PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 2, February 1989

FIBRATIONS THAT ARE COFIBRATIONS. II

JUAN M. ALONSO

(Communicated by Haynes R. Miller)

ABSTRACT. We show that fibrations that are cofibrations can be described quite explicitly (in terms of localization) when the total space of the fibration is nilpotent and that, in the absence of nilpotency, no such simple characterization exists.

1. INTRODUCTION

R. J. Milgram [6] proved that the fibration $K(\mathbf{Q}/\mathbf{Z}, n) \to K(\mathbf{Z}, n+1) \to K(\mathbf{Q}, n+1)$, $n \geq 1$, is also a cofibration, and asked which fibrations are also cofibrations. Examples include acyclic maps and localizing maps [8, 7, 3, **1**].

In [1, Corollary (2.10)] we gave the following necessary and sufficient condition (as in [1], we assume in the sequel that all spaces have the homotopy type of CW-complexes, unless explicit mention to the contrary):

Proposition. *The fibration $F \to E \xrightarrow{f} B$ with E and B path-connected, is also a cofibration if and only if there exists a set of primes P such that one of the homologies, $\tilde{H}_*(F)$ and $\tilde{H}_*(\Omega B)$, is P-torsion and the other is uniquely P-divisible.*

Remark. Note that when $P = \varnothing$ the condition reduces to $\tilde{H}_*(F) = 0$, i.e. f is an acyclic map, or $\tilde{H}_*(\Omega B) = 0$, i.e. $F \xrightarrow{\simeq} E$.

In this note we show that when E is nilpotent (see [4]), the form of these fibrations can be made quite explicit in terms of localization. In fact, we prove

Theorem 1. *Let $(*)$ $F \to E \xrightarrow{f} B$ be a fibration with E and B path-connected and E nilpotent. Then $(*)$ is also a cofibration if and only if one of the following cases obtain:*

(i) *f is a homotopy equivalence.*

(ii) *B is 1-connected and f is a P-localizing map (for some set of primes P).*

(iii) *$(*)$ is equivalent (up to homotopy) to the trivial fibration $F \to F \times B \to B$, where F is P-local and $\tilde{H}_*(\Omega B)$ is P'-torsion, for some set of primes P.*

Received by the editors December 21, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55P99, 55P60.

Remark. Compare with [8, Theorem 5 and Example 4]. Note that (iii) implies that B is 1-connected and $F = E_P$. The "reversed" fibration $B \to B \times F \to F$ need not, however, be a cofibration.

It follows from the theorem of Kan and Thurston [5] that the simple characterization in Theorem 1 depends essentially on the nilpotency of E. To obtain examples involving not necessarily acyclic maps, we extend Kan-Thurston to the following

Theorem 2. *Given a graded abelian group $(A_i)_{i\geq 1}$ and a path-connected space B, there is a (Serre) fibration $F \to E \to B$ of path-connected spaces with E a $K(\pi, 1)$ and $H_i(F) \cong A_i$ for all $i \geq 1$. The fibration is natural with respect to maps of B, and F is 2-cosimple (i.e. $\pi_1(F)$ acts trivially on $\pi_n(F)$ for $n \geq 2$).*

Remark. The theorem of Kan and Thurston is the case when all the A_i's are trivial. As in [5], it is not necessary to assume that B has the homotopy type of a CW-complex.

An immediate corollary to Theorem 2 is a result of J.-C. Hausmann [2, Theorem A] showing that in nonnilpotent spaces the higher homotopy groups are completely independent of the homology groups (see Corollary in §3 below).

2. Nilpotent total space

In this section we prove Theorem 1. When dealing with localization of nilpotent spaces, we use the notation and terminology of [4]. In particular, P' denotes the set of primes complementary to P.

($\Rightarrow$) If $\pi_1(B) \neq 0$, it follows from [1, (2.10)c] that f has to be an acyclic map. By [4, p. 62] F is nilpotent, so that $\pi_1(F)$ is both perfect and nilpotent, hence trivial. By Whitehead's theorem, f is a homotopy equivalence.

Assume now that $\pi_1(B) = 0$, and let P be the set of primes given by the Proposition, so that *either* $\tilde{H}_*(\Omega B)$ is uniquely P'-divisible and $\tilde{H}_*(F)$ is P'-torsion, *or* $\tilde{H}_*(\Omega B)$ is P'-torsion and $\tilde{H}_*(F)$ is uniquely P'-divisible.

The former possibility implies (ii): $H_n(f)$ P-localizes for all $n \geq 1$ by Lemma 1 below and, by the "Second Fundamental Theorem in NH" [4, p. 72] f itself is a P-localizing map.

