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THERE CAN BE C^*-EMBEDDED DENSE PROPER SUBSPACES IN $\beta\omega - \omega$

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ABSTRACT. Fine and Gillman have shown that CH implies that if X is a dense proper subspace of $\omega^* = \beta\omega - \omega$, then $\beta X \neq \omega^*$. Here it is shown to be consistent with MA $+\mathfrak{c} = \omega_2$ that for every $p \in \omega^*$ we have $\beta(\omega^* - \{p\}) = \omega^*$ and also that ω^* has a dense subspace X with dense complement such that $\beta X = \omega^*$.

0. INTRODUCTION

All spaces are Tychonoff and X^* *denotes* $\beta X - X$. Fine and Gillman [FG, 4.3] proved that CH implies that for each $p \in \omega^*$, $\beta(\omega^* - \{p\}) \neq \omega^*$. An easy modification of their proof yields that for all $p \in \omega^*$, if p has character ω_1 in ω^* then $\beta(\omega^* - \{p\}) \neq \omega^*$. This is a more general result since it is consistent with $\neg$CH that some $p \in \omega^*$ has character ω_1 in ω^* [K_1, remarks on p. 303] or [K_3].

The purpose of this paper is, among other things, to show that the statement "$\beta(\omega^* - \{p\}) = \omega^*$ for every $p \in \omega^*$" is consistent with MA $+\mathfrak{c} = \omega_2$. This shows that in the Fine and Gillman result CH is essential, which answers a question in [G], and that it cannot be weakened to MA.

Our proof depends on a result of Kunen concerning the nonexistence of certain gaps in $\mathscr{P}(\omega)/\text{fin}$, see [B] for details, and on a result showing that Hausdorff gaps in $\mathscr{P}(\omega)/\text{fin}$ cannot be "small" under MA (see §2).

1. CONVENTIONS AND DEFINITIONS

Cardinals are initial (von Neumann) ordinals, and get the discrete topology. We use κ^* to denote $\beta\kappa - \kappa$, the space of free ultrafilters on κ. Also, $\subset$ denotes *proper* inclusion.

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Let X be a space. As usual, $C^*(X)$ denotes the set of all continuous bounded real-valued functions on X. A subspace $Y \subseteq X$ is said to be *C^*-embedded* in X provided that every $f \in C^*(Y)$ extends to some $\mathbf{f} \in C^*(X)$.

Let X be a space and let κ be a cardinal. A subset $A \subseteq X$ is a *P_κ-set in* X provided that the intersection of fewer than κ neighborhoods of A is again a neighborhood of A. If for some $x \in X$, $\{x\}$ is a P_κ-set for $\kappa = \omega_1$, then we say that x is a *P-point* of X.

Let a and b be subsets of ω. We say that a is *almost contained* in b, $a \subseteq^* b$, if $|a - b| < \omega$. Two families A and B of subsets of ω are *orthogonal*, $A \perp B$, if for every $a \in A$ and $b \in B$ we have $|a \cap b| < \omega$. Let A and B be families of subsets of ω such that $A \perp B$. We say that A and B can be *separated* if there is a subset d of ω such that

$$\text{for every } a \in A,\ a \subseteq^* d, \quad \text{and} \quad \text{for every } b \in B,\ b \subseteq^* \omega - d.$$

If A is a family of subsets of ω and $x \subseteq \omega$ then $A \restriction x$ denotes the family $\{a \cap x : a \in A\}$.

We assume that the reader is familiar with the standard partial order terminology concerning proofs involving Martin's Axiom (abbreviated MA), see e.g. [K_3] or [R].

2. Gaps in $\mathscr{P}(\omega)/\mathrm{fin}$

We are interested in statements of the following form, where κ and λ are infinite cardinals

$G(\kappa, \lambda)$: there are a κ-sequence $\langle U_\xi : \xi < \kappa \rangle$ of clopen sets in ω^* and a λ-sequence $\langle V_\xi : \xi < \lambda \rangle$ of clopen sets in ω^* such that

(1) $U_\xi \subset U_\eta$ if $\xi < \eta < \kappa$,
(2) $V_\xi \subset V_\eta$ if $\xi < \eta < \lambda$,
(3) $(\bigcup_{\xi<\kappa} U_\xi) \cap (\bigcup_{\xi<\lambda} V_\xi) = \varnothing$,
(4) $(\bigcup_{\xi<\kappa} U_\xi)^- \cap (\bigcup_{\xi<\lambda} V_\xi)^- \neq \varnothing$.

