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APPROXIMATION OF ANALYTIC MULTIFUNCTIONS

ZBIGNIEW SLODKOWSKI

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ABSTRACT. Set-valued generalizations of analytic functions are defined by a form of local maximum principle. It is shown that they are identical with limits of decreasing sequences of multifunctions whose graphs are locally covered by graphs of single-valued analytic mappings.

INTRODUCTION

It is a natural question how to generalize the notion of an analytic function to the context of set-valued functions $z \to K(z)\colon G \to 2^{\mathbf{C}^n}$, where $G \subset \mathbf{C}^k$. In case $k = n = 1$, Oka [3] defined analytic set-valued functions (called here analytic multifunctions) by requiring that the complement of the graph of K in $G \times \mathbf{C}$ is pseudoconvex. Oka's definition does not have an obvious extension to the higher-dimensional case. The author proposed two generalizations, first in [4, Definition 5.1], and then a more restrictive one (and in case $k > 1$ actually different) in [6, §2], [9, Definition 0.2]. The existence of two definitions may raise some doubt whether any of them is the proper one. We show in this paper that analytic multifunctions in the latter sense (cf. Definition 2.1 below) can be approximated by a decreasing sequence of multifunctions which are locally union of graphs of single-valued analytic mappings (Theorem 1.3).

In addition to confirming the second definition, this result gives an insight into the structure of analytic multifunctions and allows for transparent explanation of many of their properties, previously available by other methods (see, §6).

1. RESULTS

Definition 1.1 [6, 9]. An usc compact-valued function $z \to K(z)\colon G \to 2^{\mathbf{C}^n}$, $G \subset \mathbf{C}^k$, is called analytic if for every hyperplane $L \subset \mathbf{C}^{k+n}$ with $\dim_{\mathbf{C}} L = n+1$, for every polynomial $p(z, w)$ and for every ball B, $\max(\operatorname{Re} p)|X \cap L \cap \overline{B} \le \max(\operatorname{Re} p)|X \cap L \cap \partial B$, provided $\overline{B} \cap X$ is compact, where $X = \operatorname{gr}(K) = \{(z, w)\colon z \in G, w \in K(z)\}$.

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Definition 1.2(a). We say that $K: G \to 2^{\mathbb{C}^n}$ is a *trivial analytic multifunction* if for every $z_0 \in G$, $w_0 \in K(z_0)$, there is an analytic mapping $f: G \to \mathbb{C}^n$, such that $f(z_0) = w_0$ and $f(z) \in K(z)$ for $z \in G$.
(b) We say that $K: G \to 2^{\mathbb{C}^n}$ is a *locally trivial analytic multifunction* if there is an open covering $\{G_t\}$ of G such that $K|G_t$ is a trivial analytic multifunction for every G_t.

We can formulate now the main result of this paper.

Theorem 1.3. *If $z \to K(z): G \to 2^{\mathbb{C}^n}$, $G \subset \mathbb{C}^k$, is an analytic multifunction, then there exists a countable covering $\{G_s\}_{s=1}$ of G, such that $G_s \subset G_{s+1}$, $s = 1, 2, \dots$, and a sequence of locally trivial analytic multifunctions $K_s: G_s \to 2^{\mathbb{C}^n}$, such that $K_{r+1}(z) \subset K_r(z)$ for $z \in G_s$ and $r \geq s$, and $\bigcap_{r \geq s} K_r(z) = K(z)$, $z \in G_s$.*

Remark 1.4. The last theorem can be improved if the multifunction $K(\cdot)$ is uniformly bounded (i.e., $\sup\{|w|: z \in G,\ w \in K(z)\} < +\infty$) and continuous (with respect to the Hausdorff topology in the space of all nonempty compact subsets of $\mathbb{C}^n$). Then, the conclusions of the last theorem are valid with $G_s = G$ for all s.

It is convenient to use the following terminology.

Terminology 1.5. We will say that a multifunction $K: G \to 2^{\mathbb{C}^n}$ has property (MA) on G, if it admits a sequence of locally trivial analytic multifunctions $K_s: G \to 2^{\mathbb{C}^n}$ such that $K_{s+1}(z) \subset K_s(z)$ and $\bigcap_s K_s(z) = K(z)$ for $z \in G$, $s = 1, 2, \dots$.

We will prove Theorem 1.3 in §5, after presenting, in §§2–4, auxiliary results on the (MA)-property and q-plurisubharmonic functions.

