TREELIKE CONTINUA AND EXACTLY k-TO-1 FUNCTIONS

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Abstract. To answer a question of Nadler and Ward, k-to-l maps from treelike continua onto tree-like continua are constructed, for $k > 2$. It is shown that certain arc-like continua cannot be the domain of any 2-to-1 map and that certain tree-like continua cannot be the image of any 2-to-1 map (defined on continua) but it is unknown if any indecomposable arc-like continuum can be the domain or any tree-like continuum the image of a 2-to-1 map.

Introduction

In [11], Nadler and Ward prove that any continuum not hereditarily unicoherent is the exactly k-to-1 continuous image of some continuum and they prove that any continuum whose every subcontinuum has an endpoint cannot be the exactly k-to-1 continuous image of any continuum, unless $k = 1$. In [4] it was shown that no dendrite is the (exactly) k-to-1 image of any continuum even if the function is allowed to be finitely discontinuous. These results leave unanswered a question that Nadler and Ward ask, namely can any tree-like continuum be the (exactly) k-to-1 image of a continuum? Constructed in this paper (in §1) is an example of an arc-like continuum that admits a k-to-1 continuous map onto itself for any odd integer $k$, and for each even integer $k > 2$, two tree-like continua are constructed and a k-to-1 map from one onto the other.

The difficult case is $k = 2$. Two settings are studied: 2-to-1 functions with arc-like domains in §II and 2-to-l functions with tree-like images in §III.

For the case of the tree-like domain of a 2-to-1 map there are easy examples, one from [1] is illustrated on the next page.

Also, Wayne Lewis has constructed a 2-to-l map on an arc-like continuum, see Example 3 in §II, but the following question is not yet answered.

Question 1. Can any indecomposable arc-like continuum be the domain of a 2-to-1 continuous map?

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Figure 1. “An example of a 2-to-1 map on a tree-like continuum”

In [9, (IV, 4 and 5)], Mioduszewski showed that neither the arc-like indecomposable Knaster bucket-handle continuum (described in Example 1 in §II) nor the arc-like decomposable Knaster $V$-continuum (see [6], p. 570) can be the domain of a 2-to-1 map and in §II, the class of continua which cannot be the domain of a 2-to-1 finitely discontinuous function is shown to include all finitely discontinuous 1-to-1 images of $[0, 1]$, and these continua are shown to be tree-like.

In §III, which concerns tree-like images, an example of a 2-to-1 finitely discontinuous function from a tree-like continuum onto the arc-like Knaster bucket-handle space is given, but no example is known for continuous maps.

Question 2. Can any tree-like continuum be the 2-to-1 continuous image of a continuum?

Some tree-like continua can be ruled out. It is shown here that no 1-to-1 finitely discontinuous image of $[0, 1]$, necessarily tree-like, can be such a continuous image, nor any continuum whose every subcontinuum has a finite separating set. More generally, it is shown that if every subcontinuum of a continuum $Y$ “can be pruned” (a generalization of “has a cut point”) then $Y$ is not the continuous 2-to-1 image of any continuum.

All continua are understood to be compact and metric.

§I

The examples in this section answer affirmatively for all $k > 2$ the Nadler-Ward question: can a tree-like continuum be the $k$-to-1 image of some continuum? Exactly which tree-like continua are images, however, is not known.

Example 1. An arc-like continuum $X$ that maps $k$-to-1 onto itself for each odd positive integer $k$. 
The function used will be a union of sawtooth functions as follows. If \( k \) is a positive odd integer and \( A \) is an arc, choose \( k + 1 \) points of \( A \), \( a_1 < a_2 < \cdots a_{k+1} \), with \( a_1 \) and \( a_{k+1} \) endpoints of \( A \), and define a continuous \( f \) on \( A \) so that \( f(a_i) = a_i \), \( f(a_{k+1}) = a_{k+1} \), and for each \( i \) from 1 to \( k \), \( f \) restricted to the subarc \([a_i, a_{i+1}]\) is a homeomorphism onto \( A \). Note that \( f \) is \( k \)-to-1 at each interior arc point, \( f \) is \([(k + 1)/2]\)-to-1 at each endpoint of \( A \), and if \( A \) and \( B \) are two arcs sharing only a common endpoint \( x \) with sawtooth functions defined on each, then the union of the functions is \( k \)-to-1 at \( x \).