This has a straightforward translation in terms of the existence of certain families of subsets in ω, and in terms of the existence of certain sequences in the Boolean algebra $\mathscr{P}(\omega)/\mathrm{fin}$, which we leave to the reader.

Two classical results of Hausdorff, [Ha], are that $G(\omega, \omega)$ is false but $G(\omega_1, \omega_1)$ is true. It is well known, and easy to prove, that the following holds.

2.1. **Proposition.** MA *implies* $\neg G(\kappa, \omega)$ *for each* $\kappa < \mathfrak{c}$.

Kunen has extended 2.1 by proving

2.2. **Theorem.** MA *implies that if* κ *and* λ *are regular cardinals and* $\omega \le \kappa$, $\lambda < \mathfrak{c}$, *then* $G(\kappa, \lambda)$ *holds iff* $\kappa = \lambda = \omega_1$.

He has also shown that MA $+ \neg$ CH gives no information about $G(\omega_1, \mathfrak{c})$ and $G(\mathfrak{c}, \mathfrak{c})$, by proving the next result.

2.3. **Theorem.** (A) *It is consistent with* MA (*and, necessarily,* $\neg$CH) *that* $G(\omega_1, \mathfrak{c})$ *and* $G(\mathfrak{c}, \mathfrak{c})$ *both are false.*

(B) *It is consistent with* MA+$\neg$CH *that* $G(\omega_1, \mathfrak{c})$ *and* $G(\mathfrak{c}, \mathfrak{c})$ *both are true.*

For the proofs of Theorems 2.2 and 2.3, see Baumgartner [B]. Let us also remark that *PFA* implies (A) (and $\mathfrak{c} = \omega_2$) but both (A) and (B) are consistent with $\mathfrak{c}$ being any uncountable regular cardinal.

We finish this section by discussing a definition that will be useful later. Let $SG(\kappa, \lambda)$ (S = strong) be the strengthening of $G(\kappa, \lambda)$ one obtains by replacing (4) of this section by

(4_S) $|(\bigcup_{\xi<\kappa} U_\xi)^- \cap (\bigcup_{\xi<\lambda} V_\xi)^-| = 1$.

2.4. **Theorem.** MA + $\neg$CH *implies that* $SG(\omega_1, \omega_1)$ *is false.*

Before proving this, we translate it into combinatorics. If $A = \{a_\xi : \xi < \kappa\}$ is a family of subsets of ω, we call A a κ*-tower* iff $a_\xi \subseteq^* a_\eta$ whenever $\xi \le \eta$. A *Hausdorff gap* is a pair (A, B) of ω_1-towers such that $A \perp B$ and A and B cannot be separated. To prove the theorem we must show that whenever (A, B) is a gap, we can find an $x \subseteq \omega$ such that $(A \restriction x, B \restriction x)$ and $(A \restriction (\omega \backslash x), B \restriction (\omega \backslash x))$ are both Hausdorff gaps.

We remark that our definition of tower did not imply that the a_ξ are strictly increasing modulo finite sets, or even that they are infinite; however, if (A, B) is a gap, these things must hold for some cofinal subsequence of A and B. Our definition was chosen to reduce the amount of information we must "force" to hold for x.

We now need three lemmas. The first gives a sufficient condition for (A, B) to be a gap.

2.5. **Lemma.** *If* A *and* B *are* ω_1*-towers,* $A \perp B$, $\forall\xi(a_\xi \cap b_\xi = \emptyset)$, *and* $\forall\xi, \eta(\xi < \eta \to a_\xi \cap b_\eta \ne \emptyset)$, *then* (A, B) *is a Hausdorff gap.*

We will "force" an x such that $(A \restriction x, B \restriction x)$ and $(A \restriction (\omega \backslash x), B \restriction (\omega \backslash x))$ both satisfy this condition on some cofinal set. The next lemma will be used to show that our partial order has the ccc.

2.6. **Lemma.** *If* (A, B) *is a Hausdorff gap, then for each* n *there is a* $v \subseteq \omega$ *with* $|v| \ge n$ *and* $|\{\xi : v \subseteq a_\xi\}| = |\{\eta : v \subseteq b_\eta\}| = \omega_1$.