2. Application of polynomial hulls

Lemma 2.1. *Given $\varepsilon > 0$ and $0 < r < \varepsilon^2$, let*

$$X = \{(\zeta, w): |\zeta| = r,\ \operatorname{dist}(w, \{\sqrt{\zeta}, -\sqrt{\zeta}\}) < \varepsilon\}.$$

Denote by Y the polynomial hull of X and by $Y(z)$, $|z| \leq r$, the section $Y(z) = \{w: (z, w) \in Y\}$. Then $Y(0)$ is a circle with radius $R = R(\varepsilon, r)$ bigger than ε. Furthermore,

$$\{(0, w): |w| < R\} \subset \operatorname{Int}(Y). \tag{2.1}$$

Corollary 2.2. *Denote $L_0(z) = \{\sqrt{z}, -\sqrt{z}\}$, $z \in \mathbb{C}$. Then, the multifunction $z \to L_0(z): \mathbb{C} \to 2^{\mathbb{C}}$ has property (MA).*

Proof of Lemma 2.1. Since X is invariant with respect to the group of linear transformations $(z, w) \to (\alpha^2 z, \alpha w): \mathbb{C}^2 \to \mathbb{C}^2$, where $|\alpha| = 1$, so is Y. Thus

$Y(0)$ is preserved by rotations and, being polynomially convex, is a circle. Note, further, that the set $Z = \{(z, w): |z| < r, \operatorname{dist}(w, \{\sqrt{z}, -\sqrt{z}\}) \le \varepsilon\}$ has the local maximum property relative to polynomials. Since $\overline{Z}\backslash Z \subset X$, Y contains Z. Since $Z(0)$ (=the section of Z at $0) = D(0, \varepsilon)$, we get $\varepsilon \le R$. We will show that assuming $R = \varepsilon$ leads to contradiction.

Assertion. The relative boundary of $Y\backslash X$ in $D(0, r) \times \mathbb{C}$ is covered by graphs of bounded analytic functions defined in $D(0, r)$.

If $R = \varepsilon$, then $(0, \varepsilon) \in \partial Y$ and, by the assertion, there is $f \in H^\infty(D(0, r))$ such that $f(0) = \varepsilon$ and $(z, f(z)) \in \partial Y$ for $|z| < r$. Since $Y \supset Z$,

$$(2.2) \qquad |f(z) - \sqrt{z}| \ge \varepsilon, \qquad |f(z) + \sqrt{z}| \ge \varepsilon, \qquad \text{for } |z| < r,$$

and $h(z) = f(z)^2 - z$ satisfies $|h(z)| \ge \varepsilon^2 = h(0)$, $|z| < r$. Hence $h(z) = f(z)^2 - z = \varepsilon^2$, which together with (2.2) implies $|f(z) - \sqrt{z}| = \varepsilon$, $|f(z) + \sqrt{z}| = \varepsilon$, and $f(z) = (\varepsilon^2 + z)^{1/2}$, $|z| < r$. This is a contradiction.

To prove the inclusion (2.1), note first that $D(0, r) \times \overline{D}(0, r^{1/2}) \subset X$. For any analytic function f in $D(0, r)$ such that $\operatorname{gr} f \subset Y$, denote $Z_f = \{(z, w): |z| < r, w \in \operatorname{co}(\overline{D}(0, r^{1/2}) \cup \{f(z)\})$, where co=the convex hull. Since Z_f has the local maximum property, $Z_f \subset Y$. Fix now $R_1 < R$ and consider $\delta > 0$ (to be specified later). Choose a finite δ-net $\zeta_1, \dots, \zeta_m$ in $\partial D(0, R)$. By the assertion, there are analytic functions $f_1, \dots, f_m$ such that $f_j(0) = \zeta_j$ and $\operatorname{gr} f_j \subset Y$ for $j = 1, \dots, m$. One can show easily that if $\delta > 0$, $\eta > 0$ are chosen small enough, then $\bigcup_j X_{f_j}$ contains the set $D(0, \eta) \times D(0, R_1)$. This being true for any $R_1 < R$, inclusion (2.1) follows.

The assertion follows from the main result of Forstneric [2] by considering a sequence of $C^{(2)}$-smooth manifolds $M_n \subset \partial D(0, r) \times \mathbb{C}$ such that for each $|\zeta| = r$, the sections M_n^ζ, $n = 1, 2, \dots$, are simple closed curves whose polynomial hulls form a decreasing sequence converging to $Y(\zeta)$. By [2, Theorem 3(iii)], the assertion holds for $Y_n =$ the polynomial hull of M_n, and so it holds for Y. (Use Montel theorem and $Y = \bigcap_n Y_n$). Q.E.D.