Let \( Y \) denote the classic planar Knaster bucket-handle space (see [7]) constructed as follows. Let \( C \) denote the usual deleted middle thirds Cantor set in \([0,1]\) and let \( Y = \bigcup S_i \), where \( S_1 \) is the union of all semicircles in \( \{y \geq 0\} \) with center \((1/2,0)\) and both endpoints in \( C \), and for \( i \geq 3 \), \( S_{i-1} \) is the union of all semicircles in \( \{y \leq 0\} \) with center \((5/(2 \cdot 3^i),0)\), diameter no more than \(1/3^i\), and both endpoints in \( C \).

Define \( X \) to be \( Y \cup V \), where \( V \) is \( Y \) reflected about the \( y \)-axis. The \( k \)-to-1 continuous map \( f \) from \( X \) onto \( X \) is simply the union of sawtooth functions defined on each semicircle described in the construction of \( Y \), and on each reflected semicircle in \( V \). The sawtooth functions need to be defined uniformly on each \( S_i \) collection of semicircles for \( f \) to be continuous.

**Example 2.** If \( k \) is an even integer greater than two, then there are tree-like continua \( W \) and \( Z \) and a \( k \)-to-1 continuous map from \( W \) onto \( Z \).

Let \( X = Y \cup V \) be the continuum described in Example 1. To construct \( W \), glue the \((0,0)\) points of \((k-3) + (k-3) + (k-2)(k-3) \) copies \( V_1, \ldots, V_{k-3}, Y_1, \ldots, Y_{k-3}, Y(1,1), \ldots, Y(k-2,k-3) \) of \( Y \) to \( X \) at \((0,0)\). For each \( \varepsilon > 0 \), \( W \) can be \( \varepsilon \)-mapped onto an \( n \)-od with \( n = 2 + k(k-3) \) and so in particular, \( W \) is tree-like. The image space \( Z \) is the subspace \( X \cup Y_1 \cup \cdots \cup Y_{k-3} \) of \( W \).

Define the \( k \)-to-1 map \( g \) from \( W \) onto \( Z \) as follows. On \( X \), \( g \) is the same as \( f \) in Example 1 for \( k = 3 \). On each \( V_i \), \( g \) is the natural 1-to-1 projection onto \( V \), and on each \( Y(i,j) \), \( g \) is the natural 1-to-1 projection onto \( Y_i \) if \( 1 < i < k - 3 \) and onto \( Y \) if \( i = k - 2 \). Finally, \( g \upharpoonright Y_i \) maps onto \( Y_i \), for \( 1 \leq i \leq k - 3 \), exactly as \( f \) does from \( Y \) onto \( Y \). It is straightforward to count the \( k \) point inverses for each point in \( Z \).

\( \S II \)

The collection of 1-to-1 finitely discontinuous images of \([0,1]\) includes, besides arcs and acyclic finite graphs, and \( \sin(\frac{1}{x}) \) curve plus its limiting set, a triod plus a ray spiraling down on the triod, and many other relatively simple continua. All 1-to-1 finitely discontinuous images of \([0,1]\), if continua, are shown to be tree-like in Theorem 1. It is an immediate corollary to the reference theorem below that no 1-to-1 finitely discontinuous image of \([0,1]\) can be the domain of any 2-to-1 finitely discontinuous function onto a Hausdorff space.
Reference Theorem (Heath, [5]). There is no 2-to-1 finitely discontinuous function defined on \([0, 1]\) whose image is Hausdorff.

Until recently there were no known examples of 2-to-1 maps defined on chainable continua but the following is a new result of Wayne Lewis.

Example 3 (I. W. Lewis). A chainable continuum \(X\) that admits an exactly 2-to-1 map onto a continuum.