Proof. By induction on n. It is easy for $n = 1$. If it holds for n, we prove it for $2n$ as follows. By the lemma for n, fix $v \subseteq \omega$ with $|v| \ge n$ such that $X = \{x : v \subseteq a_\xi\}$ and $Y = \{\eta : v \subseteq b_\eta\}$ both have size ω_1. Now apply the lemma for n again to the Hausdorff gap $(\{a_\xi \backslash v : \xi \in X\}, \{b_\xi \backslash v : \eta \in Y\})$.

As in many MA arguments, rather than meeting dense sets, we will apply the following lemma to our partial order.

2.7. **Lemma.** MA + $\neg$CH *implies that if* $\mathbf{P}$ *is ccc and* $p_\xi \in \mathbf{P}$ *for* $\xi < \omega_1$ *then there is a filter* $G \subseteq \mathbf{P}$ *such that* $|\{\xi : p_\xi \in G\}| = \omega_1$.

Proof. It is well known that ω_1 is a pre-caliber for $\mathbf{P}$ (which requires only that G is centered), and 2.7 is proved in exactly the same way.

Proof of Theorem 2.4. Let (A, B) be a Hausdorff gap. We may assume $\forall\xi(a_\xi \cap b_\xi = \varnothing)$ (if not, replace b_ξ by $b_\xi \backslash a_\xi$). By 2.5, it is enough to find a cofinal $Y \subseteq \omega_1$ and an $x \in \omega$ such that

$$\forall\xi, \eta \in Y(\xi < \eta \to a_\xi \cap b_\eta \cap x \neq \varnothing \,\&\, (a_\xi \cap b_\eta)\backslash x \neq \varnothing).$$

Elements of $\mathbf{P}$ will be pairs, $p = \langle s_p, y_p\rangle$ such that $s_p \in 2^{<\omega}$ (an approximation to x), $y_p \in [\omega_1]^{<\omega}$ (an approximation to Y), and

$$\forall\xi, \eta \in y_p(\xi < \eta \to a_\xi \cap b_\eta \cap s_p^{-1}\{0\}) \neq \varnothing \,\&\, a_\xi \cap b_\eta \cap s_p^{-1}\{0\} \neq \varnothing).$$

Let $p_\xi = \langle\varnothing, \{\xi\}\rangle$. Assuming $\mathbf{P}$ has ccc we may apply 2.7 to get a filter $G \subseteq \mathbf{P}$ such that $Y = \{\xi\colon p_\xi \in G\}$ has size ω_1, and let $x = \bigcup\{s_p^{-1}(1)\colon p \in G\}$. If $\mathbf{P}$ is not ccc, let $\{p_\alpha\colon \alpha < \omega_1\}$ be an antichain. By the usual Δ-system and thinning arguments, we may assume $p_\alpha = \langle s, y_\alpha\rangle$, where $s \in 2^n$ and $\alpha < \beta \to \max(y_\alpha) < \min(y_\beta)$. Let $c_\alpha = (\bigcap_{\xi\in y_\alpha} a_\xi)\backslash n$ and $d_\beta = (\bigcap_{\xi\in\cdot y_\beta} b_\xi)\backslash n$. Then $(\{c_\alpha\colon \alpha < \omega_1\}, \{d_\beta\colon \beta < \omega_1\})$ is also a gap.

By 2.6, fix $v \subseteq \omega$ with $|v| \geq 2$ and fix $\alpha < \beta$ such that $v \subseteq c_\alpha$ and $v \subseteq d_\beta$. Note that $v \cap n = \varnothing$. Fix $i, j \in v$ with $i \neq j$. Then p_α and p_β have a common extension, $\langle t, y_\alpha \cup y_\beta\rangle$, where t extends s, $t(i) = 0$, and $t(j) = 1$.

Call $U \subseteq \omega^*$ a *strict* F_κ*-set* iff U is of the form $\bigcup_{\xi<\kappa} U_\xi$, with each U_ξ clopen and $\xi < \eta \to U_\xi \subset U_\eta$. Theorem 2.4 implies immediately that under $\mathrm{MA} + \neg\mathrm{CH}$, if U and V are disjoint strict F_{ω_1}-sets with $K = U^- \cap V^- \neq \varnothing$, then K has no isolated points. A similar proof shows that in K, nonempty G_δ-sets have nonempty interiors. We do not know if K is homeomorphic to ω^*; note that Parovičenko's characterization of ω^* is false under $\neg\mathrm{CH}$ [vDvM].

