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EXTREMAL MINKOWSKI ADDITIVE SELECTIONS OF COMPACT CONVEX SETS

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ABSTRACT. A function $f:\mathscr{K}^n \to R^n$, defined on the set of all compact convex sets in R^n, is a Minkowski additive selection, provided $f(K+L) = f(K) + f(L)$ and $f(K) \in K$ for all K, $L \in \mathscr{K}^n$. The paper deals with selections which are extremal in some sense, in particular we characterize the set of all Minkowski additive selections which have the property $f(K) \in \operatorname{ext}(K)$ for all $K \in \mathscr{K}^n$, where $\operatorname{ext}(K)$ is the set of all extreme points of K.

0. INTRODUCTION

Let $\mathscr{K}^n$ be the set of all compact, convex sets in R^n equipped with the topology generated by the Hausdorff metric. A vector valued valuation or shortly a valuation is a function $f:\mathscr{K}^n \to R^n$ (see [2]). A valuation f is Minkowski additive if $f(K+L) = f(K) + f(L)$ for all K, $L \in \mathscr{K}^n$ and the class of all additive selections is the main object of study of this paper. Certainly, the most important and the best known valuation is the curvature centroid of a compact convex set K, first considered by J. Steiner (1840) in the case of a planar set. One of equivalent definitions, in the general case, is $s(K) = \int_{S^{n-1}} uh(K,u)d\nu(u)$ where $h(K,\cdot)$ is the support function of K and ν is the normalized rotation invariant measure on S^{n-1}. Perhaps the best known result about Steiner points is the following remarkable result of R. Schneider [7], see also Posicel$'$skiĭ [5], who confirmed a conjecture of Grünbaum by proving the following theorem.

Theorem 0. *If $f:\mathscr{K}^n \to R^n$ is Minkowski additive, continuous and rigid motion equivariant then $f(K)$ is Steiner point of K.*

In order to define the notion of an extreme additive selection and formulate the problem we are interested in, let us consider the set of all, not necessarily continuous, additive selections $\mathscr{S} = \{\varphi:\mathscr{K}^n \to R^n | (\forall K, L \in \mathscr{K}^n)\varphi(K+L) = \varphi(K)+\varphi(L) \& \varphi(K) \in K\}$. Let $\mathscr{S}$ be topologized as a subspace of $F(\mathscr{K}^n, R^n)$, the space of all functions from $\mathscr{K}^n$ to R^n with the weak topology i.e. the

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weakest topology which makes the maps $F(\mathscr{K}^n, R^n) \to R^n$, $\varphi \mapsto \varphi(K)$ ($K \in \mathscr{K}^n$), continuous. Obviously, $\mathscr{L}$ is a compact, convex set in the locally convex, topological linear space $F(K^n, R^n)$ so one can talk about the set $\mathscr{E}$ of extreme points in $\mathscr{L}$. In light of the fact that every element of $\mathscr{L}$ is an average (we have in mind theorems of Krein-Milman and Choquet, see [3]) of its extreme points, it is desirable to characterize or to say as much as possible about $\mathscr{E}$.

1. Regular extreme points

The problem of describing $\mathscr{E}$ in a useful way seems to be difficult so let us start with a more restricted class $\mathscr{R} \subset \mathscr{E}$ of so called regular extreme points. Precisely, $\mathscr{R} = \{\varphi \in \mathscr{E} \mid \varphi(K) \in \operatorname{ext}(K) \text{ for all } K \in \mathscr{K}^n\}$ where $\operatorname{ext}(K)$ is the set of all extreme points of K.

Theorem 1.1. *There is a* 1-1 *correspondence between* $\mathscr{R}$ *and* $O(n)$, *the set of all orthogonal, linear transformations of* R^n.

