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REPRESENTATION OF AN ABSTRACT MEASURE USING BOREL-ISOMORPHISM TYPES

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ABSTRACT. For $X \subseteq R$, the mapping t which assigns to each Borel subset of X its isomorphism type is an abstract measure. Given a monoid-valued measure m, we ask when there is an $X \subseteq R$ such that t and m are isomorphic as measures.

Given a set $X \subseteq R$, we consider the mapping $B \to t(B)$ associating to each relative Borel subset B of X its Borel-isomorphism type $t(B)$. This mapping may be considered as an abstract measure taking values in a cardinal algebra, or more generally, in a partially ordered monoid. Using techniques developed by Chuaqui [2], the author has noted [4] that in certain special circumstances, this measure may be represented by an ordinary probability measure m:

$$(*) \qquad t(B_1) \le t(B_2) \text{ iff } mB_1 \le mB_2 .$$

In this paper, the inverse idea is explored: given a (not necessarily real-valued) measure m, can one find a set X so that $(*)$ holds? This generalizes earlier work [5], which considered only real-valued measures. We employ the continuum hypothesis (CH).

Each subset X of $\mathbf{R}$ inherits a relative Borel structure (σ-field)

$$\mathscr{B}(X) = \{B \cap X : B \subseteq \mathbf{R},\ B \text{ Borel}\},$$

the elements of which we call *measurable subsets* of X. Two such subsets X_1 and X_2 are *Borel-isomorphic* if there is a one-one correspondence $f: X_1 \to X_2$ with $A \in \mathscr{B}(X_1)$ if and only if $f(A) \in \mathscr{B}(X_2)$. Each $X \subseteq R$ determines an *isomorphism type*

$$t(X) = \begin{cases} \{Y \subseteq R: Y \text{ and } X \text{ isomorphic}\}; & X \text{ uncountable} \\ 0; \quad X \text{ countable}. \end{cases}$$

Put $S = \{t(X): X \subseteq R\}$. We introduce algebraic and order structures on S as follows. Given t_1 and t_2 in S, let $X_1 \subseteq (0, 1)$ and $X_2 \subseteq (1, 2)$ be sets with

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$t(X_1) = t_1$ and $t(X_2) = t_2$. Put $t_1 + t_2 = t(X_1 \cup X_2)$. A similar device serves to define $t_1 + t_2 + \cdots$ for a sequence of types in S. Define $t_1 \leq t_2$ in case $t_2 = t + t_1$ for some $t \in S$.

It was noted by Tarski [6; pp. 234–235, *passim*] that under the above operations, S constitutes a cardinal algebra, and for each $X \subseteq R$, the subset $\{t(Y): Y \in \mathscr{B}(X)\}$ is a generalized cardinal algebra. For information about cardinal algebras, see [6].

A *partial isomorphism* of $\mathbf{R}$ is a Borel-isomorphism between sets in $\mathscr{B}(\mathbf{R})$. Given such a partial isomorphism f and any $A \subseteq \mathbf{R}$, we define

$$f^{-1}(A) = \{x \in \operatorname{dom}(f): f(x) \in A\}$$
$$f(A) = \{f(x): x \in \operatorname{dom}(f) \cap A\}.$$

Note that A need not be a subset of the domain or range of f. Let H be a collection of partial isomorphisms of $\mathbf{R}$. A set $A \subseteq \mathbf{R}$ is *complete for* H if each h in H maps $\operatorname{dom}(h) \cap A$ onto $\operatorname{range}(h) \cap A$.

We note that a Borel-isomorphism between arbitrary subsets of $\mathbf{R}$ extends to a Borel-isomorphism between Borel subsets of $\mathbf{R}$ (i.e., a partial isomorphism of $\mathbf{R}$). See [3; p. 436]. Also, any two uncountable members of $\mathscr{B}(\mathbf{R})$ are Borel-isomorphic. See [3; p. 450].

