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THE ALBERT QUADRATIC FORM FOR AN ALGEBRA OF DEGREE FOUR

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ABSTRACT. Suppose K is a field and the K-algebra A is expressed as a tensor product of two quaternion algebras $A \cong H_1 \otimes H_2$. Let N_i be the norm form on H_i and define the "Albert form" α_A to be the 6-dimensional quadratic form determined by $\alpha_A \perp \langle 1, -1\rangle \cong N_1 \perp -N_2$. In [Adv. in Math. **48** (1983), 149–165] Jacobson proved: (1) any two Albert forms for A are similar; (2) if A and B are algebras of this type, then $A \cong B$ if and only if α_A and α_B are similar.

The authors prove this result using quadratic forms and Clifford algebras, avoiding the application of Jacobson's theory of Jordan norms.

INTRODUCTION

Suppose A is a central simple (associative) algebra of degree 4 and exponent 2 over a field K. By a well-known theorem of Albert (see [A1, p. 174 or R]) such an algebra is expressible as a tensor product of two quaternion algebras. Following the idea of Jacobson [J] we define the "Albert quadratic form" α_A. First choose an expression $A \cong H_1 \otimes H_2$, where H_1 and H_2 are quaternion algebras, and let N_i be the usual norm form on H_i, defined by $N_i(x) = x\bar{x}$. Then the form $N_1 \perp -N_2$ is isotropic, so it contains a copy of the hyperbolic plane $\mathbf{H}$ as a subform. Define α_A to be the 6-dimensional quadratic form satisfying $\alpha_A \perp \mathbf{H} \approx N_1 \perp -N_2$. Our definition differs from Jacobson's in the case K has characteristic 2, correcting an error in [J] (see §3 below).

The main result is that the form α_A is well defined up to similarity, independent of the decomposition of A. Two quadratic forms α, β over K are said to be *similar* if α is isometric to a scalar multiple of β, that is, if $\alpha \approx \langle x\rangle\beta$ for some $x \in K^*$.

Jacobson's Theorem [J, THEOREM 3.12]. *Let A, B be central simple algebras of degree 4 and exponent 2 over a field K.*

(i) *Any two Albert forms for A are similar.*
(ii) *$A \cong B$ if and only if α_A and α_B are similar.*

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(iii) *The index of* A *is* 4, 2 *or* 1 *according as the Witt index of* α_A *is* 0,1 *or* 3.

Here the (Schur) index of A is the degree of the division algebra part of A. Jacobson derives this result from his theory of Jordan norms applied to the Jordan algebra of symmetric elements in A. In this note we give a different proof of the theorem using quadratic forms and Clifford algebras. We first prove the theorem in the case $\operatorname{char} K \neq 2$ using techniques of quadratic form theory as in [L and S]. When $\operatorname{char} K = 2$ we prove the theorem by establishing the analogs of each of the earlier steps.

This work forms part of the first author's doctoral dissertation [M1]. It is a pleasure to thank J. P. Tignol for many stimulating conversations and A. Wadsworth for several valuable comments.

2. Characteristic not two

Suppose K is a field with $\operatorname{char} K \neq 2$. We follow the notations for quadratic forms and Witt rings found in [L and S]. If φ is a (nonsingular) quadratic form of dimension n of K, the discriminant is $d\varphi = (-1)^{n(n-1)/2} \det \varphi$ in K^*/K^{*2}. The discriminant induces a map on the Witt ring $d\colon WK \to K^*/K^{*2}$.

The Witt invariant $c(\varphi)$ is defined to be the class in the Brauer group $\operatorname{Br}(K)$ of a certain Clifford algebra. Rules for computing $c(\varphi)$ are given in [L, p. 121 and S, p. 81]. The Witt invariant induces a map $c\colon WK \to \operatorname{Br}(K)$. For example for $a, b \in K^*$ let (a, b) denote the class of the quaternion algebra $(a, b)/K$ in $\operatorname{Br}(K)$. The norm form of this algebra is the 2-fold Pfister form $\langle\langle -a, -b\rangle\rangle = \langle 1, -a, -b, ab\rangle$, and we have $c(\langle\langle -a, -b\rangle\rangle) = (a, b)$. We follow the usual abuse of notation writing (a, b) both for the quaternion algebra and for its class in $\operatorname{Br}(K)$, and writing φ both for the quadratic form and for its class in $W(K)$.

The ideal IK of even dimensional forms in WK and its powers I^nK play a fundamental role in this theory. For example $\varphi \in I^2K$ iff $\varphi \in IK$ and $d\varphi = \langle 1\rangle$. It is also easy to check that if $\varphi \in I^3K$ then $\varphi \in I^2K$ and $c(\varphi) = 1$. Pfister [P] proved a partial converse. We state the parts of Pfister's theorem which we will use later.