Proof of Corollary 2.2. Fix $\varepsilon > 0$, $0 < r < \varepsilon^2$ and define Y as in the last lemma, with $Y(0) = \overline{D}(0, R)$. Let $K^\varepsilon(z) = \{w \in \mathbb{C}: \operatorname{dist}(w, \{z^{1/2}, -z^{1/2}\} \le \varepsilon\}$, $z \in \mathbb{C}$. Choose $\eta \in (\varepsilon/R, 1)$. Let $L^{\varepsilon, r}(z) = K^\varepsilon(z)$ for $|z| > r$ and $L^{\varepsilon, r}(z) = K^\varepsilon(z) \cup \{\eta w: w \in Y(z)\}$, for $|z| \le r$. Clearly, $L^{\varepsilon, r}: \mathbb{C} \to 2^{\mathbb{C}}$ is an usc compact-valued function; we will check that it is a locally trivial analytic multifunction. By (2.1), there is $\delta > 0$ such that $\operatorname{gr}(L^{\varepsilon, r}|D(0, \delta)) = Y \cap D(0, \delta) \times \mathbb{C}$, which is covered by graphs of analytic functions by Forstneric [2] (cf. the assertion in the previous proof), and so $L^{\varepsilon, r}|D(0, r)$ is trivial. We can choose $\rho \in (\delta, r)$ such that for $\delta < |z| \le \rho$, $\eta Y(z) \subset K^\varepsilon(z)$, and so $L^{\varepsilon, r}(z) = K^\varepsilon(z)$ for $|z| > \rho$. Clearly, K^ε, and so $L^{\varepsilon, r}$, are locally trivial for $|z| > \rho$. Finally, $L^{\varepsilon, r}(z) = K^\varepsilon(z) \cup \eta Y(z)$ for $\frac{1}{2}\delta < |z| < r$. By the above comments, both K^ε and $\eta Y(\cdot)$ are locally trivial on $\{\frac{1}{2}\delta < |z| < r\}$, and so is $L^{\varepsilon, r}$. We conclude that $z \to L^{\varepsilon, r}(z): \mathbb{C} \to 2^{\mathbb{C}}$ is a locally trivial analytic multifunction.

Denote now by $L_n: \mathbf{C} \to 2^{\mathbf{C}}$ the multifunction $L^{\varepsilon,r}$, constructed above for $\varepsilon = 2^{-n}$, and $r = r_n$, $\eta = \eta_n$ arbitrary, provided $r < \varepsilon^2$, $\eta R(r,\varepsilon) < \varepsilon$. Clearly, $L_0(z) = \bigcap_n L_n(z)$, $L_n(z) \supset L_{n+1}(z)$, $z \in \mathbf{C}$, $n = 1, 2, \ldots$ Q.E.D.

3. Limits of locally trivial analytic multifunctions

Proposition 3.1. (a) *If $K_j: G \to 2^{\mathbf{C}^n}$, $j = 1, \ldots, s$ are locally trivial analytic multifunctions (have (MA) property; cf. Terminology* 1.5), *then $K(z) = K_1(z) \cup \cdots \cup K_s(z)$ is a locally trivial analytic multifunction (has (MA) property).*

(b) If $K: G \to 2^{\mathbf{C}^n}$, $G \subset \mathbf{C}^k$, is a locally trivial analytic multifunction (has (MA) property) and $f: G^* \to G$ is an analytic mapping, then $z \to K(f(z)): G^* \to 2^{\mathbf{C}^n}$ is a locally trivial analytic multifunction (has (MA) property).

(c) If K is an in (b) *and $(z, w) \to F_z(w)$ is a $\mathbf{C}^m$-valued analytic map defined near* $\operatorname{gr}(K)$, *then $z \to F_z(K(z)): G \to 2^{\mathbf{C}^m}$ is a locally trivial analytic multifunction (has (MA) property).*

(d) If $(z, \xi) \to K(z, \xi): G \times G^* \to 2^{\mathbf{C}^n}$ is a locally trivial analytic multifunction (has (MA) property), and $F \subset G^*$ is compact, then $z \to \bigcup_{\xi \in F} K(z, \xi): G \to 2^{\mathbf{C}^n}$ is a locally trivial analytic multifunction (has (MA) property).

Proof. It is obvious that these operations preserve the class of trivial analytic multifunctions and commute with localization (restriction) and limits of decreasing sequences (as in Terminology 1.5). Q.E.D.

Lemma 3.2. *Let $z \to L(z): G \to 2^{\mathbf{C}^n}$ be an usc compact-valued multifunction. Assume that for every $a \in G$ there is a multifunction $L_a: B(a) \to 2^{\mathbf{C}^n}$ with (MA) property, such that* $\operatorname{gr}(L_a|B(a)\setminus\{a\}) \subset \operatorname{Int}\operatorname{gr}(L|B(a)\setminus\{a\})$. *Then, for every neighborhood W of* $\operatorname{gr}(L)$ *there is a locally trivial analytic multifunction $\widetilde{L}: G \to 2^{\mathbf{C}^n}$ such that $L(z) \subset \operatorname{Int}\widetilde{L}(z)$, for $z \in G$, and $\operatorname{gr}\widetilde{L} \subset W$.*

Proof. We can assume that $B(a)$ is an Euclidean ball $B(a, r)$ with $r < \operatorname{dist}(a, \partial G)$, and $L_a(z)$ is defined in a neighborhood of $\overline{B(a)}$. By the (MA) property, there is a locally trivial analytic multifunction L_n, defined near $\overline{B(a)}$, such that $\operatorname{gr} L_n|\partial B(a) \subset \operatorname{Int}(L|\partial B(a))$, $L(a) \subset L_n(a)$, and $\operatorname{gr} L_n|\overline{B(a)} \subset W$. Hence, there is $\varepsilon > 0$, such that $\{(\zeta, w): \zeta \in \partial B(a),\ \operatorname{dist}(w, L_n(\zeta)) \le \varepsilon\} \subset \operatorname{Int}(L|\partial B(a))$ and $\{(z, w): z \in \overline{B(a)},\ \operatorname{dist}(w, L_n(z)) \le \varepsilon\} \subset W$. By the continuity of $z \to L_n(z)$, the set $\{(z, w): z \in B(a),\ \operatorname{dist}(w, L_n(\zeta)) < \varepsilon\}$ is open, and so it contains $\{z\} \times L_a(z)$ for z in some open neighborhood $H(a)$ of a, $H(a) \subset G$.