3. $\omega^* - \{p\}$ CAN BE C^*-EMBEDDED IN ω^*

In this section we shall show that it is consistent with $\mathrm{MA} + \mathfrak{c} = \omega_2$ that for every $p \in \omega^*$ we have $\beta(\omega^* - \{p\}) = \omega^*$.

3.1. **Lemma.** *Suppose that* $\neg G(\kappa, \omega)$ *for every* κ *with* $\omega \leq \kappa < \mathfrak{c}$. *If there is a closed* $P_\mathfrak{c}$*-set* A *in* ω^* *such that* $\omega^* - A$ *is not* C^**-embedded in* ω^*, *then* $G(\mathfrak{c}, \mathfrak{c})$.

Proof. Since by Tietze's Theorem closed subsets of ω^* are C^*-embedded in ω^* it follows that $\omega^* - A$ is not C^*-embedded in its closure in ω^*. We can therefore find disjoint zero-sets $Z(0)$ and $Z(1)$ of $\omega^* - A$ such that $Z(0)^- \cap Z(1)^- \neq \varnothing$ [GJ, 6.4(3)]. Pick a point $p \in Z(0)^- \cap Z(1)^-$. We shall construct a $\mathfrak{c}$-sequence $\langle U_\xi\colon \xi < \mathfrak{c}\rangle$ of clopen subsets of ω^* such that

(1) $U_\xi \subset Z(0)$ for $\xi < \mathfrak{c}$,

(2) $U_\xi \subset U_\eta$ for $\xi < \eta < \mathfrak{c}$,

(3) $p \in (\bigcup_{\xi<\mathfrak{c}} U_\xi)^-$.

Once this has been done, the same construction yields a $\mathfrak{c}$-sequence $\langle V_\xi : \xi < \mathfrak{c}\rangle$ of clopen sets in $Z(1)$ having similar properties. The U_ξ's and the V_ξ's then establish $G(\mathfrak{c}, \mathfrak{c})$.

We shall now construct the sequence $\langle U_\xi : \xi < \mathfrak{c}\rangle$. Enumerate all clopen neighborhoods of p as $\langle P_\xi : \xi < \mathfrak{c}\rangle$. We shall ensure (3) by having $P_\xi \cap U_\xi \neq \varnothing$ for all $\xi < \mathfrak{c}$. Let $\gamma < \mathfrak{c}$, and assume U_ξ to be constructed for $\xi < \gamma$, with the obvious inductive hypotheses being satisfied. We claim that

(4) there is a clopen U' in ω^* with $\bigcup_{\xi<\gamma} U_\xi \subseteq U' \subseteq Z(0)$,

(5) there is a nonempty clopen U'' in ω^* with $U'' \subseteq (Z(0) \cap P_\xi) - U'$.

Then U_γ will be $U' \cup U''$. The fact that $U' \cap U'' = \varnothing$ ensures that $U_\xi \subset U_\gamma$ for $\xi < \gamma$.

We prove (4). Since A is a $P_\mathfrak{c}$-set and $\bigcup_{\xi<\gamma} U_\xi \subseteq Z(0) \subseteq \omega^* - A$ there is a clopen K in ω^* with $\bigcup_{\xi<\gamma} U_\xi \subseteq K \subseteq \omega^* - A$. Clearly $K \cap Z(0)$ is a closed G_δ-set in K, hence in ω^*. If $K \cap Z(0)$ is clopen let $U' = K \cap Z(0)$, else $\omega^* - (K \cap Z(0))$ is the union of a strictly increasing sequence of clopen sets of ω^*, hence there is a clopen U' in ω^* with $\bigcup_{\xi<\gamma} U_\xi \subseteq U' \subseteq K \cap Z(0)$ since $\neg G(\mathrm{cf}(\gamma), \omega)$ by assumption.

We prove (5). Since $(\omega^* - U') \cap P_\gamma$ is a neighborhood of p and $p \in Z(0)^-$ we can find a point $x \in (\omega^* - U') \cap P_\gamma \cap Z(0)$. Since A is closed there is a clopen neighborhood C of x that misses A. Then $T = (\omega^* - U') \cap P_\gamma \cap C \cap Z(0)$ is a nonempty G_δ in ω^*, hence has nonempty interior [GJ, 6S.8]. So for U'' just pick any nonempty clopen (in ω^*) set that is contained in T.