2.1 **Proposition** (Pfister). (i) *If* $\varphi \in I^2K$, $c(\varphi) = 1$ *and if* $\dim \varphi \leq 12$ *then* $\varphi \in I^3K$.

(ii) *If* $\dim \varphi = 6$, $d\varphi = \langle 1\rangle$ *and* $c(\varphi)$ *is quaternion then* φ *is isotropic.*

(iii) *If* $\dim \varphi = 10$, $d\varphi = \langle 1\rangle$ *and* $c(\varphi) = 1$ *then* φ *is isotropic.*

Proofs of Pfister's theorem are given in [P, Satz 14 and S, pp. 90–91]. Merkurjev has proved the general result, without restrictions on dimension: if $\varphi \in I^2K$ and $c(\varphi) = 1$ then $\varphi \in I^3K$. We will not use that difficult theorem here.

2.2 **Lemma** (Wadsworth). *Let φ and ψ be quadratic forms over K and suppose $\dim\varphi = \dim\psi = 4$ and $d\varphi = d\psi = \langle d\rangle$. Let $L = K(\sqrt{d})$. If $\varphi\otimes L$ and $\psi\otimes L$ are similar over L then φ and ψ are similar over K.*

The proof appears in [W1, Theorem 7].

2.3 **Proposition.** *Suppose φ and ψ and quadratic forms over K with $\dim\varphi = \dim\psi = 4$, $d\varphi = d\psi$ and $c(\varphi) = c(\psi)$. Then φ and ψ are similar.*

Proof. Suppose $d\varphi = d\psi = \langle d\rangle$. If $\langle d\rangle = \langle 1\rangle$ the result is easy to prove, as in [S, p. 88, Theorem 14.1]. If $\langle d\rangle \neq \langle 1\rangle$ we can extend scalars to the field $L = K(\sqrt{d})$, use the first case to conclude that $\varphi\otimes L$ and $\psi\otimes L$ are similar over L and apply the previous lemma. □

2.4 **Proposition.** *Suppose α and β are quadratic forms over K with $\dim\alpha = \dim\beta = 6$, $d\alpha = d\beta = \langle 1\rangle$ and $c(\alpha) = c(\beta)$. Then α and β are similar.*

Proof. By Proposition 2.1(i) applied to $\alpha\perp -\beta$ we see that the hypothesis implies $\alpha \equiv \beta \pmod{I^3K}$. We may scale α and β independently, so let us assume they represent 1. Say $\alpha \approx \langle 1\rangle\perp\alpha_1$ and $\beta \approx \langle 1\rangle\perp\beta_1$. Then $\alpha_1 \equiv \beta_1 \pmod{I^3K}$ so that $\alpha_1\perp -\beta_1 \in I^3K$ is 10-dimensional. By Proposition 2.1(iii) this form must be isotropic. Then α_1 and β_1 represent a common value, say $\alpha_1 \approx \langle d\rangle\perp\alpha_2$ and $\beta_1 \approx \langle d\rangle\perp\beta_2$. Then $\alpha_2 \equiv \beta_2 \pmod{I^3K}$ and $\dim\alpha_2 = \dim\beta_2 = 4$. We conclude from Proposition 2.3 that these forms are similar. Say $\alpha_2 \approx \langle x\rangle\beta_2$ for some $x \in K^*$. Then $c(\alpha_2) = c(\langle x\rangle\beta_2) = c(\beta_2)(x, d\beta_2)$. Since $c(\alpha_2) = c(\beta_2)$ and $d\beta_2 = \langle -d\rangle$ we have $1 = (x, -d)$. Then the form form $\langle\langle -x, d\rangle\rangle$ is hyperbolic so that $\langle x\rangle\langle 1, d\rangle \approx \langle 1, d\rangle$. Therefore $\langle x\rangle\beta \approx \langle x\rangle(\langle 1, d\rangle\perp\beta_2) \approx \langle 1, d\rangle\perp\alpha_2 \approx \alpha$. □

Now we can prove Jacobson's Theorem. Suppose $A \cong (a_1, b_1)\otimes(a_2, b_2)$ is the tensor product of two quaternion algebras over K. By definition the Albert form α is determined by: $\mathbf{H}\perp\alpha \approx N_1\perp -N_2$, where $N_i \approx \langle\langle -a_i, b_i\rangle\rangle$ is the norm form of the quaternion algebra. Therefore $\alpha \approx \langle -a_1, -b_1, a_1b_1, a_2, b_2, -a_2b_2\rangle$. Then $\dim\alpha = 6$, $d\alpha = \langle 1\rangle$ and $c(\alpha) = c(N_1\perp -N_2) = c(N_1)c(N_2) = (a_1, b_1)(a_2, b_2) = [A]$.