By a *commutative divisibility monoid* [1; p. 320] is meant a set $\mathscr{M}$ together with a binary operation $+$ such that:

(1) Commutative and associative laws hold for $+$.
(2) There is in $\mathscr{M}$ an identity element 0 for $+$.
(3) Define a relation $\leq$ on $\mathscr{M}$ by saying that $a \leq b$ iff $b = a + c$ for some c in $\mathscr{M}$. This relation should be a partial order on $\mathscr{M}$.

We write $a_1 + a_2 + \cdots = \sup(a_1 + \cdots + a_n)$ whenever such a supremum exists in $\mathscr{M}$. We note that every cardinal algebra as well as the nonnegative cone of any Abelian partially ordered group forms a commutative divisibility monoid.

Let $(X, \mathscr{B})$ be a measurable space and suppose that $m: \mathscr{B} \to \mathscr{M}$ is a function taking values in a commutative divisibility monoid. Say that m is a *measure* if

$$m(A_1 \cup A_2 \cup \cdots) = mA_1 + mA_2 + \cdots$$

for every pairwise disjoint sequence of sets A_n in $\mathscr{B}$; the supremum on the right-hand side is assumed to exist. Ordinary Lebesgue measure (with $\mathscr{M} = [0, \infty]$) is one example of such; another is any measure taking values in the positive cone of an ordered vector space. The most important example for this paper, however, is the following: Let X be an arbitrary subset of $\mathbf{R}$ and define $m: \mathscr{B}(\mathbf{R}) \to S$ by $mB = t(B \cap X)$. Then m is a measure taking values in the cardinal algebra $\mathscr{M} = S$.

Let $m: \mathscr{B}(\mathbf{R}) \to \mathscr{M}$ be a measure taking values in $\mathscr{M}$. Say that m is *continuous* if $m\{p\} = 0$ for each $p \in \mathbf{R}$. Call m *completely homogeneous* if whenever B_1 and B_2 are Borel subsets of $\mathbf{R}$ with $mB_1 = mB_2 > 0$, then there are Borel subsets B_1' and B_2' of $\mathbf{R}$ such that $B_1' \subseteq B_1$, $B_2' \subseteq B_2$, $m(B_1 - B_1') = m(B_2 - B_2') = 0$, and such that there is a Borel-isomorphism h of B_1' onto B_2' with $mA = mh(A)$ for each measurable $A \subseteq B_1'$. Finally, consider the condition

> (M) Whenever B_1 and B_2 are Borel subsets of $\mathbf{R}$ with $mB_1 \le mB_2$, there is some Borel $C \subseteq B_2$ with $mB_1 = mC$.

This condition seems to have arisen in some early work of Maharam.

Consider once more the measure m on $\mathscr{B}(\mathbf{R})$ defined by $mB = t(B \cap X)$. Clearly, m is continuous, completely homogeneous and satisfies condition (M). The main result below shows that (under CH) every measure on $\mathscr{B}(\mathbf{R})$ satisfying these conditions is isomorphic to one of the form $mB = t(B \cap X)$ above.

Theorem (CH). *Let m be a continuous, completely homogeneous measure on $\mathscr{B}(\mathbf{R})$ taking values in a commutative divisibility monoid $\mathscr{M}$. Then there is a set $X \subseteq \mathbf{R}$ such that*

$$t(B_1 \cap X) = t(B_2 \cap X) \quad \textit{iff} \quad mB_1 = mB_2$$

whenever B_1 and B_2 are Borel subsets of $\mathbf{R}$.

If m satisfies condition (M), *then we also have*

$$t(B_1 \cap X) \le t(B_2 \cap X) \quad \textit{iff} \quad mB_1 \le mB_2.$$

Proof. Let H be the set of all partial isomorphisms h on $\mathbf{R}$ such that $mh(A) = mA$ for each measurable set $A \subseteq \mathrm{dom}(h)$. List the elements of H in transfinite series as $h_0 h_1 \cdots h_\alpha \cdots \alpha < \omega_1$, insisting that h_0 be the identity map on $\mathbf{R}$. Next, list as $N_0 N_1 \cdots N_\alpha \cdots (\alpha < \omega_1)$ and $P_0 P_1 \cdots P_\alpha \cdots (\alpha < \omega_1)$ those Borel subsets N and P of $\mathbf{R}$ such that $mN = 0$ and $mP > 0$. Finally, consider the collection of all partial isomorphisms k such that $m(\mathrm{dom}(k)) \not\le m(\mathrm{range}(k))$. List such isomorphisms as $k_0 k_1 \cdots k_\alpha \cdots \alpha < \omega_1$.

