamma\left(s - \frac{1}{2}\right) \pi^{1/2} Z_Q^*(s) (b - a - c)^s = \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} \int_{Y>0} (\det Y)^{s-3/2} e^{-A(Y)p - B(Y)q} dY
\]
\[
= \int_{Y>0} (\det Y)^{s-3/2} \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} e^{-A(Y)p - B(Y)q} dY
\]
\[
= \int_{Y>0} (\det Y)^{s-3/2} \frac{dY}{(e^{A(Y)} - 1)(e^{B(Y)} - 1)}.
\]

Remark. Here the exchange of summation and integration is possible since the double series \( \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} e^{-A(Y)p - B(Y)q} \) is absolutely convergent and its partial sum is dominated by
\[
\frac{1}{(e^{A(Y)} - 1)(e^{B(Y)} - 1)}.
\]

Proposition 2. \( Z_Q^*(s) \) has an analytic continuation to the whole complex plane except a simple pole at \( s = 1/2 \). Furthermore, we have
\[
Z_Q^*(s) = 2\Gamma(1-s) \frac{e^{-\pi is}}{e^{2\pi is} + 1} \int_{L(\epsilon)} u^{2s-3} I(s, u) du
\]
where
\[
I(s, u) = \frac{1}{\Gamma(s-1/2)\pi^{1/2}} \int_0^1 (1-r^2)^{s-3/2} r dr \int_0^{2\pi} \frac{u^2 d\theta}{(e^{R(r, \theta) u} - 1)(e^{T(r, \theta) u} - 1)},
\]
\( L(\epsilon) \) is the contour in the complex plane consisting of the interval \( [\epsilon, + \infty) \) twice, in both directions (in and out) and the circle \( |z| = \epsilon \) in counterclockwise direction, and
\[
\begin{align*}
R(r, \theta) &= (1 + r \sin \theta) + \frac{2a}{b - a - c} (1 - r \cos \theta), \\
T(r, \theta) &= (1 - r \sin \theta) + \frac{2c}{b - a - c} (1 - r \cos \theta).
\end{align*}
\]

Proof. The first assertion was proved in [16]. Here we prove the integral expression from Proposition 1. By changing variables: \( u = (y_1 + y_2)/2 \), \( v = (y_1 - y_2)/2 \), \( w = y_{12} \), the integral expression for \( \Gamma(s)\Gamma(s-1/2)Z_Q^*(s)(b - a - c)^s \) is transformed into
\[
2 \int_{u^2 - v^2 - w^2 > 0, u > 0} (u^2 - v^2 - w^2)^{s-3/2} du dv dw
\]
where \( \delta_1 = 2a/(b - a - c) \) and \( \delta_2 = 2c/(b - a - c) \).

Let \( v = ux \), \( u = vy \) and then let \( x = r \cos \theta \), \( y = r \sin \theta \). It follows that
\[
\Gamma(s)\Gamma(s-1/2)\pi^{1/2} Z_Q^*(s)(b - a - c)^s
\]
\[
= 2 \int_0^\infty u^{2s-3} du \int_0^1 (1-r^2)^{s-3/2} r dr \int_0^{2\pi} \frac{u^2 d\theta}{(e^{R(r, \theta) u} - 1)(e^{T(r, \theta) u} - 1)}.
\]
As shown in [6], $I(s, u)$ has an analytic continuation which is a meromorphic function in $s$. The integration with respect to $u$ can be changed into a contour integral. Thus we have

$$\Gamma(s)Z^*_Q(s)(b - a - c)^s = 2(e^{4\pi is} - 1)^{-1} \int_{L(e)} u^{2s-3} I(s, u) \, du.$$ 

In light of the functional equation for the gamma function

$$\Gamma(s)\Gamma(1 - s) = \frac{2\pi i e^{\pi is}}{e^{2\pi is} - 1},$$

we then have

$$Z^*_Q(s)(b - a - c)^s = 2\Gamma(1 - s)\frac{e^{-\pi is}}{e^{2\pi is} + 1} \cdot \frac{1}{2\pi i} \int_{L(e)} u^{2s-3} I(s, u) \, du.$$ 

The contour integral is convergent for all $s$. Thus it defines the analytic continuation of $Z^*_Q(s)$.

**Proof of Theorem 1.** When $s = -m$ or $s = -m + \frac{1}{2}$ ($m > 0$), then $2s - 3$ is an integer. On the other hand, $I(s, u)$ is a holomorphic function in $u$. Consequently, the integrations along $[e, \infty)$ twice in opposite directions will cancel and the evaluation of the contour integral is reduced to the calculation of residues of $u^{2s-3} I(s, u)$ at $u = 0$ and $s = -m$ or $-m + \frac{1}{2}$.