Suppose B is another algebra decomposed as a tensor product of quaternions, and let β be the corresponding Albert form. If $A \cong B$ then $c(\alpha) = [A] = [B] = c(\beta)$ and Proposition 2.4 implies α and β are similar. Conversely, if α and β are similar, say $\alpha \approx \langle x\rangle\beta$, then $[A] = c(\alpha) = c(\langle x\rangle\beta) = c(\beta)(x, d\beta) = c(\beta) = [B]$, and we have $A \cong B$. This proves parts (i) and (ii) of Jacobson's Theorem.

The nontrivial part of (iii) is to prove that if the Albert form α_A is anisotropic then A is a division algebra. In addition to his own proof, Jacobson refers to proofs by Albert, Tamagawa and Seligman. The result was also proved by Pfister [P]. In fact it is equivalent to Proposition 2.1(ii) stated above. This completes our proof in the case $\operatorname{char} K \neq 2$. □

3. Characteristic two

Let K be a field of characteristic two. We follow most of the notations for quadratic forms, Witt groups and quaternion algebras found in [B]. If the quadratic form $\varphi = \perp_{i=1}^{n} \langle c_i \rangle [a_i, b_i]$, the Arf invariant of φ is $\Delta(\varphi) = \sum_{i=1}^{n} a_i b_i$ in $K/\wp(K)$ and the Witt invariant $c(\varphi)$ is the class in $\mathrm{Br}(K)$ of the tensor product of quaternion algebras $\bigotimes_{i=1}^{n} (b_i, a_i b_i]$. In particular if $\varphi = \langle\langle a, b]]$ is the norm form of the quaternion algebra $H = (a, b]$, then $c(\varphi) = [H]$. These invariants induce maps on the Witt group $Wq(K)$. We follow the usual abuse of notation writing $(b, a]$ both for the quaternion algebra and for its class in $\mathrm{Br}(K)$, and writing φ both for the quadratic form and its class in $Wq(K)$.

3.1 **Lemma.** *The sequences*

$$0 \to IWq(K) \to Wq(K) \xrightarrow{\Delta} K/\wp(K)$$
$$0 \to I^2 Wq(K) \to IWq(K) \xrightarrow{c} \mathrm{Br}(K)$$

are exact.

Proof. See [Sa, Theorem 2, p. 152]. □

The next Lemma is the characteristic 2 analogue of Wadsworth's Lemma 2.2.

3.2 **Lemma.** *Let φ and ψ be quadratic forms over K and suppose* $\dim \varphi = \dim \psi = 4$ *and $\Delta(\varphi) = \Delta(\psi) = \Delta$. Let $L = K(\wp^{-1}(\Delta))$. If $\varphi \otimes L$ is similar to $\psi \otimes L$ over L then φ and ψ are similar over K.*

Proof. See [M1, p. 79 or M2]. □

3.3 **Proposition.** *Suppose φ and ψ are quadratic forms over K with* $\dim \varphi = \dim \psi = 4$ *and* $\varphi \equiv \psi \pmod{I^2 Wq(K)}$. *Then φ and ψ are similar.*

Proof. The congruence condition and Lemma 3.1 imply that $\Delta(\varphi) = \Delta(\psi)$ and $c(\varphi) = c(\psi)$. Let $L = K(\wp^{-1}(\Delta))$ where $\Delta = \Delta(\varphi)$. Since the Arf invariants become 0 over L we can express $\varphi \otimes L \approx \langle c_1 \rangle \langle\langle b_1, a_1]]$ and $\psi \otimes L \approx \langle c_1 \rangle \langle\langle b_2, a_2]]$, for some $a_i, b_i, c_i \in L^*$. Computing the Witt invariants over L we find that $(b_1, a_1] = c(\varphi \otimes L) = c(\psi \otimes L) = (b_2, a_2]$ in $\mathrm{Br}(L)$. Then those quaternion algebras are isomorphic so their norm forms are isometric: $\langle\langle b_1, a_1]] \approx \langle\langle b_2, a_2]]$ over L. Therefore $\varphi \otimes L \approx \langle c_1 c_2 \rangle \psi \otimes L$, and Lemma 3.2 completes the proof. □

3.4 **Lemma.** *Suppose* $\dim \varphi = 6$, $\Delta(\varphi) = 0$. *Then $c(\varphi)$ is quaternion iff φ is isotropic.*

Proof. This is part of the characteristic 2 analog of Pfister's Theorem 2.1 above, proved by Baeza. See [B, Theorem 4.18, p. 129]. □

3.5 **Proposition.** *Suppose α and β are quadratic forms over K with* $\dim \alpha = \dim \beta = 6$, *$\Delta(\alpha) = \Delta(\beta) = 0$ and $c(\alpha) = c(\beta)$. Then α and β are similar.*