Let $L_a^*(z) = \{w \in \mathbf{C}^n: \operatorname{dist}(w, L_n(z)) \le \varepsilon\}$ for $z \in \overline{B(a)}$. Clearly, $L_a^*: \overline{B(a)} \to 2^{\mathbf{C}^n}$ is compact-valued and usc, and $L_a^*|B(a)$ is a locally trivial analytic multifunction. By our construction,

$$L(z) \subset L_a^*(z), \qquad z \in H(a), \tag{3.1}$$

$$\operatorname{gr}(L_a^*|\partial B(a)) \subset \operatorname{gr} L, \qquad \operatorname{gr} L_a^* \subset W. \tag{3.2}$$

Seeing that $H(a) \subset B(a)$ and $\bigcup_a H(a) = G$, we can select a countable covering $\{H(a(n))\}_{n=1}^{\infty}$ from $\{H(a)\}$, so that $\{\overline{B(a(n))}\}_{n=1}^{\infty}$ is a locally finite family of compact subsets of G. The argument is identical with that in [11, Proof of Lemma 4.7]. Hence, the compact sets $\operatorname{gr}(L^*_{a(n)}\overline{B(a(n))})$, $n = 1, 2 \ldots$, form a locally finite family relative to $G \times \mathbb{C}^n$, and so their union, which contains $\operatorname{gr} L$, by (3.1) is the graph of an usc, compact-valued multifunction, say $\widetilde{L}\colon G \to 2^{\mathbb{C}^n}$, such that $L(z) \subset \widetilde{L}(z)$, $z \in G$. By (3.2), $\operatorname{gr} \widetilde{L} \subset W$.

Finally, $\widetilde{L}$ is a locally trivial analytic multifunction. Indeed, for $z_0 \in G$ let $M = \{n\colon z_0 \in B(a(n))\}$. Then, M is a finite set and $H = \bigcap_{n \in M} H(a(n))$ is open. By (3.2), $\widetilde{L}(z) = \bigcup_{n \in M} L^*_{a(n)}(z)$, for $z \in H$. By Proposition 3.1(a), $\widetilde{L}|H$ is locally trivial. Q.E.D.

4. Application of q-plurisubharmonic functions

Lemma 4.1. *Let $\varphi(z, w)$ be a continuous $(n-1)$-plurisubharmonic function in an $(n-1)$-pseudoconvex open set $U \subset \mathbb{C}^k \times \mathbb{C}^n$ (cf. [7, Definition 4.1]). Fix $\varepsilon > 0$, $\chi > 0$, $\delta > 0$. Then, there is a continuous function $u\colon U \to R$, such that*

$$(4.1) \qquad \varphi(x) + \varepsilon|x|^2 < u(x) < \varphi(x) + (\varepsilon + \chi)|x|^2 + \delta, \qquad x \in U,$$

and for every $p \in U$, there is a ball $B(p) = \{|x - p| < r\}$, with $r < \operatorname{dist}(p, \partial u)$, and a continuous function $u_p(x)$ on $\overline{B(p)}$ with the following properties.

(i) *If $p = (a, b) \in U$, then $u_p = u$ on $\overline{B(p)} \cap \{a\} \times \mathbb{C}^n$, while $u_p < u$ on $B(p) \backslash \{a\} \times \mathbb{C}^n$.*

(ii) *Every u_p, $p \in U$, has the representation $u_p(x) = \max(f_1(x), \ldots, f_n(x))$, $x \in \overline{B(p)}$, $m = m(p)$, where each f_j is of the form*

$$(4.2) \qquad f(x) = \min(\operatorname{Re} l_1(x), \ldots, \operatorname{Re} l_{n-1}(x), \operatorname{Re} h(x)), \qquad x \in \mathbb{C}^{k+n},$$

where $l_1(z, w), \ldots, l_{n-1}(z, w)$ are $\mathbb{C}$-affine functions and $h(z, w)$ is a second-degree polynomial which satisfy the next condition:

($*$) *for every set of constants $\eta_1, \eta_2, \ldots, \eta_n$, and for every $z \in \mathbb{C}^k$, the systems of equations*

$$(4.3) \qquad l_i(z, w) = \eta_i, \qquad i = 1, 2, \ldots, n-1, \quad h(z, w) = \eta_n,$$

has at most two solutions w.