Note that

$$\frac{u}{e^{R(r, \theta)u} - 1} = \frac{1}{R(r, \theta)} - \frac{1}{2} + \sum_{m=1}^{\infty} \frac{B_{2m}u^{2m} R(r, \theta)^{2m-1}}{(2m)!}, \quad |R(r, \theta)u| < 2\pi,$$

$$\frac{u}{e^{T(r, \theta)u} - 1} = \frac{1}{T(r, \theta)} - \frac{1}{2} + \sum_{m=1}^{\infty} \frac{B_{2m}u^{2m} T(r, \theta)^{2m-1}}{(2m)!}, \quad |T(r, \theta)u| < 2\pi.$$

By considering the coefficients of $u^{2m+2}$ ($s = -m$) and $u^{2m+1}$ ($s = -m + \frac{1}{2}$) in the power expansion of

$$\frac{u^2}{(e^{R(r, \theta)u} - 1)(e^{T(r, \theta)u} - 1)}$$

at $u = 0$, we get our assertion for $Z^*_Q(-m)$ and $Z^*_Q(-m + \frac{1}{2})$ as listed in Theorem 1 of the Introduction.

**Remark.** Here we use the following identities, which can be verified in an elementary way:

$$\lim_{s \to -m} 2\Gamma(1 - s)\frac{e^{-\pi is}}{e^{2\pi is} + 1} \cdot \frac{1}{\Gamma(s - \frac{1}{2}) \pi^{1/2}} = -\frac{(2m + 1)!}{2^{2m}} \cdot \frac{1}{2\pi},$$

$$\lim_{s \to -m + \frac{1}{2}} 2\Gamma(1 - s)\frac{e^{-\pi is}}{e^{2\pi is} + 1} \cdot \frac{1}{\Gamma(s - \frac{1}{2}) \pi^{1/2}} = -\frac{(2m)!}{2^{2m}\pi}.$$
3. The proof of theorem 2

The evaluation of \( N_1(-m) \) and \( N_2(-m + \frac{1}{2}) \) can be done by the same arguments as in [3]. However, as the values of \( Z_Q(s) \) at nonpositive integers were given in [16], it is unnecessary to compute \( N_1(-m) \) (though it is possible). Here we only compute the value of \( N_2(-m + \frac{1}{2}) \).

Proof of Theorem 2. Note that

\[
1 + r \sin \theta + \delta_1 (1 - r \cos \theta) = 1 + \delta_1 + \sqrt{1 + \delta_1^2 r \sin(\theta - \phi)}
\]

with \( \phi = \tan^{-1}(1/\delta_1) \). Hence

\[
\int_0^{2\pi} R(r, \theta)^{2m-1} d\theta = \int_0^{2\pi} [1 + \delta_1 + \sqrt{1 + \delta_1^2 r \sin(\theta - \phi)}]^{2m-1} d\theta
\]

\[
= \int_0^{2\pi} [1 + \delta_1 + \sqrt{1 + \delta_1^2 r \sin \theta}]^{2m-1} d\theta
\]

\[
= \sum_{l=0}^{m-1} \left( \frac{2m - 1}{2l} \right) (1 + \delta_1)^{2m-1-2l} (1 + \delta_1^2)^l \int_0^{2\pi} r^{2l} \sin^{2l} \theta d\theta.
\]

For sufficiently large \( s \), we have

\[
\int_0^{1} \int_0^{2\pi} (1 - r^2)^{s-3/2} r^{2l+1} \sin^2 \theta d\theta dr = \frac{[1 \cdot 3 \cdots (2l - 1)] 2\pi}{(2s - 1)(2s + 1) \cdots (2s - 1 + 2l)}.
\]

Thus the contribution from \( R(r, \theta)^{2m-1} \) to \((1/2\pi)N_2(-m)\) is given by

\[
\sum_{l=0}^{m-1} \left( \frac{2m - 1}{2l} \right) (1 + \delta_1)^{2m-1-2l} (1 + \delta_1^2)^l \frac{[1 \cdot 3 \cdots (2l - 1)]}{[(-2m)(-2m + 2) \cdots (-2m + 2l)]} \frac{[1 \cdot 3 \cdots (2l - 1)]}{[m(m-1) \cdots (m-l)](-2)^l}.
\]

In the same way, we get the contribution from \( T(r, \theta)^{2m-1} \) to \((1/2\pi)N_2(-m)\).

Corollary. Let \( m \) be a positive integer. Then

\[
Z_Q \left( -m + \frac{1}{2} \right) = -\frac{B_{2m}}{2^{2m}} \sum_{l=0}^{m-1} \left( \frac{2m - 1}{2l} \right) \frac{[1 \cdot 3 \cdots (2l - 1)]}{[m(m-1) \cdots (m-l)](-2)^l} \cdot \{(1 + \delta_1)^{2m-1-2l} (1 + \delta_1^2)^l \cdot (b-a-c)^{m-1/2} - \frac{1}{2} (a^{m-1/2} + c^{m-1/2}) \frac{B_{2m}}{2^{2m}} \}
\]

where \( \delta_1 = 2a/(b-a-c), \delta_2 = 2c/(b-a-c). \)
REFERENCES

3. Minking Eie and Chong-hsio Fang, On the residues and values of a zeta function at negative integers and negative half-integers, manuscript, 1987.
9. I. Satake, Special values of zeta functions associated with self-dual homogeneous cones, manuscript, 1981.

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