3.2. **Lemma.** *Let $p \in X$. If there is a regularly open set U in X such that $p \in U^- - U$, such that* $\mathrm{bd}\, U = U^- - U$ *has no isolated points and such that both U and $X - U^-$ are C^*-embedded in X, then $X - \{x\}$ is C^*-embedded in X.*

Proof. Consider any continuous bounded function $f : X - \{x\} \to \mathbf{R}$. Since U is C^*-embedded in U^-, $f \restriction (U^- - \{p\})$ extends to a continuous function $\mathbf{f_0} : U^- \to \mathbf{R}$. Since $X - U^-$ is C^*-embedded in $(X - U^-)^-$, and since U is regularly open, so that $(X - U^-)^- = X - U$, $f \restriction ((X - U^-) - \{p\})$ extends to a continuous function $\mathbf{f_1} : (X - U) \to \mathbf{R}$. By construction, $\mathbf{f_i} \restriction (\mathrm{bd}\, U - \{p\}) = f \restriction (\mathrm{bd}\, U - \{p\})$ for $i \in \{0, 1\}$. Hence $\mathbf{f_0} \restriction \mathrm{bd}\, U = \mathbf{f_1} \restriction \mathrm{bd}\, U$ since $\mathrm{bd}\, U$ has no isolated points. Therefore $\mathbf{f} = \mathbf{f_0} \cup \mathbf{f_1}$ is a function $X \to \mathbf{R}$. By construction $\mathbf{f}$ extends f. Also, $\mathbf{f}$ is continuous since both $\mathbf{f} \restriction U^- = \mathbf{f_0}$ and $\mathbf{f} \restriction (X - U) = \mathbf{f_1}$ are continuous.

3.3. **Lemma.** *Let U be an open F_σ in ω^* such that $\omega^* - U^-$ is C^*-embedded in ω^*. If $p \in \mathrm{bd}\, U$ then $\omega^* - \{p\}$ is C^*-embedded in ω^*.*

Proof. U is C^*-embedded since every open F_σ of ω^* is C^*-embedded [GJ, 14.27], and U is regularly open since every open F_σ of ω^* is regularly open [FG, 3.1]. Also, since U is C^*-embedded in U^-, bd $U = U^*$, but U is σ-compact, hence realcompact [GJ, 8.2], and therefore U^* has no isolated points [GJ, 9D.1]. Hence the lemma follows from Lemma 3.2.

3.4. **Corollary.** *Suppose that $\neg G(\kappa, \omega)$ for every κ with $\omega \leq \kappa < \mathfrak{c}$. If $p \in \omega^*$ is not a P-point and if $\omega^* - \{p\}$ is not C^*-embedded in ω^*, then $G(\mathfrak{c}, \mathfrak{c})$.*

Proof. Since p is not a P-point, there is an open F_σ $U \subseteq \omega^*$ such that $p \in U^- - U$. By Lemma 3.2 we conclude that $\omega^* - U^-$ is not C^*-embedded in ω^*. Since we can write U as the union of a strictly increasing ω-sequence of clopen sets in ω^*, our assumption $\neg G(\kappa, \omega)$ for every κ with $\omega \leq \kappa < \mathfrak{c}$ easily implies that U^- is a $P_\mathfrak{c}$-set (prove that $G(\kappa, \omega)$ is equivalent to the following "unordered" version: $UG(\kappa, \omega)$: There are collections $\mathscr{U}$ and $\mathscr{V}$ of clopen sets in ω^* with $|\mathscr{U}| \leq \omega$ and $|\mathscr{V}| \leq \kappa$ such that $\bigcup\mathscr{U} \cap \bigcup\mathscr{V} = \emptyset$ but $(\bigcup\mathscr{U})^- \cap (\bigcup\mathscr{V})^- \neq \emptyset$). The desired result follows now from Lemma 3.1.