Proof (Sketch). This part of the argument relies heavily on the results and terminology of [11, §4]. Denote by $\widetilde{F}$ the class of all functions of the form (4.2) with $h(z, w)$ polynomial of degree two and $l_1(z, w), \ldots, l_{n-1}(z, w)$ complex affine functions. Denote by F the subset of those f in $\widetilde{F}$ for which functions $l_1, \ldots, l_{n-1}$, h satisfy condition ($*$).

Assertion 1. $\widetilde{F} = \overline{F}$, the closure of F in the uniform convergence on compact sets.

Assertion 2. F^d, the dual class of functions to F in the sense of [10, Definition 1.11] is equal to the class of all $(n-1)$-plurisubharmonic functions on $\mathbb{C}^{k+n}$.

The proof of Assertion 1 is elementary. The proof of Assertion 2 is similar to those of [11, Remark 5.1 and Theorem 5.8] and is omitted. The assertions mean that F and the class P consisting of all $(n-1)$-plurisubharmonic functions on $\mathbb{C}^{k+n}$ satisfy all the assumptions of [11, Lemma 4.3 and Corollary 4.8]. Applying the last fact to the continuous function φ, we obtain that there is a continuous (and $(n-1)$-plurisubharmonic) function v on U, such that $\varphi(x)+\varepsilon|x|^2 < v(x) < \varphi(x)+\varepsilon|x|^2+\delta$, $x \in U$, such that for every $p \in U$ there is a ball $\overline{B(p)} \subset U$ such that $v|B(p) = \max(f_{1_2}, \dots, f_m)$, $m = m(x)$, where $f_1, \dots, f_{m(x)} \in F$. Define $u(z,w) = v(z,w) + \chi|z|^2$, $(z,w) \in U$, and, for $p = (a,b) \in U$ and $(z,w) \in B(p)$, let $u_p(z,w) = v(z,w) + \chi|z|^2 - \chi|z-a|^2$. Clearly, condition (i) holds. Since $u_p(z,w) = v(z,w) + \operatorname{Re} l(z)$, $(z,w) \in B(p)$, where $l(z) = 2\chi a^t z - \chi|a|^2$, we get that $u_p = \max(f_1 + \operatorname{Re} l, \dots, f_m + \operatorname{Re} l)$ in $B(p)$. Since l depends only on z, the representation (4.2) for $f_j + \operatorname{Re} l$ still satisfies $(*)$, and so condition (ii) holds. Q.E.D.

5. Proof of Theorem 1.3

Outline of proof. The function $x \to -\log \operatorname{dist}(x, \operatorname{gr} K)$ is $(n-1)$-plurisubharmonic near $\operatorname{gr}(K)$ and, by lemma 4.1, it can be approximated by a special $(n-1)$-plurisubharmonic function $u(z,w)$. The fact that this u is made up from simple polynomials $l_1, \dots, l_{n-1}, h$ satisfying conditions (i) and (ii) of Lemma 4.1, will allow us to approximate the sets $\operatorname{gr} K \cup \{u \geq C\}$ by locally trivial analytic multifunctions which are constructed, with use of Lemma 3.2, from simple multifunctions with (MA) property. The latter are produced from $l_1, \dots, l_{n-1}, h$ as in the next lemma.

Lemma 5.1. *Let $l_1, \dots, l_{n-1}, h$ satisfy condition* (ii) *of Lemma* 4.1 *and $F \subset \mathbb{C}^n$ be compact. Let $K(z) = \{w \in \mathbb{C}^n : (l_1(z,w), \dots, l_{n-1}(z,w), h(z,w)) \in F\}$. Then, the multifunction $z \to K(z) : \mathbb{C}^k \to 2^{\mathbb{C}^n}$ has the (MA) property.*

Proof of Theorem 1.3.

Assertion. Let U_1 be a neighborhood of $\operatorname{gr}(K)$ in $G \times \mathbb{C}^n$ and $G^* \subset G$ be bounded and open. Let $U = U_1 \setminus \operatorname{gr}(K)$. Assume that $u : U \to R$ satisfies conditions (i) and (ii) of Lemma 4.1, and $\lim_{x \to x_0} u(x) = +\infty$ for $x_0 \in \partial(\operatorname{gr} K)$, $x \in U$. Assume that C is such that the set $X = (\{x \in U : u(x) \geq C\} \cup \operatorname{gr} K) \cap G^* \times \mathbb{C}^n$ is bounded and relatively closed in $G^* \times \mathbb{C}^n$. Then, for every neighborhood W of X there is a locally trivial analytic multifunction $z \to \widetilde{L}(z) : G^* \to 2^{\mathbb{C}^n}$, such that $X \subset \operatorname{gr}(\widetilde{L}) \subset \mathrm{W}$.

