TREE-LIKE CONTINUA AND EXACTLY $k$-TO-1 FUNCTIONS

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ABSTRACT. To answer a question of Nadler and Ward, $k$-to-1 maps from tree-like continua onto tree-like continua are constructed, for $k > 2$. It is shown that certain arc-like continua cannot be the domain of any 2-to-1 map and that certain tree-like continua cannot be the image of any 2-to-1 map (defined on continua) but it is unknown if any indecomposable arc-like continuum can be the domain or any tree-like continuum the image of a 2-to-1 map.

INTRODUCTION

In [11], Nadler and Ward prove that any continuum not hereditarily unicoherent is the exactly $k$-to-1 continuous image of some continuum and they prove that any continuum whose every subcontinuum has an endpoint cannot be the exactly $k$-to-1 continuous image of any continuum, unless $k = 1$. In [4] it was shown that no dendrite is the (exactly) $k$-to-1 image of any continuum even if the function is allowed to be finitely discontinuous. These results leave unanswered a question that Nadler and Ward ask, namely can any tree-like continuum be the (exactly) $k$-to-1 image of a continuum? Constructed in this paper (in §1) is an example of an arc-like continuum that admits a $k$-to-1 continuous map onto itself for any odd integer $k$, and for each even integer $k > 2$, two tree-like continua are constructed and a $k$-to-1 map from one onto the other.

The difficult case is $k = 2$. Two settings are studied: 2-to-1 functions with arc-like domains in §II and 2-to-1 functions with tree-like images in §III.

For the case of the tree-like domain of a 2-to-1 map there are easy examples, one from [1] is illustrated on the next page.

Also, Wayne Lewis has constructed a 2-to-1 map on an arc-like continuum, see Example 3 in §II, but the following question is not yet answered.

Question 1. Can any indecomposable arc-like continuum be the domain of a 2-to-1 continuous map?

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In [9, (IV, 4 and 5)], Mioduszewski showed that neither the arc-like indecomposable Knaster bucket-handle continuum (described in Example 1 in §II) nor the arc-like decomposable Knaster $V$-continuum (see [6], p. 570) can be the domain of a 2-to-1 map and in §II, the class of continua which cannot be the domain of a 2-to-1 finitely discontinuous function is shown to include all finitely discontinuous 1-to-1 images of $[0,1]$, and these continua are shown to be tree-like.

In §III, which concerns tree-like images, an example of a 2-to-1 finitely discontinuous function from a tree-like continuum onto the arc-like Knaster bucket-handle space is given, but no example is known for continuous maps.

**Question 2.** Can any tree-like continuum be the 2-to-1 continuous image of a continuum?

Some tree-like continua can be ruled out. It is shown here that no 1-to-1 finitely discontinuous image of $[0,1]$, necessarily tree-like, can be such a continuous image, nor any continuum whose every subcontinuum has a finite separating set. More generally, it is shown that if every subcontinuum of a continuum $Y$ "can be pruned" (a generalization of "has a cut point") then $Y$ is not the continuous 2-to-1 image of any continuum.

All continua are understood to be compact and metric.

§I

The examples in this section answer affirmatively for all $k > 2$ the Nadler-Ward question: can a tree-like continuum be the $k$-to-1 image of some continuum? Exactly which tree-like continua are images, however, is not known.

**Example 1.** An arc-like continuum $X$ that maps $k$-to-1 onto itself for each odd positive integer $k$. 

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**Figure 1.** "An example of a 2-to-1 map on a tree-like continuum"
The function used will be a union of sawtooth functions as follows. If $k$ is a positive odd integer and $A$ is an arc, choose $k + 1$ points of $A$, $a_1 < a_2 < \cdots < a_{k+1}$, with $a_1$ and $a_{k+1}$ endpoints of $A$, and define a continuous $f$ on $A$ so that $f(a_1) = a_1$, $f(a_{k+1}) = a_{k+1}$, and for each $i$ from 1 to $k$, $f$ restricted to the subarc $[a_i, a_{i+1}]$ is a homeomorphism onto $A$. Note that $f$ is $k$-to-1 at each interior arc point, $f$ is $[(k + 1)/2]$-to-1 at each endpoint of $A$, and if $A$ and $B$ are two arcs sharing only a common endpoint $x$ with sawtooth functions defined on each, then the union of the functions is $k$-to-1 at $x$.