We are now in the position to prove the following

3.5. **Theorem.** *If there is a point $p \in \omega^*$ such that $\omega^* - \{p\}$ is not C^*-embedded in ω^* then at least one of the following statements is true*:

(1) *There is a κ with $\omega \leq \kappa < \mathfrak{c}$ such that $G(\kappa, \omega)$.*
(2) *$G(\mathfrak{c}, \mathfrak{c})$.*
(3) *There are regular cardinals κ, λ with $\omega_1 \leq \kappa, \lambda \leq \mathfrak{c}$ such that $SG(\kappa, \lambda)$.*

Proof. Suppose that (1) is not true. If p is not a P-point then by Corollary 3.4, $G(\mathfrak{c}, \mathfrak{c})$. So we may assume that p is a P-point. We shall establish (3).

Claim. If Z is a noncompact zero-set of $\omega^* - \{p\}$ then there are a regular cardinal $\omega_1 \leq \kappa \leq \mathfrak{c}$ and a κ-sequence $\langle U_\xi : \xi < \kappa \rangle$ of clopen subsets of ω^* such that

(a) $U_\xi \subset Z$ for $\xi < \kappa$,
(b) $U_\xi \subset U_\eta$ if $\xi < \eta < \kappa$,
(c) $p \in (\bigcup_{\xi<\kappa} U_\xi)^-$.

The proof of this claim is similar to the proof of 4.1. For the reader's convenience we shall give most of the details. Enumerate all clopen neighborhoods of p as $\langle P_\xi : \xi < \mathfrak{c} \rangle$. By transfinite induction we shall construct for every $\xi < \mathfrak{c}$ a clopen set U_ξ in ω^* such that

(d) $U_\xi \subset Z$ for $\xi < \mathfrak{c}$,
(e) if $p \notin (\bigcup_{\eta<\xi} U_\eta)^-$ then $U_\eta \subset U_\xi$ for every $\eta < \xi$, and $U_\xi \cap P_\xi \neq \emptyset$,
(f) if $p \in (\bigcup_{\eta<\xi} U_\eta)^-$ then $U_\xi = \emptyset$.

Let $\xi < \mathfrak{c}$, and assume U_η to be constructed for every $\eta < \xi$. If $p \in (\bigcup_{\eta<\xi} U_\eta)^-$ then (f) tells us that $U_\xi = \emptyset$. So suppose that $p \notin (\bigcup_{\eta<\xi} U_\eta)^-$. We claim that

(g) there is a clopen U' in ω^* with $\bigcup_{\eta<\xi} U_\eta \subseteq U' \subseteq Z$,

(h) there is a nonempty clopen U'' in ω^* with $U'' \subseteq (Z \cap P_\xi) - U'$.

Then U_ξ will be $U' \cup U''$. The fact that $U' \cap U'' = \varnothing$ ensures that $U_\eta \subset U_\xi$ for $\eta < \xi$.

We prove (g). Since $p \notin (\bigcup_{\eta<\xi} U_\eta)^-$ there is a clopen K in ω^* with $\bigcup_{\eta<\xi} U_\eta \subseteq K \subseteq \omega^* - \{p\}$. Clearly, $K \cap Z$ is a closed G_δ-set in K, hence in ω^*. Since by assumption we have $\neg G(\omega, \kappa)$ for every κ with $\omega \leq \kappa < \mathfrak{c}$, we can find U' precisely such as in the proof of 4.1.

We prove (h). Since $(\omega^* - U') \cap P_\xi$ is a neighborhood of p and $p \in Z^-$, $T = (\omega^* - U') \cap P_\xi \cap Z$ contains a nonempty G_δ in ω^*, hence it has nonempty interior [GJ, 6S.8]. So for U'' just pick any nonempty clopen (in ω^*) set that is contained in T.

Now if $U_\xi \neq \varnothing$ for every $\xi < \mathfrak{c}$, let $\kappa = \mathfrak{c}$. Observe that (e) implies that $p \in (\bigcup_{\xi<\kappa} U_\xi)^-$. If there is a $\xi < \mathfrak{c}$ with $U_\xi = \varnothing$, let α be the first ξ having this property and let $\kappa = \mathrm{cf}(\alpha)$. Observe that (f) implies that in this case also $p \in (\bigcup_{\xi<\kappa} U_\xi)^-$. That $U_\eta \subset U_\xi$ for all η and ξ with $\eta < \xi < \kappa$ follows trivially from (e). That $\mathrm{cf}(\kappa) \geq \omega_1$ is clear since p is a P-point.