We show first, how this implies Theorem 1.3. Since $\operatorname{gr} K$ has the local maximum property of order $(k-1)$ (cf. [7, Definition 2.1]), its complement in $G \times \mathbb{C}^n$ is $(n-1)$-pseudoconvex relative to $G \times \mathbb{C}^n$, by [7, Theorem 4.2]. In other words (cf. [7, Definition 4.1]), there is an open set U_1 with $\operatorname{gr} K \subset U_1 \subset G \times \mathbb{C}^n$, such that the function $\varphi(x) = -\log \operatorname{dist}(x, \operatorname{gr} K)$ is $(n-1)$-plurisubharmonic in the set $U = U_1 \setminus \operatorname{gr}(K)$. Since φ is continuous, by Lemma 4.1 there is a function $u \colon U \to R$, such that

$$(5.1) \qquad \varphi(x) + \tfrac{1}{2}|x|^2 < u(x) < \varphi(x) + |x|^2 + 1, \qquad x \in U,$$

and $u(z, w)$ satisfies conditions (i), and (ii) of Lemma 4.1.

Choose a countable monotone covering $\{G_s\}_{s=1}$ of G, such that $\overline{G}_s \subset G_{s+1}$ and $\overline{G}_s$ are compact. We can assume without loss of generality that $(G_s \times \mathbb{C}^n) \subset G_s \times B(0, R_s)$, $R_s < +\infty$. By (5.1), we get

$$(5.2) \qquad \text{if } u(x) \geq C, \quad \text{then } \operatorname{dist}(x, \operatorname{gr} K) \leq e^{-C} e^{|x|^2+1}.$$

Now, for $x \in (G_s \times \mathbb{C}^n) \cap U$ the term $|x|^2$ is uniformly bounded, Hence, we can choose a sequence $C_s \nearrow +\infty$, such that for every $s = 1, 2 \ldots$, the set $X_s = \operatorname{gr}(K|G_s) \cup \{(z, w) \in U \colon z \in G_s,\ u(z, w) \geq C_s\}$ is closed in $G_s \times \mathbb{C}^n$. (Note that, by (5.1), $\lim u(x) = +\infty$ as $U \ni x \to x_0 \in \operatorname{gr} K$.) By the same reason

$$(5.3) \qquad \bigcap_{r \geq s} X_r \cap G_s \times \mathbb{C}^n = \operatorname{gr} K|G_s, \qquad s \geq 1.$$

Denote $W_s = \operatorname{gr}(K) \cup \{(z, w) \in U \colon u(z, w) > C_s\}$, $s = 1, 2, \ldots$ and $W_0 = U$. Then,

$$(5.4) \qquad W_s \cap (G_s \times \mathbb{C}^n) \subset X_s \subset W_{s-1}, \qquad s \geq 1.$$

By the assertion, for every $s \geq 1$, there is a locally trivial analytic multifunction $K_s \colon G_s \to 2^{\mathbb{C}^n}$, such that $X_s \subset \operatorname{gr}(K_s) \subset W_{s-1}$, $s \geq 1$. This and (5.4) imply that $K_{r+1}(z) \subset K_r(z)$, for $z \in G_s$ and $r \geq s$. By (5.3) and (5.4), $\bigcap_{r \geq s} K_r(z) = K(z)$, $z \in G_s$, as required.

Concerning the assertion, note first that X is the graph of some compact-valued multifunction $L \colon G^* \to 2^{\mathbb{C}^n}$. We will prove the assertion by applying Lemma 3.2 to L; that is for each $a \in G^*$ we will construct a neighborhood $B(a)$ and a minorant multifunction $L_a \colon B(a) \to 2^{\mathbb{C}^n}$ satisfying assumptions of Lemma 3.2.

Consider an arbitrary $b \in \{w \in L(a) \colon u(a, w) = C\}$. Denote $p = (a, b)$, and let $B = B(p)$ be a ball and $u_p = \max(f_1, \ldots, f_m)$ be a function in B, satisfying conditions (i) and (ii) of Lemma 4.1. We can assume without loss of generality that $B \cap \operatorname{gr} K = \varnothing$ and $B \subset U$. Clearly,

$$(5.5) \qquad \{u_p \geq C\} \cap B = \bigcap_{j=1}^{m} \{f_j \geq C\} \cap B.$$

In turn, if $f_j = \min(\operatorname{Re} l_1, \ldots, \operatorname{Re} l_{n-1}, \operatorname{Re} h)$ as in Lemma 4.1(ii), then $\{f_j \geq C\} \cap B = B \cap \bigcap_{i=1}^{n-1} \{\operatorname{Re} l_i \geq C\} \cap \} \operatorname{Re} h \geq C\}$. (We suppress in the notation the dependence of l_i, and h on j and b.) We will now select a part of the latter set which is the graph of an analytic multifunction $L^{p,j} \colon B_{p,j} \to 2^{\mathbb{C}^n}$, where $a \in B_{p,j} \subset G^*$. (From these $L_{p,j}$'s the minorant L_a will be eventually constructed.) We distinguish two cases.