Let $Y$ denote the classic planar Knaster bucket-handle space (see [7]) constructed as follows. Let $C$ denote the usual deleted middle thirds Cantor set in $[0,1]$ and let $Y = \bigcup S_i$, where $S_1$ is the union of all semicircles in $\{y \geq 0\}$ with center $(\frac{1}{2}, 0)$ and both endpoints in $C$, and for $i \geq 3$, $S_{i-1}$ is the union of all semicircles in $\{y \leq 0\}$ with center $5/(2 \cdot 3^i)$, diameter no more than $1/3^i$, and both endpoints in $C$.

Define $X$ to be $Y \cup V$, where $V$ is $Y$ reflected about the $y$-axis. The $k$-to-1 continuous map $f$ from $X$ onto $X$ is simply the union of sawtooth functions defined on each semicircle described in the construction of $Y$, and on each reflected semicircle in $V$. The sawtooth functions need to be defined uniformly on each $S_i$ collection of semicircles for $f$ to be continuous.

Example 2. If $k$ is an even integer greater than two, then there are tree-like continua $W$ and $Z$ and a $k$-to-1 continuous map from $W$ onto $Z$.

Let $X = Y \cup V$ be the continuum described in Example 1. To construct $W$, glue the $(0, 0)$ points of $(k-3)+(k-3)+(k-2)(k-3)$ copies $V_1, \ldots, V_{k-3}$, $Y_1, \ldots, Y_{k-3}, Y(1, 1), \ldots, Y(k-2, k-3)$ of $Y$ to $X$ at $(0,0)$. For each $\varepsilon > 0$, $W$ can be $\varepsilon$-mapped onto an $n$-od with $n = 2 + k(k-3)$ and so in particular, $W$ is tree-like. The image space $Z$ is the subspace $X \cup Y_1 \cup \cdots \cup Y_{k-3}$ of $W$.

Define the $k$-to-1 map $g$ from $W$ onto $Z$ as follows. On $X$, $g$ is the same as $f$ in Example 1 for $k = 3$. On each $V_i$, $g$ is the natural 1-to-1 projection onto $V$, and on each $Y(i, j)$, $g$ is the natural 1-to-1 projection onto $Y_i$ if $1 < i < k-3$ and onto $Y$ if $i = k-2$. Finally, $g \upharpoonright Y_i$ maps onto $Y_i$, for $1 \leq i \leq k-3$, exactly as $f$ does from $Y$ onto $Y$. It is straightforward to count the $k$ point inverses for each point in $Z$.

§II

The collection of 1-to-1 finitely discontinuous images of $[0, 1]$ includes, besides arcs and acyclic finite graphs, and $\sin(\frac{1}{x})$ curve plus its limiting set, a triod plus a ray spiraling down on the triod, and many other relatively simple continua. All 1-to-1 finitely discontinuous images of $[0, 1]$, if continua, are shown to be tree-like in Theorem 1. It is an immediate corollary to the reference theorem below that no 1-to-1 finitely discontinuous image of $[0, 1]$ can be the domain of any 2-to-1 finitely discontinuous function onto a Hausdorff space.
Reference Theorem (Heath, [5]). There is no 2-to-1 finitely discontinuous function defined on \([0,1]\) whose image is Hausdorff.

Until recently there were no known examples of 2-to-1 maps defined on chainable continua but the following is a new result of Wayne Lewis.

Example 3 (I. W. Lewis). A chainable continuum \(X\) that admits an exactly 2-to-1 map onto a continuum.

Construction. Lewis has shown in [8] that for each \(n \geq 2\) there is a chainable continuum \(C\) admitting a homeomorphism \(h\) of period \(n\) such that \(h\) has exactly one fixed point \(p\) and every other point of \(C\) has minimal period \(n\). Furthermore, there is a point \(q\) in \(C\) such that for each \(\epsilon > 0\) there is an \(\epsilon\)-chain covering \(C\) with \(p\) and \(q\) in opposite end-links. The continuum \(C\) will be indecomposable and may be chosen to be the pseudo-arc.