The proof of this theorem consists of two lemmas. Before they are formulated let us define the function $\pi: O(n) \to \mathscr{R}$ which will establish the 1-1 correspondence mentioned in the theorem. First, let us identify $O(n)$ with the set of all orthogonal frames, where an orthogonal frame, denoted by $[a_1, \dots, a_n]$, is a set of n mutually orthogonal unit vectors in R^n. For $u \in R^n$ let $\psi_u: \mathscr{K}^n \to \mathscr{K}^n$ be defined by $\psi_u(K) = K \cap \{z \in \mathscr{R}^n | \langle u, z\rangle = h(K, u)\}$ i.e. $\psi_u(K)$ is the intersection of K with the supporting hyperplane orthogonal to u. $\psi_u(K)$ will be called the extremal face of K in the direction of u. Let us note that $\psi_u(K+L) = \psi_u(K) + \psi_u(L)$. Indeed, if $h(K+L, u) = \alpha$, $h(K, u) = \beta$, $h(L, u) = \gamma$ then $\alpha = \beta + \gamma$ and $\psi_u(K+L) = \{x + y | x \in K, y \in L, \langle u, x+y\rangle = \alpha\} = \{x + y | x \in K, y \in L, \langle u, x\rangle = \beta, \langle u, y\rangle = \gamma\} = \psi_u(K) + \psi_u(L)$. If $g = [a_1, \dots, a_n] \in O(n)$ let us define $\pi_g \in \mathscr{R}$ by $\pi_g(K) = \psi_{a_n} \circ \psi_{a_{n-1}} \circ \cdots \circ \psi_{a_1}(K)$. Since $\psi_u(K)$ lies in a hyperplane orthogonal to u, we see that $\pi_g(K)$ is a single point whereas the additivity of π_g follows from the additivity of ψ_{a_i}, $1 \le i \le n$. Since obviously $\pi_g(K) \in \operatorname{ext}(K)$ we have just established the following lemma.

Lemma 1.1. *For every* $g \in O(n)$, π_g *is a regular extreme point.*

π is obviously a 1-1 map so it remains to be shown that it is onto.

Lemma 1.2. *For every* $\varphi \in \mathscr{R}$ *there exists* $g \in O(n)$ *such that* $\varphi = \pi_g$.

Proof. Let $V_1 = \{x \in R^n | \, \|x\| \le 1\}$ be the unit ball and $a_1 = \varphi(V_1)$. If $V_2 = \{x \in V_1 | \langle x, a_1\rangle = 0\}$ let $a_2 = \varphi(V_2)$. Inductively, if $a_i \in R^n$ and $V_i \subset R^n$ are defined, $i < n$, let $V_{i+1} = \{x \in V_i | \langle x, a_i\rangle = 0\}$ and $a_{i+1} = \varphi(V_{i+1})$. Since all a_i, $1 \le i \le n$, are unit vectors by the regularity of φ, and mutually orthogonal by the construction, let $g = [a_1, \dots, a_n] \in O(n)$. We claim that $\varphi(K) = \pi_g(K)$ for every $K \in \mathscr{K}^n$. First of all this is true for all sets $V_1, V_2, \dots, V_n$ by the construction. Now, let us suppose $\varphi(K) \ne \pi_g(K)$ for some $K \in \mathscr{K}^n$. Further, let K_i, $0 \le i \le n$, be defined inductively by $K_0 := K$, $K_{i+1} = \psi_{a_{i+1}}(K_i)$,

$0 \le i \le n-1$, so in particular $K_n = \{\pi_g(K)\}$. Obviously, there exists $0 \le i < n$ such that $\varphi(K) \in K_i \backslash K_{i+1}$. Since $\varphi(K + mV_{i+1}) = \varphi(K) + m\varphi(V_{i+1})$, $m \in N$, we observe that for m big enough $\varphi(K + mV_{i+1})$ is in the relative interior of the set $\pi_g(K) + mV_{i+1} \subset K + mV_{i+1}$ which contradicts the assumption $\varphi(L) \in \text{ext}(L)$ for all $L \in \mathscr{K}^n$. Hence $\varphi(K) = \pi_g(K)$ for all $K \in \mathscr{K}^n$.

Let us note that π_g cannot be continuous for any $g \in O(n)$. This can be seen directly but we prefer to prove a more general result.