So we assume that the latter holds and prove (iii). We have the homotopy commutative diagram of fibrations [4, p. 77] .

$$\begin{array}{ccccc} F & \xrightarrow{j} & E & \xrightarrow{f} & B \\ \downarrow{\scriptstyle e_F} & & \downarrow{\scriptstyle e_E} & & \downarrow{\scriptstyle e_B} \\ F_P & \xrightarrow[j_P]{} & E_P & \xrightarrow[f_P]{} & B_P \end{array}$$

where the e's are P-localizing maps. Since F is P-local, we can take

$F = F_P$ and $1_F = e_F$. Next, $\tilde{H}_*(\Omega B)$ is P'-torsion so that $0 = \{\tilde{H}_*(\Omega B)\}_P \cong \tilde{H}_*\{(\Omega B)_P\}$; this, together with the fact that $(\Omega B)_P \simeq \Omega(B_P)$ [4, p. 78], implies that $\Omega(B_P)$ is acyclic, and it follows easily that $B_P \simeq *$.

Thus, the bottom fibration in the above diagram is $F \xrightarrow{j_P} E_P \to *$ so j_P is a homotopy equivalence. Then the map $\phi\colon E \to F \times B$ defined by $\phi(x) = ((j_P)^{-1} e_E(x), f(x))$ induces isomorphisms of the homotopy groups and is the required homotopy equivalence. Moreover, the diagram

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d, equal] & E \arrow[r, "f"] \arrow[d, "\phi"] & B \arrow[d, equal] \\ F \arrow[r] & F \times B \arrow[r] & B \end{tikzcd}$$

is homotopy commutative and shows that $(*)$ is equivalent to the trivial fibration.

($\Leftarrow$) This is trivial for (i), and for (ii) it follows from [7 or 1, (3.2)]. For (iii), use the Proposition.

To complete the proof of Theorem 1 we have

Lemma 1. *Let $F \to E \xrightarrow{f} B$ be a fibration with $\tilde{H}_*(F)$ P'-torsion and $\tilde{H}_*(\Omega B)$ uniquely P'-divisible, for some set of primes P. Then $H_n(f)\colon H_n(E) \to H_n(B)$ P-localizes for all $n \geq 1$.*

Remark. We assume no nilpotency condition here.

Proof. By the Proposition the fibration is also a cofibration, so that we have the long exact homology sequence:

$$\cdots \to \tilde{H}_{n+1}(B) \to \tilde{H}_n(F) \to \tilde{H}_n(E) \xrightarrow{H_n(f)} \tilde{H}_n(B) \to \cdots.$$

We have to show that $H_n(f)$ is a P-isomorphism and that $H_*(B)$ is P-local. Suppose first that $P' = \varnothing$. Then F is acyclic so that $H_n(f)$ is an isomorphism and $H_*(B)$ is Π-local ($\Pi =$ all primes).

When $P' \neq \varnothing$ the hypothesis imply that F is path-connected and that B is 1-connected, since $\tilde{H}_0(-)$ is either trivial or free abelian. The above sequence together with the fact that $\tilde{H}_*(F)$ is P'-torsion, imply immediately that $H_n(f)$ is a P-isomorphism for $n \geq 1$. That $H_*(B)$ is P-local follows easily from [4, p. 76].

3. Nonnilpotent total space

We need the following easy fact:

Lemma 2. *Given a fibration $E \xrightarrow{p} B$ with fibre F, and a space X, the map $E \times X \xrightarrow{\mathrm{pr}} E \xrightarrow{p} B$ is a fibration with fibre $F \times X$.*

Proof of Theorem 2. Choose a connected space Y with $H_i(Y) \cong A_i$ for $i \geq 1$. By Kan and Thurston there is a group G_Y and a fibration (natural with respect to maps of Y) $UY \to TY \to Y$, such that UY is acyclic and TY is a $K(G_Y, 1)$.

Apply Kan and Thurston again, this time to the given space B, to obtain a fibration $UB \to TB \to B$ with UB acyclic and TB a $K(G_B, 1)$. Using Lemma 2 we obtain the fibration $UB \times TY \to TB \times TY \to B$. Then $E = TB \times TY$ is a $K(G_B \times G_Y, 1)$ and $F = UB \times TY$ satisfies $H_i(F) \cong A_i$ for $i \geq 1$. The construction is clearly natural with respect to maps of B.