Since $\omega^* - \{p\}$ is not C^*-embedded in ω^*, there are disjoint zero-sets $Z(0)$ and $Z(1)$ of $\omega^* - \{p\}$ with $Z(0)^- \cap Z(1)^- \neq \varnothing$ [GJ, 6.4(3)]. Clearly $\{p\} = Z(0)^- \cap Z(1)^-$. A straightforward application of the claim therefore proves $SG(\kappa, \lambda)$ for certain regular uncountable cardinals κ and λ with $\omega_1 \leq \kappa$, $\lambda \leq \mathfrak{c}$.

3.6. **Corollary.** *If* $\mathrm{MA} + \mathfrak{c} = \omega_2 + \neg G(\omega_1, \mathfrak{c}) + \neg G(\mathfrak{c}, \mathfrak{c})$ *then* $\beta(\omega^* - \{p\}) = \omega^*$ *for every* $p \in \omega^*$.

Proof. We shall prove that under $\mathrm{MA} + \mathfrak{c} = \omega_2 + \neg G(\omega_1, \mathfrak{c}) + \neg G(\mathfrak{c}, \mathfrak{c})$ the statements (1), (2) and (3) of Theorem 3.5 are false.

That (1) is false is clear by 2.1. That (2) is false is also clear. For (3), first observe that $\kappa = \lambda = \mathfrak{c}$ is not possible since $SG(\mathfrak{c}, \mathfrak{c})$ implies $G(\mathfrak{c}, \mathfrak{c})$. Since both κ and λ have uncountable cofinality there are two possibilities, namely (a) $\kappa = \omega_1$ and $\lambda = \omega_2$ (or vice versa), and (b) $\kappa = \lambda = \omega_1$. However, (a) is impossible because of $\neg G(\omega_1, \mathfrak{c})$, and (b) is also impossible because of Theorem 2.4.

We can now present our main result.

3.7. **Theorem.** *It is consistent with* $\mathrm{MA} + \mathfrak{c} = \omega_2$ *that* $\beta(\omega^* - \{p\}) = \omega^*$ *for every* $p \in \omega^*$.

Proof. Theorem 2.3(A) and Corollary 3.6.

4. A DENSE C^*-EMBEDDED SUBSET OF ω^* HAVING DENSE COMPLEMENT

It is well known that ω^* is not extremally disconnected [GJ, 6R.1], and therefore not every dense subspace of ω^* is C^*-embedded [GJ, 6M.2]. In view of Theorem 3.7 it now is natural to ask whether there can be a small dense

C^*-embedded subspace of ω^*. In this section we shall answer this question affirmative if we interpret "small" to mean "with dense complement". We do not know whether ω^* can have a dense C^*-embedded subspace of cardinality less than $2^{\mathfrak{c}}$, or of cardinality $\mathfrak{c}$. We also do not know whether ω^* can have two disjoint dense C^*-embedded subspaces.

4.1. **Theorem.** *Let bY be a compactification of a space Y such that*

(A) *every countable subset of $bY - Y$ is closed in $bY - Y$,*
(B) *$\beta(bY - \{y\}) - bY$ for $y \in bY - Y$.*

Then $\beta Y = bY$.

Proof. Let $f: Y \to \mathbf{R}$ be a bounded continuous function. By a classical result of Lavrentieff, cf. [E, 4.3.20], there is a G_δ-subset G of bY with $G \supseteq Y$ such that f can be extended to a continuous $\mathbf{f}: G \to \mathbf{R}$. By (A) every countable subset of $bY - Y$ is relatively discrete. Hence $bY - Y$ has no infinite compact subsets. It follows that $bY - G$, being a σ-compact subset of $bY - Y$, is at most countable. Now since $bY - G$ is relatively discrete, by applying (B), it is easy to extend $\mathbf{f}$ to a function $\mathbf{f}': bY \to \mathbf{R}$ which is continuous at all points of $bY - Y$. The details of checking this are left to the reader. It follows that $\mathbf{f}'$ is a continuous extension of f, which is as required.

So in view of 3.7 we need only to construct a dense set $X \subseteq \omega^*$ such that (1) $\omega^* - X$ is also dense, and (2) every countable subset of $\omega^* - X$ is closed in $\omega^* - X$. This is easy. We can for example let X be the set of all non-weak P-points in ω^*. Then X is dense, and so is $\omega^* - X$, [K_2].

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