Proposition 1.1. *If $f:\mathscr{K}^n \to R^n$ is a continuous selection then $f(K) = s(K)$ for some $K \in \mathscr{K}^n$.*

Proof. Actually the following proof shows that the equality $f(K) = s(K)$ holds quite often. Let $K \in \mathscr{K}^n$ be a centrally symmetric convex body such that $s(K) = 0$. Let $\mathfrak{G}$ be the Grassman manifold of all hyperplanes containing the origin and $\xi: E \xrightarrow{p} \mathfrak{G}$ the canonical $(n-1)$-plane bundle over $\mathfrak{G}$, i.e. $E = \{(l, v) | v \in l\} \subset \mathfrak{G} \times R^n$. It is well known that this bundle cannot have a nowhere zero section which follows from the fact that the total Stiefel-Whitney class of this bundle has the form $w(\xi) = 1 + u + \cdots + u^{n-1}$, $u \in H^1(\mathfrak{G}, Z_2)$, i.e. this bundle does not split since the total Stiefel-Whitney class is multiplicative and $w(\xi)$ has a nonzero u^{n-1}-term (see [4] for details). $\mathfrak{G}$ can be embedded in $\mathscr{K}^n$ by the map $I: \mathfrak{G} \to \mathscr{K}^n$, $l \mapsto l \cap K$, which induces a bundle $\xi': E' \to I(\mathfrak{G})$ isomorphic to ξ. Since $s(l \cap K) = 0$ for all $l \in \mathfrak{G}$ (K is centrally symmetric) we see that $f \restriction I(\mathfrak{G})$ can be thought of as a section of this bundle so in particular $f(l \cap K) = 0$ for some $l \in \mathfrak{G}$.

Let us observe that $\pi: O(n) \to \mathscr{R}$ is not continuous either. Otherwise the map $O(n) \xrightarrow{\pi} \mathscr{R} \xrightarrow[\text{ev}(\cdot, K)]{} \text{ext}(K)$, where $\text{ev}(\varphi, K) =: \varphi(K)$, for a fixed polyhedron K would be onto and continuous which is a contradiction with the finiteness of $\text{ext}(K)$. On the other hand the map $\text{ev}(\cdot, K) \circ \pi: O(n) \to R^n$ is measurable for every $K \in \mathscr{K}^n$. This follows from the fact that the set $A = \{e \in S^{n-1} | (\exists_1 u \in K) h(K, e) = \langle u, e \rangle$ has ν-measure 1 where ν is the normalized, rotation invariant measure on S^{n-1}. Let μ be the normalized Haar-measure on the group $O(n)$ and let $\alpha: O(n) \to S^{n-1}$ be the map $[a_1, \dots, a_n] \mapsto a_1$. Then the following holds.

Proposition 1.2. $s(K) = \int_{O(n)} \pi_g(K)\, d\mu(g)$.

Proof. The proof immediately follows from Theorem 0. Alternatively, one can directly prove, first for smooth K, that

$$\int_{O(n)} \pi_g(K)\, d\mu(g) = \int_{S^{n-1}} u h(K, u)\, d\nu(u)$$

using the fact that π_g is γ-a.e. constant function along fibers of α and then extend the result by continuity. The last proposition is interesting because it says that, in order to give a Choquet [3] representation of Steiner selection s

it is possible to concentrate the representing measure on the set $\mathscr{R}$ of regular extreme points.

2. Nonregular extreme points

Can it be that $\mathscr{E} = \mathscr{R}$? In light of examples of R. Schneider [8], pp. 76 and G. T. Sallee [6] it is very likely that the following conjecture holds.

Conjecture. $\mathscr{E}\backslash\mathscr{K} \neq \emptyset$. A natural way to establish the conjecture above is following. One has to find a valuation $\varphi \in \mathscr{L}$ and a continuous linear functional $\Lambda: F(\mathscr{K}^n, R^n) \to R$ such that $\Lambda(\varphi) > \Lambda(\pi_g)$ for all $g \in O(n)$. It is not difficult to see, since $F(\mathscr{K}^n, R^n)$ is a topological product of 1-dimensional spaces, that every continuous linear functional has the form $\Lambda(\varphi) = \langle\varphi(K_1), v_1\rangle + \cdots + \langle\varphi(K_r), v_r\rangle$ for some choice of $K_1, \dots, K_r \in \mathscr{K}^n$ and $v_1, \dots, v_r \in R^n$. Now, a natural choice for $\varphi \in \mathscr{L}\backslash \mathrm{cl}(\mathrm{conv}\,\mathscr{R})$ could be either the valuation defined by Sallee or the one defined by Schneider but we do not know if either of them can be a selection, i.e. whether $\varphi(K) \in K$ holds for all $K \in \mathscr{K}^n$.

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