Construction. Lewis has shown in [8] that for each \(n \geq 2\) there is a chainable continuum \(C\) admitting a homeomorphism \(h\) of period \(n\) such that \(h\) has exactly one fixed point \(p\) and every other point of \(C\) has minimal period \(n\). Furthermore, there is a point \(q\) in \(C\) such that for each \(e > 0\) there is an \(e\)-chain covering \(C\) with \(p\) and \(q\) in opposite end-links. The continuum \(C\) will be indecomposable and may be chosen to be the pseudo-arc.

For \(i = 1, 2, 3, \ldots\) let \(C_i\) denote a copy of \(C\) for \(n = 2\) with diameter less than \(1/2^i\), and let \(p_i\) and \(q_i\) denote the copies of \(p\) and \(q\) respectively in \(C_i\). To construct \(X\), string together the \(C_i\) by identifying \(q_i\) and \(p_{i+1}\) for \(i = 1, 2, \ldots\), and add a limiting point, \(p_w\), of \(\{C_i\}\). Consider the space \(X'\) obtained from \(X\) by identifying each point \(x\) of \(C_i\) with \(h_i(x)\) (where \(h_i\) is the period 2 homeomorphism of \(C_i\) with fixed point \(p_i\)) and also identifying \(p_1\) with \(p_w\). Then there is a 2-to-1 map from \(X\) onto \(X'\), namely the map determined by the identifications.

Theorem 1. If the continuum \(Y\) is a 1-to-1 finitely discontinuous image of \([0, 1]\), then \(Y\) is tree-like and each subcontinuum of \(Y\) has a cut-point.

Proof. If \(f : [0, 1] \rightarrow Y\) for some continuum \(Y\), where \(f\) is 1-to-1 and finitely discontinuous, certain "end" sets in \(Y\) can be defined. If \((a, b)\) is a component of \((0, 1)\) minus the discontinuities and \(c\) is in \((a, b)\), then \(\text{End}(a, c)\) denotes those points in \(Y\) that are limits of some \(\{f(x_i)\}\) sequence, where \(x_i\) is in \((a, c)\) and \(x_i \rightarrow a\). Similarly \(\text{End}(c, b)\) is defined. Each end is compact and connected.

Now suppose there are continua that satisfy the hypothesis but not the conclusion, and let \(Y\) denote one of the counter-examples with the least number of nondegenerate ends, let \(f\) denote the corresponding 1-to-1 finitely discontinuous function and let \(\text{DIS}\) denote the set of discontinuities of \(f\) plus the endpoints \(\{0, 1\}\).

Case 1. Suppose \(Y\) has no nondegenerate end. Then for each component \((a, b)\) of \([0, 1] - \text{DIS}\), \(\overline{f((a, b))}\) is a finite graph and so \(Y\) is a graph. Let \(J\) denote \(f(\text{DIS})\) plus the (degenerate) ends in \(Y\). Then each component of \([0, 1] - f^{-1}(J)\) maps onto a component of \(Y - J\). Suppose \([0, 1] - f^{-1}(J)\) has \(m\) components. Then \(m = |f^{-1}(J)| - 1 = |J| - 1\). Since \(f\) is 1-to-1, \(Y - J\) has \(m\) components also. Consider \(J\) as the vertices of the graph \(Y\). Then the
Euler number of $Y$ is $m - |J| = -1$ and so $Y$ has no simple closed curve. Each acyclic graph satisfies the conclusion of the theorem, so case 1 cannot hold.

**Case 2.** Suppose $Y$ has at least one nondegenerate end. We will show that for some component $(a, b)$ of $[0, 1] - \operatorname{DIS}$ and some $\delta > 0$, either $f((a, a + \delta))$ or $f((b - \delta, b))$ is open in $Y$.

**Claim.** Suppose $\{x_i\} \to x_0 \in \operatorname{DIS}$, each $x_i < x_0$, and $M$ is a continuum in $Y$ containing each $f(x_i)$. Then there is an $\varepsilon > 0$ such that $M$ contains $f((x_0 - \varepsilon, x_0))$.