Case 1. The system of equations

$$(5.6)\quad l_1(a,w) = l_1(a,b), \dots, l_{n-1}(a,w) = l_{n-1}(a,b), \qquad h(a,w) = h(a,b)$$

has two distinct solutions in w, namely b and b^*. Clearly, there exist a neighborhood $B_{p,j}$ of a and disjoint neighborhoods V and V^* of b and b^*, respectively, and a compact neighborhood F_0 of b, such that $F_0 \subset V$, $B_{p,j} \times (V \cup V^*) \subset B_p$ and for every $\beta \in F_0$ and every $z \in B_{p,j}$ the system

$$(5.7)\quad l_1(z,w) = l_1(a,\beta), \dots, l_{n-1}(z,w) = l_{n-1}(a,\beta), \quad h(z,w) = h(a,\beta)$$

has exactly two solutions w, one in V and one in V^*.

Case 2. The system (5.6) has only one solution $w = b$. Then, there exist: a neighborhood $B_{p,j}$ of a, a compact neighborhood F_0 of b and an open neighborhood V of b, such that $F_0 \subset V$ and $B_{p,j} \times V \subset B_p$, and for every $\beta \in F_0$ and for every $z \in B_{p,j}$ all solutions w of the system (5.7) belong to V. For conformity of notation with the Case 1, we let $V^* = \emptyset$ in Case 2.

In either case, we denote $F = F_0 \cap \{w \in \mathbb{C}^n : f_j(a,w) \geq C\}$. For $z \in B_{p,j}$ let $L^{p,j}(z)$ to be the set of all solutions w of (5.7) corresponding to $\beta \in F$. Since F is compact, the construction implies easily that $z \to L^{p,j}(z) : B_{p,j} \to 2^{\mathbb{C}^n}$ is an usc compact-valued multifunction. In Case 1, it is obvious that $L^{p,j}$ is a trivial analytic multifunction. In Case 2, multifunction $L^{p,j}(\cdot)$ has (MA) property by Lemma 5.1.

With still fixed $p = (a,b)$, define now $L^p(z) = \bigcup_{j=1}^{m(p)} L^{p,j}(z)$ for $z \in B_p = \bigcap_{j=1}^{m(p)} B_{p,j}$. By Proposition 3.1(a), $z \to L^p(z) : B_p \to 2^{\mathbb{C}^n}$ has the (MA) property. By (5.5), the section $L^p(a)$ is a neighborhood of b relative to $L(a)$. Since b varies through the compact set $\{w : u(a,w) = C\}$, we can choose a finite family $\{L^p(a) : p \in R\}$, where $R = \{(a,b_1), \dots, (a,b_r)\}$, which covers the set $\{w \in \mathbb{C}^n : C \leq u(a,w) < C + \varepsilon\}$, for some $\varepsilon > 0$. Denote $F^* = K(a) \cup \{w \in \mathbb{C}^n : C + \frac{1}{2}\varepsilon \leq u(a,w)\}$; clearly, F^* is compact and there is a neighborhood B^* of a in $\mathbb{C}^k$, such that $F^* \subset K(z) \cup \{w \in \mathbb{C}^n : u_p(z,w) > C\}$ for every $z \in B^*$. Let, finally, $B(a) = B^* \cap \bigcap_{p \in R} B(p)$ and $L_a(z) = F^* \cup \bigcup_{p \in R} L^p(z)$, $z \in B(a)$. By our construction, $L_a(a) = L(a)$. By Proposition 3.1(a), the usc multifunction $z \to L_a(z) : B(a) \to 2^{\mathbb{C}^n}$ has (MA) property. Finally, $\operatorname{gr} L_a \subset \operatorname{gr} K|B(a) \cup \{(z,w) : u_p(z,w) \geq C\}$. Since $u_p(z,w) < u(z,w)$ for $z \neq a$ (cf. Lemma 4.1(i)), we get $\operatorname{gr} L_a|B(a)\backslash\{a\} \subset \operatorname{Int}(\operatorname{gr} L|B(a)\backslash\{a\})$, as required in Lemma 3.2, which completes the proof of the assertion. Q.E.D.

Proof of Lemma 5.1 (Sketch). The assumptions on $l_1, l_2, \dots, l_{n-1}, h$ imply that there is an additional $\mathbb{C}$-affine function $l(z,w)$, such that $(z_1, z_2, \dots, z_k, \xi_1, \dots, \xi_{n-1}, \zeta)$ with $\xi_j = l_j(z,w)$, $\zeta = l(z,w)$ form a biholomorphically equivalent coordinate system on $\mathbb{C}^k \times \mathbb{C}^n$. If the representation of the polynomial $h(z, w)$ in the new coordinates is $h_0\zeta^2 + 2\zeta h_1(z, \xi') + h_2(z, \xi')$, where $\xi' = (\xi_1, \dots, \xi_{n-1})$, then the solution to the system

$$(5.8) \qquad (l_1(z, w), \dots, l_{n-1}(z, w), h(z, w)) = (\eta_1, \dots, \eta_n)$$

