A NOTE ON ROE'S CHARACTERIZATION OF THE SINE FUNCTION

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Abstract. Let \( f^{(n)} n = 0, \pm 1, \pm 2, \ldots \) be a sequence of complex valued functions on the real line with \( (d/dx)f^{(n)} = f^{(n+1)} \) and satisfying inequalities \( |f^{(n)}(x)| \leq M_n(1 + |x|)^k \) where as \( n \to \infty \) the growth conditions \( \lim M_n(1 + \varepsilon)^{-n} = 0 \) and \( \lim M_n(1 + \varepsilon)^{-n} = 0 \) hold for all \( \varepsilon > 0 \). Then \( f^{(0)}(x) = p(x)e^{ix} + q(x)e^{-ix} \) where \( p \) and \( q \) are polynomials of degree at most \( k \).

In his paper [1] J. Roe proves

Theorem (Roe [1]). Let \( \{f^{(n)}\}_{n=-\infty}^{\infty} \) be a two way infinite sequence of real valued functions defined on the real line \( R \). Assume \( f^{(n+1)}(x) = (d/dx)f^{(n)}(x) \) and that there is a constant \( M \) so that \( |f^{(n)}(x)| \leq M \) for all \( n \) and \( x \). Then \( f^{(0)}(x) = a \sin(x + \phi) \) for some real constants \( a \) and \( \phi \).

This gives a rather striking characterization of the sine functions \( a \sin(x + \phi) \) in terms of the size of their derivatives and antiderivatives. In this note we show that the bounds \( |f^{(n)}(x)| \leq M \) can be relaxed to \( |f^{(n)}(x)| \leq M_n(1 + |x|)^\alpha \) with \( 0 \leq \alpha < 1 \) and where the constants only need to have supexponential growth. More generally:

Theorem. Let \( \{f^{(n)}\}_{n=-\infty}^{\infty} \) be a sequence of complex valued functions defined on the real numbers with

\( f^{(n+1)}(x) = \frac{d}{dx}f^{(n)}(x) \)

and so that there are constants \( M_n \geq 0, \alpha \in [0, 1], \) and a nonnegative integer \( k \) satisfying

\[ |f^{(n)}(x)| \leq M_n(1 + |x|)^{k+\alpha}. \]

If

\[ \lim_{n \to \infty} \frac{M_n}{(1 + \varepsilon)^n} = 0 \quad \forall \varepsilon > 0 \]

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and

\[(4) \quad \lim_{n \to \infty} \frac{M_{-n}}{(1 + \varepsilon)^n} = 0 \quad \text{all } \varepsilon > 0,\]

then

\[(5) \quad f^{(0)}(x) = p(x)e^{ix} + q(x)e^{-ix}\]

where \(p(x)\) and \(q(x)\) are polynomials of degree at most \(k\).

**Remark 1.** The conclusion of the theorem can be sharpened by giving a more precise description of the functions \(f^{(n)}\). If \(k = 0, 1, 2, \ldots\) and \(n = 0, \pm 1, \pm 2, \ldots\) then define

\[
p_{n,k}(x) = (1 + \frac{d}{dx})^n x^k = \sum_{m=0}^{k} \frac{n(n-1) \cdots (n+m-k+1)}{(k-m)!} k(k-1) \cdots (m+1)x^m
\]

where for negative \(n\) we expand \((1 + d/dx)^n\) formally by use of Taylor’s theorem. Then \(p_{n,k}(x)\) is a polynomial of degree \(k\) and

\[
p_{n,k}(x) + p'_{n,k}(x) = \left(1 + \frac{d}{dx}\right) p_{n,k}(x) = p_{n+1,k}(x).
\]

When \(n \geq 0\) these are Laguerre polynomials. This is because

\[
(-1)^k p_{n,k}(-x) = \left(1 - \frac{d}{dx}\right)^n x^k = e^x \frac{d^n}{dx^n}(x^k e^{-x}).
\]

See for example [0, p. 204]. This last equation implies that if \(\lambda\) is a complex number and

\[
f_{k,\lambda}^{(n)}(x) = \lambda^n p_{n,k}(\lambda x)e^{\lambda x}
\]

then \(\{f_{k,\lambda}^{(n)}\}_{n=-\infty}^{\infty}\) satisfies equation (1). Then if \(\{f_{k}^{(n)}\}_{n=-\infty}^{\infty}\) is a sequence of functions satisfying the hypothesis of the theorem then there are complex numbers \(a_0, \ldots, a_k, b_0, \ldots, b_k\) so that

\[
f^{(n)} = \sum_{m=0}^{k} (a_m f_{m,i}^{(n)} + b_m f_{m,-i}^{(n)})
\]

where \((i)^2 = -1\). The proof of this from the theorem is done by induction on \(k\). The details are left to the reader.

