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SUMS AND PRODUCTS OF HILBERT SPACES

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ABSTRACT. Let H be a Hilbert space. We prove that the locally convex sum $\bigoplus_I H$ is a subspace of the product H^J if and only if I is countable, H is infinite dimensional, and card $J \geq 2^{\aleph_0}$.

Notations. For the general terminology on locally convex spaces we refer to [1, 2].

If E is a locally convex space, $U(E)$ will denote a fundamental system of absolutely convex closed neighborhoods of 0. If p_U is the associated seminorm of $U \in U(E)$, we note $\widehat{E}_U$ for the completion of the normed space $(E/\operatorname{Ker} p_U \| \, \|_U)$, where $\|\phi_U(x)\|_U = p_U(x)$, ϕ_U being the quotient map. The spaces $\widehat{E}_U$ will be referred as the associated Banach spaces. If $U, V \in U(E)$, $V \subset U$, the canonical linking map $\widehat{T}_{VU}$ is the extension to the completions of the operator $T_{VU} \in L(E_V, E_U)$ defined by $T_{VU}\phi_V x = \phi_U x$.

K denotes the real or complex scalar field. If I is a set of cardinality d then φ_d denotes the locally convex sum $\bigoplus_I K$ that is, the space $\bigoplus_I K$ endowed with the strongest locally convex topology. When $I = \mathbf{N}$ we simply write φ. Analogously, if E is a locally convex space, $\bigoplus_I E$ denotes the sum space endowed with the strongest locally convex topology making all the inclusions $E \to \bigoplus_I E$ continuous.

Let T be an operator acting between the Banach spaces $T: X \to Y$. Let Z be a Banach space. By a subfactorization of T through Z we mean two operators $A: X \to Z$ and $B: \overline{\operatorname{Im} A} \to Y$ such that $T = BA$. Note that B need not be defined on all of Z, but only on the closure of the range of A in Z. When B is defined on the whole Z then we have a factorization of T through Z.

The spaces $l_p(I)$, $0 < p \leq +\infty$, are defined to be the Banach (p-Banach if $0 < p < 1$) spaces

$$l_p(I) = \{(x_i)_{i\in I} \in K^I : \|(x_i)\|_p\} = \left\{ \sum_{i\in I} |x_i|^p \right\}^{1/p} < +\infty$$

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if $p < +\infty$, and

$$l_\infty(I) = \{(x_i)_{i \in I} \in K^I : \|(x_i)\|_\infty\} = \sup_{i \in I} |x_i| < +\infty$$

if $p = +\infty$.

We write

$$l^+_\infty(I) = \{x \in l_\infty(I) : x_i > 0 \ \forall_i \in I\}$$

and recall the well-known fact that any Hilbert space is isometric with some $l_2(I)$.

Let X be a Banach space, we will also consider the vector valued sequence spaces

$$l_p(X) = \{(x_n) \in X^N : (\|x_n\|) \in l_p\}, \qquad 1 \le p < +\infty$$

and

$$c_0(X) = \{(x_n) \in X^N : (\|x_n\|) \in c_0\}$$

which in fact are Banach spaces.

Main results

Theorem. *Let H be an infinite-dimensional Hilbert space. Then the locally convex sum $\bigoplus_I H$ is a subspace of some product $\prod_J H$ if and only if I is countable and card $J \ge 2^{\aleph_0}$.*

Obviously H needs to be infinite-dimensional since φ, cannot be a subspace of any product K^J. On the other hand $\operatorname{card} J \ge 2^{\aleph_0}$ is required since $\bigoplus_N H$ is not metrizable.

Proposition 1. *$\bigoplus_N l_2$ is a subspace of $\prod_J l_2$*, $\operatorname{card} J \ge 2^{\aleph_0}$.

Proof. Since the locally convex sum topology coincides with the so-called box-topology (see [1]) on countable sums, a fundamental system of neighborhoods of 0 is given by the sets: $U(z) = \prod_N z_n B \cap \bigoplus_N l_2$, where B is the unit ball of l_2, and $z \in c_0$. We may suppose $z_n \neq 0$ for all $n \neq N$. Its associated seminorm is: $p_z((x_n)) = \sup_n z_n^{-1} \|x_n\|_2$, and the associated Banach space is clearly seen to be the completion of $\bigoplus_N l_2$ endowed with the norm p_z, that is:

$$\left\{ (x_n) \in l_2^N : \sup_n z_n^{-1} \|x_n\|_2 \to 0 \right\}.$$

This space is isometric with $c_0(l_2)$. Under this isometry, if $k \in c_0$ and $0 < k_n \le z_n$, then the linking map between the associated Banach spaces to p_k and p_z is precisely the "diagonal" operator $D_\sigma : c_0(l_2) \to c_0(l_2)$, $D_\sigma((x_n)) = (\sigma_n x_n)$, where $\sigma_n = z_n^{-1} | k_n^{-1}$.