First note that if $K$ is a continuum in $Y$, $[a, b] \subset I - \operatorname{DIS}$ and neither $f(a)$ nor $f(b)$ belongs to $K$ but $f(c)$ is in $K$ for some $c$ in $(a, b)$, then either $K \subset f([a, b])$ or for some discontinuity $x$ and some $\delta > 0$, $K$ contains either $f((x, x + \delta))$ or $f((x - \delta, x))$. Otherwise there are at least two but no more than countably many disjoint closed intervals $I_1, I_2, \ldots$ in $[0, 1] - \operatorname{DIS}$ such that $K$ is the union of countably many disjoint compacta $f(\operatorname{DIS}) \cap K$ and $f(I_j) \cap K$, $j = 1, 2, \ldots$. This contradicts Theorem 56 [9, p. 23].

Now suppose $\{x_i\}, x_0$, and $M$ satisfy the hypothesis of the claim.

If no such $\varepsilon$ exists, then there is another sequence $\{x'_i\} \to x_0$ such that for each $i$, $x'_i < x'_{i+1} < x_0$, $f(x'_i)$ is not in $M$, and some $x_j$ lies between $x'_i$ and $x'_{i+1}$. For each integer $n$ there are open sets $U_1, \ldots, U_n$ in $M$ with disjoint closures such that $U_i$ contains the compactum $f([x'_i, x'_{i+1}]) \cap M$ for $i = 1, 2, \ldots, n$. Since the continuum $M$ is not in $U_1$, the component $L_1$ of $U_1$ containing some $f(x_j)$ for $x'_i < x_j < x'_2$ has a limit point $p_1$ in $M$ on the boundary of $U_1$ ("to the boundary" theorem, for instance in [7]). Since $\overline{L_1}$ contains neither $f(x'_1)$ nor $f(x'_2)$ but does contain $f(x_j)$ and a point $p$ not in $f([x'_1, x'_2])$ it follows from the earlier observation that $\overline{L_1}$ contains an end. Similarly, each $\overline{U_j}$ contains an end, contradicting the fact that there are only finitely many ends in $Y$.

Now, let $E$ denote the union of all the nondegenerate ends in $Y$. Write $[0, 1] - \operatorname{DIS} = C_1 \cup C_2 \cup \cdots$, where each $C_i$ is a closed interval. Then $E$ is the countable union of the closed sets $E \cap f(DIS), E \cap f(C_1), E \cap f(C_2), \ldots$. Since $E - f(DIS)$ is locally compact, it satisfies the Baire category theorem, so some $p$ in some $E \cap f(C_j)$ is not a limit point of the union of the other compacta. Since $p \in E$, $p$ belongs to a nondegenerate end, say $\operatorname{End}(x - \delta, x)$. Thus there is a sequence $\{x_i\} \to x$, $x_i < x$, with $\{f(x_i)\} \to p$ and each $x_i \notin E$. Thus no end contains $f((x - \varepsilon, x))$ for any $\varepsilon > 0$ and it follows from the claim (since each end is a continuum) that for each end there is an $\varepsilon'$ so that the end misses $f((x - \varepsilon', x))$. There are only finitely many ends, degenerate or nondegenerate, so some $\delta' > 0$ exists such that no point of $f((x - \delta', x))$ belongs to any end.

Since $f((x - \delta', x))$ is open in $Y$, its complement is compact. Let $Y' = Y - f((x - \delta', x)) \cup A$, where $A$ is an open arc added to $Y$ so that the endpoints
of \( A \) are \( q \) and \( q' = f(x - \delta') \). Then \( Y' \) is a continuum. Now define \( \hat{f} \) from \([0,1]\) onto \( Y' \) by \( \hat{f} = f \upharpoonright ([0,1] - (x - \delta',x)) \cup f_1 \) where \( f_1 \) is a homeomorphism from \([x - \delta',x]\) onto \( \overline{A} - \{q\} \). Then \( \hat{f} \) is also 1-to-1 and finitely discontinuous, and \( \hat{f} \) has one less (than \( f \)) nondegenerate end in \( Y' \) so \( Y' \) is tree-like by assumption.

Note that \( Y' - A = H \cup K \), two disjoint continua, since no component of \( Y' - A \) can have both \( q \) and \( q' \) in its closure (because \( Y' \) is tree-like and hence unihoherent). This means that \( Y - f((x - \delta',x)) = H \cup K \), so \( Y \) is the union of two disjoint tree-like continua \( H \cup K \) plus a ray from one to the other, and \( Y \) must be tree-like also.