**Remark 2.** The functions \(f_{k,\lambda}^{(n)}(x)\) just defined satisfy

\[
|f_{k,\lambda}^{(n)}(x)| \leq (k + 1)!n^k \max(|\lambda|^n, |\lambda|^{n+k})(1 + |x|)^k e^{\Re(\lambda)x}.
\]

By giving \(\lambda\) pure imaginary values close to \(i\) or \(-i\) we see that there is no obvious weakening of the growth conditions (2), (3), or (4).
Remark 3. It is impossible to replace the interval $(-\infty, \infty)$ by a half infinite interval. The functions $f^{(n)}(x) = (-1)^n e^{-x}$ on $(0, \infty)$ yield a counterexample. (This observation is due to David Richman.)

Proof of the theorem. Let $f(x) = f^{(0)}(x)$. The, following [1], we will show the support of the Fourier transform $\hat{f}$ of $f$ contained in the set $\{1, -1\}$. As the integral defining the Fourier transform may diverge, we define it as a distribution, that is as a linear functional on the vector spaces $\mathcal{S}'$ of rapidly decreasing functions on $R$. Explicitly the value of $\hat{f}$ on $\phi \in \mathcal{S}'$ is

$$\langle \hat{f}, \phi \rangle = \langle \hat{B}, \hat{\phi} \rangle = \int_{-\infty}^{\infty} f(x) \hat{\phi}(x) \, dx.$$  

Here we follow the notation of [2, Chapter 7].

Suppose it has been shown that the support of $\hat{f}$ is contained in $\{1, -1\}$. Then a standard result [2, Theorem 6.25, p. 150] implies there is an $m \geq 0$ and complex numbers $a_j, b_j \quad 0 \leq j \leq m$ so that

$$\hat{f} = \sum_{j=0}^{m} a_j \delta_{1}^{(j)} + \sum_{j=0}^{m} b_j \delta_{-1}^{(j)}$$

where $\delta_1$ (resp. $\delta_{-1}$) is the delta function at 1 (resp. at $-1$) and $\delta_{1}^{(j)}$ is the $j$th distributional derivative of $\delta_1$. This Fourier transform can be inverted to give that $f(x) = f^{(0)}(x)$ has the form given by (5) with $p(x)$ and $g(x)$ polynomials (of degree at most $m$). But $|f(x)| \leq |f^{(0)}(x)| \leq M_0(1 + |x|)^{k+\alpha}$. This implies the polynomials have degree at most $k$.

This reduces the proof to showing

Lemma 1. The conditions (2) and (3) imply the support of $\hat{f}$ is disjoint from $(1, \infty)$ and $(-\infty, -1)$.

Lemma 2. The conditions (2) and (4) imply the support of $\hat{f}$ is disjoint from $(-1, 1)$.

Proof of Lemma 1. We will only show the support of $\hat{f}$ is disjoint from $(1, \infty)$, the proof for $(-\infty, -1)$ being identical. Let $\phi$ be a smooth function with its support, $\text{spt}(\phi)$, in $(1, \infty)$. Then $\text{spt}(\phi) \subseteq \{r, \infty\}$ for some $r > 1$. We now need to show $\langle \hat{f}, \phi \rangle = 0$. Let $n \geq 0$ and

$$\psi_n(t) = \frac{\phi(t)}{(-it)^n}.$$  

This is smooth as $\phi = 0$ near $t = 0$. Thus differentiating under the integral gives

$$\psi_n^{(n)}(x) = \frac{d^n}{dx^n} \int_{-\infty}^{\infty} \frac{\phi(t)}{(-it)^n} e^{-itx} \, dx = \hat{\phi}(x).$$
So
\[
(f, \phi) = (f, \hat{\phi}) = (f, \psi_n^{(n)}) = (-1)^n (f^{(n)}, \psi_n) = (-1)^n \int_{-\infty}^{\infty} f^{(n)}(x) \psi_n(x) \, dx.
\]

By (2) this implies
\[
|\langle \hat{f}, \phi \rangle| \leq M_n \int_{-\infty}^{\infty} (1 + |x|^{k+\alpha} |\psi_n(x)|) \, dx.
\]

We now estimate $|\psi_n(x)|$. First using that $\text{spt}(\phi) \subseteq [r, \infty)$,
\[
|\psi_n(x)| \leq \int_{r}^{\infty} \frac{|\phi(t)|}{t^n} \, dt \leq \frac{1}{r^n} \|\phi\|_{L^1}.
\]