That F is 2-cosimple is an easy consequence of the fact that the homotopy sequence of $F \xrightarrow{i} E \xrightarrow{f} B$ is a sequence of $\pi_1(E)$-modules, where $\pi_1(E)$ operates on $\pi_n(B)$ through $f_\#$, and $\xi \cdot \alpha = (i_\# \xi) \cdot \alpha$ for $\xi \in \pi_1(F)$ and $\alpha = \pi_n(F)$. In fact, since $\partial : \pi_{n+1}(B) \to \pi_n(F)$ is an isomorphism for $n \geq 2$, given any $\alpha \in \pi_n(F)$ there is a unique $\beta \in \pi_{n+1}(B)$ with $\partial(\beta) = \alpha$. Now $\partial\{(i_\#\xi) \cdot \beta\} = \xi \cdot \alpha$; but on the other hand $(i_\#\xi) \cdot \beta = (f_\# i_\# \xi) \cdot \beta = \beta$, so that $\alpha = \xi \cdot \alpha$.

Corollary [2, Theorem A]. *Let $(A_i)_{i \geq 1}$ and $(C_j)_{j \geq 2}$ be two arbitrary graded abelian groups. Then there exists a 2-cosimple space F such that $\pi_j(F) \cong C_j$ for $j \geq 2$, and $H_i(F) \cong A_i$ for $i \geq 1$.*

Proof. Choose a connected space B with $\pi_{j+1}(B) \cong C_j$ for $j \geq 2$, and apply Theorem 2 to $(A_i)_{i \geq 1}$ and B. This completes the proof.

Finally, we show that if the total space is not nilpotent, then for every set of primes P as in the Proposition, there are fibrations that are also cofibrations which are not of the type of those in Theorem 1.

Examples. *Case* (i). It suffices to take B to be any path-connected space with nontrivial fundamental group, B not a $K(\pi, 1)$, and apply Kan and Thurston to B: this gives an acyclic map which is not a homotopy equivalence.

Case (ii). This case obtains when $\tilde{H}_*(\Omega B)$ is uniquely P'-divisible and $\tilde{H}_*(F)$ is P'-torsion, where $P' \neq \varnothing$. So let $P \neq \Pi$ be a set of primes, and let B be a 1-connected space with $\pi_n(B) \neq 0$ for some $n \geq 2$, and $\pi_i(B)$ uniquely P'-divisible for all $i \geq 2$. Let $(A_i)_{i \geq 1}$ be any sequence of P'-torsion groups, and apply Theorem 2 to obtain a fibration $(*)$ $F \to E \xrightarrow{f} B$ with E a $K(\pi, 1)$ and $H_i(F) \cong A_i$.

Then $\tilde{H}_*(\Omega B)$ is also uniquely P'-divisible because ΩB is P-local, since $B \simeq B_P$ implies that $(\Omega B)_P \simeq \Omega(B_P) \simeq \Omega B$. Thus $(*)$ is also a cofibration and, by Lemma 1, $H_i(f)$ is a P-localizing map for $i \geq 1$. However, $\pi_i(f)$ does not P-localize since $\pi_n(E) = 0$ but $\pi_n(B) \neq 0$.

Case (iii). This obtains when $\tilde{H}_*(\Omega B)$ is P'-torsion and $\tilde{H}_*(F)$ is uniquely P'-divisible, with $P' \neq \varnothing$ (if $P' = \varnothing$ then $B \simeq *$ and $F \simeq E$). Similarly as above, one obtains a fibration and a cofibration $F \to E \to B$ with E a $K(\pi, 1)$. Thus the only case when this fibration can be trivial is when both F and B are $K(\pi, 1)$'s, so it suffices to make sure that B has a nontrivial higher homotopy group.

REFERENCES

1. J. M. Alonso, *Fibrations that are cofibrations*, Proc. Amer. Math. Soc. **87** (1983), 749–753.
2. J.-C. Hausmann, *On the homotopy of nonnilpotent spaces*, Math. Z. **178** (1981), 115–123.
3. J.-C. Hausmann and D. Husemoller, *Acyclic maps*, Enseign. Math. **25** (1979), 53–75.
4. P. J. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, Math. Studies No. 15, North-Holland, Amsterdam, 1975.
5. D. Kan and W. Thurston, *Every connected space has the homology of a* $K(\pi, 1)$, Topology **15** (1976), 253–258.
6. R. J. Milgram, *Surgery with coefficients*, Ann. of Math. **100** (1974), 194–248.
7. S. J. Schiffman, *A* $\operatorname{mod} p$ *Whitehead theorem*, Proc. Amer. Math. Soc. **82** (1981), 139–144.
8. Z. Wojtkowiak, *On fibrations which are also cofibrations*, Quart. J. Math. Oxford (2) **30** (1979), 505–512.

STOCKHOLMS UNIVERSITET, MATEMATISKA INSTITUTIONEN, BOX 6701, 113 85 STOCKHOLM, SWEDEN

Current address: Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720