If \( M \) is a subcontinuum of \( Y \) that intersects \( f((x - \delta',x)) \), then each interior point of \( f((x - \delta',x)) \cap M \) is a cut point of \( M \) and if \( M \subset H \) or \( M \subset K \) then \( M \) has a cut point since each subcontinuum of \( Y' \) has a cut point by assumption.

§III

Nadler and Ward showed in [11] that if each subcontinuum of \( Y \) has an endpoint then \( Y \) is not the \( k \)-to-1 image of any continuum, and in [1] D. Fox proved that if each subcontinuum of \( Y \) has a cut point then \( Y \) is not a 2-to-1 continuous image of any continuum (see Corollary 1). As a direct result (Corollary 2), it follows from Theorem 1 that no 1-to-1 finitely discontinuous image of \([0,1]\) can be the 2-to-1 image of any continuum. The class of tree-like continua which are known not to be 2-to-1 images is further expanded in Corollary 3 and Theorem 3 to include continua whose subcontinua either have finite separating sets or can be "pruned". An example of a tree-like continuum which can be pruned but has no end point nor finite separating set is the Cantor parquet set, CP:

Let \( C \) be the Cantor set in \([0,1]\) and define

\[
\text{CP} = \{(x,y) : x \in C, \ 0 \leq y \leq 1\} \cup \{(x,y) : y \in C, \ -1 \leq x \leq 0\}
\]

\[
\cup \{(x,y) : -x \in C, \ -1 \leq y \leq 0\} \cup \{(x,y) : -y \in C, \ 0 \leq x \leq 1\}
\]

And lastly, Example 4 shows there is a tree-like continuum, in fact an arc-like continuum, that is the 2-to-1 finitely discontinuous image of a continuum.

Reference Theorem (D. Fox [1, paraphrase of Theorems 1 and 3]). If \( f \) maps the continuum \( X \) onto the continuum \( Y \) then there is a subcontinuum \( Y' \) of \( Y \) such that (1) \( f^{-1}(Y') \) is a continuum but for no proper subcontinuum \( Y'' \) of \( Y' \) is \( f^{-1}(Y'') \) a continuum, and (2) if \( Y' \) satisfies (1) and the point \( p \) separates \( Y' \) into \( n \) disjoint separated sets, then \( f^{-1}(p) \) has more than \( n \) components.

Corollary 1 (Also proved in [2] and in [10]). If each subcontinuum of the continuum \( Y \) has a cut point, then \( Y \) is not the 2-to-1 continuous image of any continuum.

Corollary 2. If \( Y \) is the 1-to-1 finitely discontinuous image of \([0,1]\) then \( Y \) is not the 2-to-1 continuous image of any continuum.
Corollary 3. If each subcontinuum of the continuum $Y$ has a finite separating set and $Y$ is hereditarily unicoherent, then $Y$ is not the 2-to-1 image of any continuum.

Proof. Suppose $M$ is a subcontinuum of $Y$ and the finite set $F \subset M$ minimally separates $M$. Then $M - F = U \cup V$, two disjoint open sets. If the point $p$ of $F$ belongs to $\overline{U} - U$ and does not belong to $\overline{V} - V$ then $M - (F - \{p\}) = (U \cup \{p\}) \cup V$, two disjoint open sets. This contradicts the minimality of $F$. Hence $F = [\overline{U} - U] \cap [\overline{V} - V] = \overline{U} \cap \overline{V}$. Now suppose $\overline{U}$ is not connected, $\overline{U} = A \cup B$, two separated closed sets. Since $M$ is connected and $M = A \cup (B \cup V) = B \cup (A \cup V)$, $A$ and $B$ both intersect $F$. But $M - (F \cap A) = [A - F] \cup [V \cup B]$, two separated sets and $F \cap A$ is a proper subset of $F$ (since $F \cap B$ exists). Hence $\overline{U}$ is connected, and likewise $\overline{V}$. Since $m$ is unicoherent and $\overline{V} \cap \overline{U}$ is $F$, $F$ must be a single point. Thus each subcontinuum of $Y$ has a cut point and Corollary 3 follows from Corollary 1.