For each countable ordinal α and $x \in \mathbf{R}$, let $\mathscr{O}_\alpha(x)$ be the smallest subset of R containing x and complete for the collection $\{h_0 h_1 \cdots h_\alpha\}$. Fix α and define F to be the set of all partial isomorphisms of the form

$$f = h_{\beta_1}^{n_1} \circ h_{\beta_2}^{n_2} \circ \cdots \circ h_{\beta_k}^{n_k},$$

where $n_1 \cdots n_k$ are integers and $\beta_1 \cdots \beta_k$ are ordinals not greater than α. Then F is countable and may be listed as $f_1 f_2 \cdots$. It is not hard to verify that

$$\mathscr{O}_\alpha(x) = \{f_n(x) : x \in \mathrm{dom}(f_n), n \ge 1\}.$$

Given $A \subseteq \mathbf{R}$, define $\mathscr{O}_\alpha(A) = \bigcup\{\mathscr{O}_\alpha(x) : x \in A\}$. We see that $\mathscr{O}_\alpha(A)$ is the smallest subset of R containing A and complete for $\{h_0 \cdots h_\alpha\}$. From this follows

Claim 1. If A is countable [resp. $mA = 0$], then $\mathscr{O}_\alpha(A)$ is countable [resp. $m\mathscr{O}_\alpha(A) = 0$]. We also make

Claim 2. Suppose that k is a partial isomorphism on $\mathbf{R}$ with $m(\mathrm{dom}(k)) \nleq m(\mathrm{range}(k))$. Then $m\{x \in \mathrm{dom}(k): k(x) \notin \mathscr{O}_\alpha(x)\} > 0$.

Proof of claim. Define sets $A_1 A_2 \cdots$ by setting

$$A_1 = \{x \in \mathrm{dom}(f_1) \cap \mathrm{dom}(k): k(x) = f_1(x)\},$$

$$A_n = \{x \in \mathrm{dom}(f_n) \cap \mathrm{dom}(k): k(x) = f_n(x)\} - \bigcup_1^{n-1} A_k.$$

Also put $A_0 = \{x \in \mathrm{dom}(k): k(x) \notin \mathscr{O}_\alpha(x)\}$. Noting that h_0 (and therefore one of the f_n's) is the identity map on $\mathbf{R}$, we see that $\mathbf{R} = \bigcup \mathrm{dom}(f_n)$ and

$$\mathrm{dom}(k) = A_0 \cup A_1 \cup A_2 \cup \cdots$$

as a disjoint union. Thus A_0 is measurable. If now $mA_0 = 0$, then

$$\begin{aligned} m(\mathrm{range}(k)) &\geq mk(A_1) + mk(A_2) + \cdots \\ &= mf_1(A_1) + mf_2(A_2) + \cdots \\ &= mA_0 + mA_1 + mA_2 + \cdots \\ &= m(\mathrm{dom}(k)) \end{aligned}$$

is a contradiction. The claim is proved.

Now choose points x_α and y_α for $\alpha < \omega_1$ so that

$$\begin{aligned} x_\alpha \in \mathrm{dom}(k_\alpha) &- \mathscr{O}_\alpha(N_0 \cup \cdots \cup N_\alpha) \\ &- k_\alpha^{-1}(\mathscr{O}_\alpha\{x_\beta, y_\beta: \beta < \alpha\}) \\ &- k_\alpha^{-1}(\mathscr{O}_\alpha(x_\alpha)) \\ &- \mathscr{O}_\alpha\{k_\beta(x_\beta): \beta < \alpha\} \end{aligned}$$

and

$$\begin{aligned} y_\alpha \in P_\alpha &- \mathscr{O}_\alpha(N_0 \cup \cdots \cup N_\alpha) \\ &- \mathscr{O}_\alpha\{k_\beta(x_\beta): \beta \leq \alpha\}. \end{aligned}$$

Claims 1 and 2 guarantee the possibility of such choices. Define $X = \bigcup\{\mathscr{O}_\alpha(x_\alpha, y_\alpha): \alpha < \omega_1\}$.