For \(i = 1, 2, 3, \ldots\) let \(C_i\) denote a copy of \(C\) for \(n = 2\) with diameter less than \(1/2^i\), and let \(p_i\) and \(q_i\) denote the copies of \(p\) and \(q\) respectively in \(C_i\). To construct \(X\), string together the \(C_i\) by identifying \(q_i\) and \(p_{i+1}\) for \(i = 1, 2, \ldots, \) and add a limiting point, \(p_w\), of \(\{C_i\}\). Consider the space \(X'\) obtained from \(X\) by identifying each point \(x\) of \(C_i\) with \(h_i(x)\) (where \(h_i\) is the period 2 homeomorphism of \(C_i\) with fixed point \(p_i\)) and also identifying \(p_1\) with \(p_w\). Then there is a 2-to-1 map from \(X\) onto \(X'\), namely the map determined by the identifications.

Theorem 1. If the continuum \(Y\) is a 1-to-1 finitely discontinuous image of \([0,1]\), then \(Y\) is tree-like and each subcontinuum of \(Y\) has a cut-point.

Proof. If \(f: [0,1] \to Y\) for some continuum \(Y\), where \(f\) is 1-to-1 and finitely discontinuous, certain "end" sets in \(Y\) can be defined. If \((a, b)\) is a component of \((0,1)\) minus the discontinuities and \(c\) is in \((a, b)\), then \(\text{End}(a, c)\) denotes those points in \(Y\) that are limits of some \(\{f(x_i)\}\) sequence, where \(x_i\) is in \((a, c)\) and \(\{x_i\} \to a\). Similarly \(\text{End}(c, b)\) is defined. Each end is compact and connected.

Now suppose there are continua that satisfy the hypothesis but not the conclusion, and let \(Y\) denote one of the counter-examples with the least number of nondegenerate ends, let \(f\) denote the corresponding 1-to-1 finitely discontinuous function and let \(\text{DIS}\) denote the set of discontinuities of \(f\) plus the endpoints \(\{0, 1\}\).

Case 1. Suppose \(Y\) has no nondegenerate end. Then for each component \((a, b)\) of \([0,1] - \text{DIS}\), \(f((a, b))\) is a finite graph and so \(Y\) is a graph. Let \(J\) denote \(f(\text{DIS})\) plus the (degenerate) ends in \(Y\). Then each component of \([0,1] - f^{-1}(J)\) maps onto a component of \(Y - J\). Suppose \([0,1] - f^{-1}(J)\) has \(m\) components. Then \(m = |f^{-1}(J)| - 1 = |J| - 1\). Since \(f\) is 1-to-1, \(Y - J\) has \(m\) components also. Consider \(J\) as the vertices of the graph \(Y\). Then the
Euler number of $Y$ is $m - |J| = -1$ and so $Y$ has no simple closed curve. Each acyclic graph satisfies the conclusion of the theorem, so case 1 cannot hold.

**Case 2.** Suppose $Y$ has at least one nondegenerate end. We will show that for some component $(a, b)$ of $[0, 1] - \text{DIS}$ and some $\delta > 0$, either $f((a, a + \delta))$ or $f((b - \delta, b))$ is open in $Y$.

**Claim.** Suppose $\{x_i\} \to x_0 \in \text{DIS}$, each $x_i < x_0$, and $M$ is a continuum in $Y$ containing each $f(x_i)$. Then there is an $\varepsilon > 0$ such that $M$ contains $f((x_0 - \varepsilon, x_0))$.

First note that if $K$ is a continuum in $Y$, $[a, b] \subset I - \text{DIS}$ and neither $f(a)$ nor $f(b)$ belongs to $K$ but $f(c)$ is in $K$ for some $c$ in $(a, b)$, then either $K \subset f([a, b])$ or for some discontinuity $x$ and some $\delta > 0$, $K$ contains either $f((x, x + \delta))$ or $f((x - \delta, x))$. Otherwise there are at least two but no more than countably many disjoint closed intervals $I_1, I_2, \ldots$ in $[0, 1] - \text{DIS}$ such that $K$ is the union of countably many disjoint compacta $f(\text{DIS}) \cap K$ and $f(I_j) \cap K$, $j = 1, 2, \ldots$. This contradicts Theorem 56 [9, p. 23].