Proof. By Lemma 3.1 applied to $\alpha\perp-\beta$ we see that the hypothesis implies $\alpha\equiv\beta \pmod{I^2Wq(K)}$. We may scale α to assume $\alpha=[1,a]\perp\alpha_1$. Let $L=K(\wp^{-1}(a))$ so that $\alpha\otimes L$ is isotropic. By Lemma 3.4 applied twice we find that $\beta\otimes L$ is isotropic too. By [B, Theorem 4.2, p. 121] we know that β contains a subform isometric to $\langle y\rangle[1,a]$, for some $y\in K^*$. Then $\langle y\rangle\beta\approx[1,a]\perp\beta_1$ for some form β_1. Then $\alpha\equiv\beta\equiv\langle y\rangle\beta \pmod{I^2Wq(K)}$ implies that $\alpha_1\equiv\beta_1 \pmod{I^2Wq(K)}$, and Proposition 3.3 shows that α_1 and β_1 are similar. Say $\alpha_1\approx\langle x\rangle\beta_1$ for some $x\in K^*$. Then $c(\alpha_1)=c(\langle x\rangle\beta_1)=c(\beta_1)(x,\Delta(\beta_1)]$. Since $c(\alpha_1)=c(\beta_1)$ and $\Delta(\beta_1)=a$ we have $1=(x,a]$. This implies that the form $[1,a]$ represents x, so that $\langle x\rangle[1,a]\approx[1,a]$. Therefore $\langle xy\rangle\beta\approx\langle x\rangle([1,a]\perp\beta_1)\approx[1,a]\perp\alpha_1\approx\alpha$. □

Now we can prove Jacobson's Theorem. Suppose $A\cong(b_1,a_1]\otimes(b_2,a_2]$ is the tensor product of two quaternion algebras over K. The Albert quadratic form α is defined from the equation $\mathbf{H}\perp\alpha\approx N_1\perp-N_2$ were $N_1\approx\langle\langle b_i,a_i]]$ is the norm of the quaternion algebra. Cancelling $\mathbf{H}$ shows that $\alpha\approx[1,a_1+a_2]\perp\langle b_1\rangle[1,a_1]\perp\langle b_2\rangle[1,a_2]$. Then $\dim\alpha=6$, $\Delta(\alpha)=0$ and $c(\alpha)=c(N_1\perp-N_2)=c(N_1)c(N_2)=(b_1,a_1](b_2,a_2]=[A]$. Note that there is an error in [J] concerning the calculation of α. On p. 155 line 18 when $\operatorname{char}\Phi=2$ the space $\mathscr{L}$ has dimension 5, not 6.

The proof of parts (i) and (ii) of Jacobson's Theorem is nearly identical with the case done in §2 above. Part (iii) is essentially done in Baeza's Lemma 3.4 above, as well as in the reference mentioned in [J]. □

Final remark. A. Wadsworth has informed us that the characteristic 2 case of Jacobson's Theorem can be deduced directly from the characteristic 0 case using a valuation argument in the style of his work in [W2]. Here is a brief outline of these ideas.

Let F be a Henselian valued field with valuation v, valuation ring V, residue field $\overline{F}$ and value group Γ. If q' is a nondegenerate quadratic form over $\overline{F}$, let q be a "lift" of q' considered as a quadratic form over F. Any two such lifts are isometric by [K, Satz 3.3].

1. **Lemma.** (1) *If q' is anisotropic then q is anisotropic and for any $c\in D_F(q)$ we have $v(c)\in2\Gamma$.*

(2) *If q_1' and q_2' are similar nondegenerate quadratic forms over $\overline{F}$, then their lifts q_1 and q_2 are similar over F.*

Proof. (1) uses a standard valuation argument, and (2) follows from (1). □

2. **Proposition.** *Let D' be a finite dimensional $\overline{F}$-central division algebra. Then D' has a unique "inertial lift" D which is an F-central division algebra satisfying $[D:F]=[D':\overline{F}]$ and (using the unique extension of v to D), $\overline{D}\cong D'$.*

Proof. See [JW, §2]. □

Now we can deduce Jacobson's Theorem in characteristic 2. Suppose A' and B' are K-algebras which are expressed as tensor products of two quaternion algebras, where K is a field of characteristic 2. For instance let $A \cong Q_1' \otimes Q_2'$. Choose a Henselian valued field F of characteristic 0 such that $\overline{F} = K$. If Q_i is the inertial lift of Q_i' then $A = Q_1 \otimes Q_2$ is the inertial lift of A'. It also follows that the lift of the Albert form $\alpha_{A'}$ is just α_A. A similar analysis can be done for B'. Assuming the characteristic 0 version of Jacobson's Theorem we find: $A' \cong B'$ over K iff $A \cong B$ over F iff α_A and α_B are similar over F iff $\alpha_{A'}$ and $\alpha_{B'}$ are similar over K.

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