If we choose k such that σ belongs to l_2, then D_σ factorizes through $l_2(l_2)$:

$$\begin{array}{ccc} c_0(l_2) & \xrightarrow{D_\sigma} & c_0(l_2) \\ & {\searrow}_{D_\sigma} \quad {\nearrow}_{\text{inclusion}} & \\ & l_2(l_2) & \end{array}$$

$$\|D_\sigma x\|_{l_2(l_2)} = \left(\sum \|\sigma_n x_n\|_2^2\right)^{1/2} = \left(\sum |\sigma_n|^2 \|x_n\|_2^2\right)^{1/2}$$
$$\leq \sup_n \|x_n\|_2 \left(\sum |\sigma_n|^2\right)^{1/2} = \|x\|_{c_0(l_2)} \|\sigma\|_2 .$$

The continuity of the inclusion is obvious. But the space $l_2(l_2)$ is isometric with l_2. Thus, the space $\bigoplus_N l_2$, as a projective limit of l_2 is a closed subspace of the topological product $\prod_I l_2$ [2, 19.10.3].

Remark. Since $l_p(l_p)$ is isometric with l_p, $1 \leq p < +\infty$, the preceding proof serves for the spaces l_p, and with minor modications for the nonseparable spaces $l_p(I)$, $1 \leq p < +\infty$. Therefore it covers the situation for all Hilbert spaces.

Proposition 2. *Let I be uncountable. Then $\bigoplus_I l_2$ is not a subspace of any product $\prod_J l_2$.*

Proof. The latter space has separable associated Banach spaces while the former does not. □

Proposition 3. *Let H be a Hilbert space, and I an uncountable set. Then $\bigoplus_I H$ is not a subspace of any product of copies of H.*

Proof. We may write $H = l_2(I)$ with I uncountable, by the remarks previous to Proposition 1, and Proposition 2. Let I be uncountable with $d = \operatorname{card} I$.

Step 1. Let $A \in L(l_2(I), l_1(I))$ represented by a matrix $(a_{i,j})_{(i,j)\in I\times I}$ in the form:

$$A(x_j) = (y_i) \quad \text{with } y_i = \sum_{j\in I} a_{ij} x_j .$$

Suppose that A has (a) a row of zeros or (b) a column of zeros. Then A cannot be part of a factorization of a diagonal operator $D_\sigma : l_1(I) \to l_1(I)$, $\sigma \in l_\infty^+(I)$ through $l_2(I)$. In case (a) since then all the vectors in IM A would have some coordinate zero, and IM $AB \neq$ IM D_σ. In case (b) it is $A' : l_\infty(I) \to l_2(I)$ which has a row of zeros and cannot be injective; since $D_{\sigma^{-1}}$ is injective, the factorization $D_{\sigma^{-1}} = D'_\sigma = B'A'$ is impossible, and thus $D_\sigma = AB$ is impossible too.

From all this it follows that a nonzero element must exist in each row and in each column of A. Therefore the set $\{i, j) \in I \times I : a_{ij} \neq 0\}$ is uncountable, and we may assume $a_{ij} > 0$ for uncountable many pairs (i, j). Thus an $\varepsilon > 0$ must exist such that $a_{ij} \geq \varepsilon$ for an uncountable set $Z \subset I \times I$. Moreover these indexes of Z need to be scattered through infinitely many rows and columns of $I \times I$; because if we suppose that they are "concentrated" in, let us say, a single column, then those vectors of $l_2(I)$ with the corresponding index zero have zero as the image by A. Since B can be considered surjective (see Step 2), A would not be a part of a factorization of D_σ. If they are "concentrated" on a row we obtain the same result by transposition. Therefore we can choose a countable set $Z_0 = \{(i_n, j_n) \in Z,\ n \in N\}$ such that $i_n \neq i_m$ and $j_n \neq j_m$

whenever $n \neq m$. Choose then an element $(z_j) \in l_2(I) \backslash l_1(I)$ with $z_j \geq 0$ $\forall j$ and $z_j \neq 0$ if some couple $(i, j) \in Z_0$. If $Az = y$, then we find that for each pair $(i, j) \in Z_0$:

$$y_i = \sum_{K \in I} a_{ik} z_k \geq \varepsilon z_j$$

whence

$$\sum_{i \in I} |y_i| \geq \varepsilon \sum_{j \in I} |z_j| = +\infty$$

and A cannot be an operator from $l_2(I)$ into $l_1(I)$. In this way we have essentially proved that:

Step 2. The diagonal operator $D_\sigma: l_1(I) \to l_1(I)$, $\sigma \in l_\infty^+(I)$, cannot be subfactorized through $l_2(I)$: the above manipulations settle the case of factorization. For subfactorizations we use orthogonal projection onto $\overline{\mathrm{IM}\,B}$ to obtain a factorization through a Hilbert space. If this is nonseparable the calculations of step 1 apply. If it is l_2 then $D_\sigma = AB$, $B \in L(l_1(I), l_2)$ and $A \in L(l_2, l_1(I))$ is clearly false since the image of D_σ cannot be contained in any $l_1(N)$— N a countable subset of I.

Step 3. It is not hard to check that φ_d has a fundamental system of neighborhoods of 0 with associated Banach spaces isometric with $l_1(I)$. Under this isometry the linking maps are the diagonal operators D_σ, $\sigma \in l_\infty^+(I)$.

Step 4. Let us assume that φ_d is a subspace of some product $l_2(I)^J$. There is a fundamental system of neighborhoods of 0 in $l_2(I)^J$, $\mathscr{U}$, with associated Banach spaces isometric with $l_2(I)$. Thus an embedding of φ_d into $l_2(I)^J$ would imply for $U \in \mathscr{U}$ the subfactorization

$$\begin{array}{ccccc} l_1(I) & \longrightarrow & (\widehat{\varphi}_d)_{\mathscr{U} \cap \varphi_d} & \longrightarrow & l_1(I) \\ & & \downarrow & & \\ & & l_2(I) & & \end{array}$$

of D_σ, which we know is not possible.

Step 5. We complete the proof of our Proposition 3. Since the embedding of φ_d into H^J is not possible when d is uncountable, the embedding of $\bigoplus_I H$ into H^J is impossible too. □

Remark. The result in Step 4 also holds when φ_d and $l_2(J)$ with different index sets are considered. It is obviously true when $\operatorname{card} J < d$ and, reasoning as in Step 2, when $\operatorname{card} J > d$.

Remark. For general operators $T: l_p(I) \to l_q(J)$, I, J uncountable and $p > q$ we can obtain (compare with the final part of Step 1):

(1) $\forall j \in I \ \operatorname{card}\{i \in I : a_{ij} \neq 0\} \leq \aleph_0$,

(2) $\forall i \in I \ \operatorname{card}\{j \in I : a_{ij} \neq 0\} \leq \aleph_0$ if $p > 1$.

From this it follows that if $\operatorname{card}\{(i,j) \in I \times I : a_{ij} \neq 0\} > \aleph_0$ then these indexes cannot be "concentrated" in a countable number of rows or columns. We could then proceed as in Step 1 to obtain a contradiction. Therefore:

$$(*) \qquad \operatorname{card}\{(i,j) \in I \times I : a_{ij} \neq 0\} \leq \aleph_0$$

and then $\operatorname{IM} T \subset l_q(N)$. We have

Lemma. *Let* I, J, *be uncountable sets,* $p > q \leq 1$ *and* $T : l_p(I) \to l_q(J)$ *a continuous operator. Then* $\operatorname{IM} T \subset l_q(N)$.

In the advance of this paper [4] the above lemma was incorreclty stated due to the omission of the hypothesis $p > 1$. The next counterexample shows that in that way it is no longer true: consider a partition $I = \bigcup_{n=1}^{\infty} I_n$, with I_n uncountable for all n. Take I_0 a countable subset of I, and define

$$a_{ij} = \begin{cases} i^{-4} & \text{when } i \in I_0 \text{ and } j \in I_i, \\ 0 & \text{otherwise}. \end{cases}$$

This matrix defines an operator from $l_1(I)$ to $l_{1/2}(I)$ for which $(*)$ does not hold.

References

1. H. Jarchow, *Locally convex spaces*, Teubner, Stuttgart, 1981.
2. G. Kothe, *Topological vector spaces*. I, Springer-Verlag, 1969.
3. ——, *Topological vector spaces*. II, Springer-Verlag, 1979.
4. Jesús M. F. Castillo, *The sum problem for Hilbert spaces*, Extracta Math. **3** (1988), 26–27.

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