Now suppose $\{x_i\}, x_0$, and $M$ satisfy the hypothesis of the claim.

If no such $\varepsilon$ exists, then there is another sequence $\{x'_i\} \to x_0$ such that for each $i$, $x'_i < x_{i+1}' < x_0$, $f(x'_i)$ is not in $M$, and some $x_j$ lies between $x_i'$ and $x_{i+1}'$. For each integer $n$ there are open sets $U_1, \ldots, U_n$ in $M$ with disjoint closures such that $U_i$ contains the compactum $f([x'_i, x'_{i+1}]) \cap M$ for $i = 1, 2, \ldots, n$. Since the continuum $M$ is not in $U_1$, the component $L_1$ of $U_1$ containing some $f(x_i)$ for $x'_i < x_j < x'_2$ has a limit point $p_1$ in $M$ on the boundary of $U_1$ ("to the boundary" theorem, for instance in [7]). Since $\overline{L_1}$ contains neither $f(x'_1)$ nor $f(x'_2)$ but does contain $f(x_j)$ and a point $p$ not in $f([x'_1, x'_2])$ it follows from the earlier observation that $\overline{L_1}$ contains an end. Similarly, each $\overline{U_j}$ contains an end, contradicting the fact that there are only finitely many ends in $Y$.

Now, let $E$ denote the union of all the nondegenerate ends in $Y$. Write $[0, 1] - \text{DIS} = C_1 \cup C_2 \cup \cdots$, where each $C_i$ is a closed interval. Then $E$ is the countable union of the closed sets $E \cap f(C_1), E \cap f(C_2), \ldots$. Since $E - f(\text{DIS})$ is locally compact, it satisfies the Baire category theorem, so some $p$ in some $E \cap f(C_i)$ is not a limit point of the union of the other compacta. Since $p \in E$, $p$ belongs to a nondegenerate end, say $\text{End}(x - \delta, x)$. Thus there is a sequence $\{x_i\} \to x$, $x_i < x$, with $\{f(x_i)\} \to p$ and each $x_i \notin E$. Thus no end contains $f((x - \varepsilon, x))$ for any $\varepsilon > 0$ and it follows from the claim (since each end is a continuum) that for each end there is an $\varepsilon'$ so that the end misses $f((x - \varepsilon', x))$. There are only finitely many ends, degenerate or nondegenerate, so some $\delta' > 0$ exists such that no point of $f((x - \delta', x))$ belongs to any end.

Since $f((x - \delta', x))$ is open in $Y$, its complement is compact. Let $Y' = Y - f((x - \delta', x)) \cup A$, where $A$ is an open arc added to $Y$ so that the endpoints
of $A$ are $q$ and $q' = f(x - \delta')$. Then $Y'$ is a continuum. Now define $\hat{f}$ from $[0,1]$ onto $Y'$ by $\hat{f} = f \upharpoonright ([0,1] - (x - \delta', x)) \cup f_1$ where $f_1$ is a homeomorphism from $[x - \delta', x)$ onto $A - \{q\}$. Then $\hat{f}$ is also 1-to-1 and finitely discontinuous, and $\hat{f}$ has one less (than $f$) nondegenerate end in $Y'$ so $Y'$ is tree-like by assumption.

Note that $Y' - A = H \cup K$, two disjoint continua, since no component of $Y' - A$ can have both $q$ and $q'$ in its closure (because $Y'$ is tree-like and hence unicoherent). This means that $Y - f((x - \delta', x)) = H \cup K$, so $Y$ is the union of two disjoint tree-like continua $H \cup K$ plus a ray from one to the other, and $Y$ must be tree-like also.