Definition. The closed, totally disconnected set $K$ in the nondegenerate continuum $Y$ prunes $Y$ if $Y - K$ is disconnected and every component of $Y - K$ except one has exactly one point of $K$ in its closure.

Lemma. If $X$ and $Y$ are continua, $f: X \to Y$ is 2-to-1 and continuous and if some set $K$ prunes $Y$, then some proper subcontinuum of $Y$ has connected inverse.

Proof. Let $D$ be a component of $Y - K$ with one point, $p(D)$, of $K$ in its closure. If some component of $X - f^{-1}(K)$ that maps into $D$ has both points of $f^{-1}(p(D))$ in its closure then $f^{-1}(D)$ is connected. Hence we may assume that each component $V$ of $X - f^{-1}(K)$ with $f(V) \subset D$ has only one point, $q(V)$, of $f^{-1}(p(D))$ in its closure.

Since $Y$ is connected, some component $W$ of $Y - K$ has $K$ in its closure. Either $f^{-1}(W)$ is connected and the theorem is proved, or $f^{-1}(W) = N \cup M$, two disjoint closed sets in $X$. Since $k$-to-1 maps preserve dimension [3], $f^{-1}(K)$ is also closed and totally disconnected, so there is a disconnection $f^{-1}(K) = K_1 \cup K_2$ so that $N \cup K_1$ and $M \cup K_2$ are disjoint closed sets. Define $N' = N \cup K_1 \cup (\bigcup \{V: q(V) \in K_1 \text{ and } V \in \mathcal{V}\})$ and $M' = M \cup K_2 \cup (\bigcup \{V: q(V) \in K_2 \text{ and } V \in \mathcal{V}\})$, where $\mathcal{V}$ is the collection of components of $X - f^{-1}(K)$ whose image has only one limit point in $K$. Then $X = N' \cup M'$ is a disconnection of $X$, a contradiction.

Theorem 2. If every subcontinuum of the continuum $Y$ can be pruned then $Y$ is not the 2-to-1 continuous image of any continuum.

Proof. Theorem 2 follows directly from the Lemma and part (1) of the D. Fox reference theorem.

Example 4. There is a 2-to-1 function with only one discontinuity from a hereditarily decomposable tree-like continuum onto the indecomposable arc-like Knaster bucket-handle space (described in Example 1).
Let $F(1), F(2), \ldots$ denote disjoint copies of a Cantor fan such that for each $i$, $F(i)$ has diameter less than $1/i$ and $x_i$ denotes its only cut point. (A Cantor fan is a copy of $C \times [0,1]$ with the points of $C \times \{1\}$ identified, where $C$ is a Cantor set.) Let $W$ be the union of the $F(i)$ with the $x_i$ points identified as a single point $x$. The point $x$ will be the only discontinuity for $g : W \to Y$, where $Y$ is the bucket-handle space. Each $F(i) - \{x\}$ is a copy of $C \times [0,1)$, and recall that $Y = \bigcup S_i$ (see Example 1) where $S_i$ is a collection of semicircles homeomorphic to $C \times [0,1]$. Let $f_i'$ denote a homeomorphism from $C \times [0,1]$ onto $S_i$, for each $i$, and let $h$ denote the reversing homeomorphism from $C \times \{0,1\}$ onto $C \times (0,1)$ defined by $h(c,r) = (c, 1-r)$. Now define $g_i$ from $(F(2i) \cup F(2i-1)) - \{x\}$ onto $S_i$ by $g_i(p) = f_i'(p)$ if $p$ is in $F(2i)$, and $g_i(p) = f_i'h(p)$ if $p$ is in $F(2i-1)$. Note that $g_i$ is 1-to-1 at the points of $S_i$ on the x-axis, i.e. on $f_i'^{-1}(C \times \{0,1\})$, and $g_i$ is 2-to-1 at the other points of $S_i$. Finally, let $g$ be the union of the $g_i$ maps plus the ordered pair $(x,(0,0))$.

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