Also if $x \neq 0$ then integration by parts $(k + 2)$ times yields
\[
|\psi_n(x)| = \left| \int_{r}^{\infty} \left( \frac{d^{k+2}}{dt^{k+2}} \frac{\phi(t)}{(-it)^n} e^{-i\pi x} \right) dt \right|
\]
\[
\leq \frac{1}{|x|^{k+2}} \int_{r}^{\infty} \left| \frac{d^{k+2}}{dt^{k+2}} \frac{\phi(t)}{t^n} \right| dt
\]
\[
\leq \frac{1}{|x|^{k+2}} \sum_{j=0}^{k+2} \binom{k+2}{j} n(n+1) \cdots (n+j-1) \|\phi^{(k+2-j)}\|_{\infty} \int_{r}^{\infty} \frac{dt}{t^{n+j}}
\]
\[
= \frac{1}{|x|^{k+2}} \sum_{j=0}^{k+2} \binom{k+2}{j} n(n+1) \cdots (n+j-1) \frac{\|\phi^{(k+2-j)}\|_{\infty}}{(n+j-1)r^{n+j-1}}
\]
\[
\leq \frac{c_1(k, \phi)n^{k+1}}{|x|^{k+2}r^{n-1}}
\]

where $c_1(k, \phi)$ is a constant depending only on $k$ and $\phi$. This can be combined with (8) to give
\[
|\psi_n(x)| \leq \begin{cases} 
1/r^n \|\phi\|_{L^1} & |x| \leq 1, \\
\frac{c_1(k, \phi)n^{k+1}}{|x|^{k+2}r^{n-1}} & |x| > 1.
\end{cases}
\]

Using this in (7) gives an estimate of the form
\[
|\langle \hat{f}, \phi \rangle| \leq c_2(k, \alpha, \phi) \frac{n^{k+1}}{r^{n-1}} M_n.
\]

Let $\varepsilon$ be so that $1 < 1 + \varepsilon < r$. Then for large $n$
\[
\frac{n^{k+1}}{r^{n-1}} < \frac{1}{(1 + \varepsilon)^n}.
\]
Using this and the condition (3) and (11)

\[ |\langle \hat{f}, \phi \rangle| \leq c_2(k, \alpha, \phi) \lim_{n \to \infty} \frac{M_n}{(1 + \varepsilon)^n} = 0. \]

Therefore \( \langle \hat{f}, \phi \rangle = 0 \) for all \( \phi \) with support in \((1, \infty)\), i.e., the support of \( \hat{f} \) is disjoint from \((1, \infty)\) as claimed.

**Proof of Lemma 2.** The proof is very similar to the proof of Lemma 1. Let \( \phi \) be a smooth function with support in \((-1, 1)\). Then for some \( r < 1 \) the inclusion \( \text{spt}(\phi) \subseteq [-r, r] \) holds.

\[ \langle \hat{f}, \phi \rangle = \langle f^{(0)}, \hat{\phi} \rangle = \langle \frac{d^n}{dx^n} f^{(-n)}, \hat{\phi} \rangle \]

\[ = (-1)^n \langle f^{(-n)}, \phi^{(n)} \rangle \]

\[ = (-1)^n \int_{-\infty}^{\infty} f^{(-n)}(x) \hat{\phi}^{(n)}(x) \, dx. \]

By (2) this implies

\[ |\langle \hat{f}, \phi \rangle| \leq M_{-n} \int_{-\infty}^{\infty} (1 + |x|)^{k+\alpha} |\hat{\phi}^{(n)}(x)| \, dx. \tag{12} \]

Differenting under the integral and using \( \text{spt}(\phi) \subseteq [-r, r] \)

\[ \hat{\phi}^{(n)}(x) = \int_r^{-r} (-it)^n \phi(t) e^{-itx} \, dt. \tag{13} \]

Thus

\[ |\hat{\phi}^{(n)}(x)| \leq \int_r^{-r} |t|^n |\phi(t)| \, dt \leq 2r^n \|\phi\|_{L^1}. \tag{14} \]

Also for \( x \neq 0 \), integration by parts \((k + 2)\) times and calculations similar to those of inequality (9) yield

\[ |\hat{\phi}^{(n)}(x)| = \left| \int_r^{-r} \left( \frac{d^{k+2}}{dt^{k+2}} (-it)^n \phi(t) \right) \frac{e^{-itx}}{(-ix)^{k+2}} \, dt \right| \]

\[ \leq \left| \frac{1}{|x|^{k+2}} \int_r^{-r} \left| \frac{d^{k+2}}{dt^{k+2}} (t^n \phi(t)) \right| \, dt \right| \]

\[ \leq \frac{c_3(k, \phi) h^{k+2}}{|x|^{k+2}} r^{n-k-2}. \tag{15} \]

Putting the last two estimates together

\[ |\hat{\phi}^{(n)}(x)| \leq \begin{cases} 2r^n \|\phi\|_{L^1} & |x| \leq 1, \\ c_3(k, \phi) h^{k+2} r^{n-k-2} & |x| > 1. \end{cases} \tag{16} \]

Putting this in (12) gives an estimate

\[ |\langle \hat{f}, \phi \rangle| \leq c_4(k, \alpha, \phi) M_{-n} h^{k+2} r^{n-k-2}. \]
The proof is now completed in the same manner as the proof of Lemma 1.

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References

0. E. D. Rainville, Special functions, Chelsea, New York, 1971.

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