has the representation $\xi_1 = \eta_1, \dots, \xi_{n-1} = \eta_{n-1}$, and the set of values of ζ is $L(z, \eta) = -h_0^{-1}h_1(z, \eta') + L_0(\Delta(z, \eta))$, with $\Delta(z, \eta) = h_0^{-2}h_1(z, \eta')^2 - h_0^{-1}(h_2(z, \eta') - \eta_n)$, where L_0 is the multifunction with (MA) property from Corollary 2.2. Applying Corollary 2.2 and Proposition 3.1(b) and (c), we obtain that the multifunction $L\colon \mathbb{C}^{2n} \to 2^{\mathbb{C}}$ has (MA) property. Returning to original coordinates, which amounts to applying Proposition 3.1(b) and (c) once again, we get that $K'\colon C^{2n} \to 2^{\mathbb{C}^n}$ has (MA) property, where $K'(z, \eta) =$ the set of w satisfying (5.8). Hence, $K(z) = \bigcup_{\eta\in F} K'(z, \eta)$ has (MA) property by Proposition 3.2(d). Q.E.D.

6. Concluding remarks

A. *Further properties of analytic multifunctions.* With Theorem 1.3 proven, we can conclude now that properties (a), (b), (c), and (d) of Proposition 3.1 hold for arbitrary analytic multifunctions. In the same way, we can easily obtain the following properties (starting with the obvious observation that they are satisfied by trivial analytic multifunctions).

(a) If $K\colon G \to 2^{\mathbb{C}^n}$ and $L\colon G \to 2^{\mathbb{C}^m}$ are analytic multifunctions, then $z \to K(z) \times L(z)\colon G \to 2^{\mathbb{C}^{m+n}}$ is analytic multifunction.

(b) If $K\colon G \to 2^H$, and $L\colon H \to 2^{\mathbb{C}^s}$ are analytic multifunctions, then so is their composition $L \circ H\colon G \to 2^{\mathbb{C}^s}$, where $(L \circ K(z) = \bigcup\{L(w)\colon w \in K(z)\}$.

(c) If $K\colon G \to 2^{\mathbb{C}^n}$ is an analytic multifunction, then $z \to \operatorname{co}(z))\colon G \to 2^{\mathbb{C}^n}$ is an analytic multifunction, where co = the convex hull.

(d) If $\psi(z, w)$ is an r-plurisubharmonic function, $r \geq 0$, and $K\colon G \to 2^{\mathbb{C}^n}$ is an analytic multifunction, then $\varphi(z) = \max\{\psi(z, w)\colon w \in K(z)\}$ is r-plurisubharmonic on the set $\{z\colon \{z\} \times K(z) \subset \operatorname{Dom}\psi\}$.

(e) If $K\colon G \to 2^{\mathbb{C}^n}$ is an analytic multifunction, then

$$\varphi(z, w) = -\log \operatorname{dist}(w, K(z))$$

is an $(n-1)$-plurisubharmonic function in $G \times \mathbb{C}^n \setminus \operatorname{gr}(K)$.

Remark. The above properties can be also obtained by applying results on k-maximum sets, cf. [7].

B. *Proof of Remark* 1.4 (Sketch). The method is essentially the same as in the case of Theorem 1.3, so we only indicate the necessary modifications. The assertion remains unchanged; it is, however, applied now to function $\varphi(z, w)$ as defined in property (e) above, because $\varphi(z, w)$ is continuous, due to the continuity of $z \to K(z)$. This and the uniform boundedness of $K(z)$ allows us to use $G_s = G$, $s = 1, 2, \dots$ in the proof of Theorem 1.3. Q.E.D.

C. The use of Forstneric result [2] in the proof of Lemma 2.1 shortens the argument but is not essential. One can also apply the theorem of Alexander and Wermer [1] and the author [8] about hulls with convex fibers to the image of the set Y under the map $(z, w) \to (z, w^{-1})$ (plus the zero section $D(0, r) \times \{0\}$), which has, indeed, convex vertical sections. (Some additional argument is still needed.)

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Note added July 17, 1988. The referee has pointed out that a result closely related to the case $k = n = 1$ of Theorem 1.3 above was stated by E. Vesentini in [13], pp. 66 (bottom) and 67 (top). Vesentini asserts that, for dimensions $k = n = 1$, decreasing approximations K_r (like in Theorem 3.1) can be found so that graphs of multifunctions K_r satisfy a certain condition (L). E. Vesentini credits H. Yamaguchi [14, p. 420–421] with this theorem. However, according to Vesentini, a set $F \subset \mathbb{C}^2$ has the property (L) if it is covered (locally) by graphs of single-valued analytic functions, while Yamaguchi defines that F has the property (L) if it is covered by open pieces of nonsingular analytic varieties. If we are right in assuming that Yamaguchi bases himself on Oka [12, Lemma II, p. 171], only the latter version seems to be actually proved. Since nonsingular analytic variety in C^2 is locally given by one of the equations $w = f(z)$ or $z = g(w)$, the result is sufficient for Yamaguchi and Vesentini's applications (see [5] for a more general situation of this kind), but it does not *directly* imply the full strength of Theorem 1.3 of the present paper.

Department of Mathematics, University of Illinois at Chicago, Chicago, Illinois 60680