If $M$ is a subcontinuum of $Y$ that intersects $f((x - \delta', x))$, then each interior point of $f((x - \delta', x)) \cap M$ is a cut point of $M$ and if $M \subset H$ or $M \subset K$ then $M$ has a cut point since each subcontinuum of $Y'$ has a cut point by assumption.

§III

Nadler and Ward showed in [11] that if each subcontinuum of $Y$ has an endpoint then $Y$ is not the $k$-to-$l$ image of any continuum, and in [1] D. Fox proved that if each subcontinuum of $Y$ has a cut point then $Y$ is not a 2-to-1 continuous image of any continuum (see Corollary 1). As a direct result (Corollary 2), it follows from Theorem 1 that no 1-to-1 finitely discontinuous image of $[0,1]$ can be the 2-to-1 image of any continuum. The class of tree-like continua which are known not to be 2-to-1 images is further expanded in Corollary 3 and Theorem 3 to include continua whose subcontinua either have finite separating sets or can be "pruned". An example of a tree-like continuum which can be pruned but has no end point nor finite separating set is the Cantor parquet set, CP:

Let $C$ be the Cantor set in $[0,1]$ and define

$$CP = \{(x,y): x \in C, \; 0 \leq y \leq 1\} \cup \{(x,y): y \in C, \; -1 \leq x \leq 0\}$$

$$\cup \{(x,y): -x \in C, \; -1 \leq y \leq 0\} \cup \{(x,y): -y \in C, \; 0 \leq x \leq 1\}.$$

And lastly, Example 4 shows there is a tree-like continuum, in fact an arc-like continuum, that is the 2-to-1 finitely discontinuous image of a continuum.

**Reference Theorem** (D. Fox [1, paraphrase of Theorems 1 and 3]). *If $f$ maps the continuum $X$ onto the continuum $Y$ then there is a subcontinuum $Y'$ of $Y$ such that (1) $f^{-1}(Y')$ is a continuum but for no proper subcontinuum $Y''$ of $Y'$ is $f^{-1}(Y'')$ a continuum, and (2) if $Y'$ satisfies (1) and the point $p$ separates $Y'$ into $n$ disjoint separated sets, then $f^{-1}(p)$ has more than $n$ components.*

**Corollary 1** (Also proved in [2] and in [10]). *If each subcontinuum of the continuum $Y$ has a cut point, then $Y$ is not the 2-to-1 continuous image of any continuum.*

**Corollary 2.** *If $Y$ is the 1-to-1 finitely discontinuous image of $[0,1]$ then $Y$ is not the 2-to-1 continuous image of any continuum.*
Corollary 3. If each subcontinuum of the continuum $Y$ has a finite separating set and $Y$ is hereditarily unicoherent, then $Y$ is not the 2-to-1 image of any continuum.

Proof. Suppose $M$ is a subcontinuum of $Y$ and the finite set $F \subset M$ minimally separates $M$. Then $M - F = U \cup V$, two disjoint open sets. If the point $p$ of $F$ belongs to $\overline{U} - U$ and does not belong to $\overline{V} - V$ then $M - (F - \{p\}) = (U \cup \{p\}) \cup V$, two disjoint open sets. This contradicts the minimality of $F$. Hence $F = [\overline{U} - U] \cap [\overline{V} - V] = \overline{U} \cap \overline{V}$. Now suppose $\overline{U}$ is not connected, $\overline{U} = A \cup B$, two separated closed sets. Since $M$ is connected and $M = A \cup (B \cup V) = B \cup (A \cup V)$, $A$ and $B$ both intersect $F$. But $M - (F \cap A) = [A - F] \cup [V \cup B]$, two separated sets and $F \cap A$ is a proper subset of $F$ (since $F \cap B$ exists). Hence $\overline{U}$ is connected, and likewise $\overline{V}$. Since $m$ is unicoherent and $\overline{V} \cap \overline{U}$ is $F$, $F$ must be a single point. Thus each subcontinuum of $Y$ has a cut point and Corollary 3 follows from Corollary 1.

Definition. The closed, totally disconnected set $K$ in the nondegenerate continuum $Y$ prunes $Y$ if $Y - K$ is disconnected and every component of $Y - K$ except one has exactly one point of $K$ in its closure.

Lemma. If $X$ and $Y$ are continua, $f: X \to Y$ is 2-to-1 and continuous and if some set $K$ prunes $Y$, then some proper subcontinuum of $Y$ has connected inverse.

Proof. Let $D$ be a component of $Y - K$ with one point, $p(D)$, of $K$ in its closure. If some component of $X - f^{-1}(K)$ that maps into $D$ has both points of $f^{-1}(p(D))$ in its closure then $f^{-1}(D)$ is connected. Hence we may assume that each component $V$ of $X - f^{-1}(K)$ with $f(V) \subset D$ has only one point, $q(V)$, of $f^{-1}(p(D))$ in its closure.

Since $Y$ is connected, some component $W$ of $Y - K$ has $K$ in its closure. Either $f^{-1}(W)$ is connected and the theorem is proved, or $f^{-1}(W) = N \cup M$, two disjoint closed sets in $X$. Since $k$-to-1 maps preserve dimension [3], $f^{-1}(K)$ is also closed and totally disconnected, so there is a disconnection $f^{-1}(K) = K_1 \cup K_2$ so that $N \cup K_1$ and $M \cup K_2$ are disjoint closed sets. Define $N' = N \cup K_1 \cup (\bigcup\{V: q(V) \in K_1\text{ and } V \in \mathcal{V}\})$ and $M' = M \cup K_2 \cup (\bigcup\{V: q(V) \in K_2\text{ and } V \in \mathcal{V}\})$, where $\mathcal{V}$ is the collection of components of $X - f^{-1}(K)$ whose image has only one limit point in $K$. Then $X = N' \cup M'$ is a disconnection of $X$, a contradiction.

Theorem 2. If every subcontinuum of the continuum $Y$ can be pruned then $Y$ is not the 2-to-1 continuous image of any continuum.

Proof. Theorem 2 follows directly from the Lemma and part (1) of the D. Fox reference theorem.

Example 4. There is a 2-to-1 function with only one discontinuity from a hereditarily decomposable tree-like continuum onto the indecomposable arc-like Knaster bucket-handle space (described in Example 1).
Let \( F(1), F(2), \ldots \) denote disjoint copies of a Cantor fan such that for each \( i \), \( F(i) \) has diameter less than \( 1/i \) and \( x_i \) denotes its only cut point. (A Cantor fan is a copy of \( C \times [0, 1] \) with the points of \( C \times \{0, 1\} \) identified, where \( C \) is a Cantor set.) Let \( W \) be the union of the \( F(i) \) with the \( x_i \) points identified as a single point \( x \). The point \( x \) will be the only discontinuity for \( g : W \rightarrow Y \), where \( Y \) is the bucket-handle space. Each \( F(i) - \{x\} \) is a copy of \( C \times [0, 1] \), and recall that \( Y = \bigcup S_i \) (see Example 1) where \( S_i \) is a collection of semicircles homeomorphic to \( C \times [0, 1] \). Let \( f_i \) denote a homeomorphism from \( C \times [0, 1] \) onto \( S_i \), for each \( i \), and let \( h \) denote the reversing homeomorphism from \( C \times \{0, 1\} \) onto \( C \times (0, 1) \) defined by \( h(c, r) = (c, 1 - r) \). Now define \( g_i \) from \( (F(2i) \cup F(2i - 1)) - \{x\} \) onto \( S_i \) by \( g_i(p) = f_i(p) \) if \( p \) is in \( F(2i) \), and \( g_i(p) = f_i h(p) \) if \( p \) is in \( F(2i - 1) \). Note that \( g_i \) is 1-to-1 at the points of \( S_i \) on the \( x \)-axis, i.e. on \( f_i^{-1}(C \times \{0, 1\}) \), and \( g_i \) is 2-to-1 at the other points of \( S_i \). Finally, let \( g \) be the union of the \( g_i \) maps plus the ordered pair \((x, (0, 0))\).

References

4. J. Heath, \textit{There is no k-to-l function from any continuum onto \([0, 1]\), or any dendrite, with only finitely many discontinuities}, Trans. Amer. Math. Soc. 306 no. 1 (1988), 293–305.