Claim 3. For each countable α, we have $X \cap N_\alpha \subseteq \bigcup\{\mathscr{O}_\beta(x_\beta, y_\beta): \beta < \alpha\}$. Thus, $X \cap N_\alpha$ is countable.

Proof of claim. Suppose that $u \in N_\alpha \cap \mathscr{O}_\beta(x_\beta, y_\beta)$ for some $\beta \geq \alpha$. Then

$$u \in \mathscr{O}_\beta(x_\beta, y_\beta) \cap \mathscr{O}_\beta(N_0 \cup \cdots \cup N_\alpha \cup \cdots \cup N_\beta)$$

implies that either x_β or y_β is a member of $\mathscr{O}_\beta(N_0 \cup \cdots \cup N_\beta)$, a contradiction which proves the claim.

Claim 4. For each α, the intersection $X \cap P_\alpha$ is uncountable.

Proof of claim. Suppose not. Then because $m(P_\alpha - (X \cap P_\alpha)) + m(X \cap P_\alpha) = mP_\alpha > 0$, we have $P_\alpha - (X \cap P_\alpha) = P_\beta$ for some β. But $X \cap P_\beta$ is nonvoid (it contains y_β), a contradiction.

Claim 5. Suppose that B_1 and B_2 are Borel subsets of R with $mB_1 = mB_2$. Then $t(B_1 \cap X) = t(B_2 \cap X)$.

Proof of claim. If $mB_1 = mB_2 = 0$, then Claim 3 implies that $B_1 \cap X$ and $B_2 \cap X$ are countable, so that $t(B_1 \cap X) = t(B_2 \cap X) = 0$. Now suppose that $mB_1 = mB_2 > 0$. Complete homogeneity for m implies that there is some α with $\mathrm{dom}(h_\alpha) \subseteq B_1$, $\mathrm{range}(h_\alpha) \subseteq B_2$, and

$$m(B_1 - \mathrm{dom}(h_\alpha)) = m(B_2 - \mathrm{range}(h_\alpha)) = 0.$$

The set $X_0 = \bigcup\{\mathscr{O}_\beta(x_\beta, y_\beta) : \beta \geq \alpha\}$ is complete for h_α, so that

$$\begin{aligned} h_\alpha(X_0 \cap B_1 \cap X) &= h_\alpha(X_0 \cap B_1) \\ &= X_0 \cap B_2 \\ &= X_0 \cap (B_2 \cap X). \end{aligned}$$

Now $(B_1 \cap X) - X_0$ and $(B_2 \cap X) - X_0$, being contained in $X - X_0$, are countable. Thus $t(B_1 \cap X) = t(B_2 \cap X)$.

Claim 6. Suppose that B_1 and B_2 are Borel subsets of $\mathbf{R}$ with $t(B_1 \cap X) = t(B_2 \cap X)$. Then $mB_1 = mB_2$.

Proof of claim. We prove that $mB_1 \leq mB_2$ and appeal to symmetry. Proceeding *absurdi causa*, suppose that $mB_1 \nleq mB_2$. Then $mB_1 > 0$, so that (by Claim 4) $B_1 \cap X$ is uncountable. Thus also $B_2 \cap X$ is uncountable. Let f be a Borel-isomorphism of $B_1 \cap X$ onto $B_2 \cap X$. Then f is the restriction to $B_1 \cap X$ of some Borel-isomorphism k such that

(i) $C_1 = \mathrm{dom}(k)$ is a Borel subset of B_1,
(ii) $C_2 = \mathrm{range}(k)$ is a Borel subset of B_2.

Then $mB_i = mC_i + m(B_i - C_i) = mC_i$ for $i = 1, 2$, using Claim 4. Since $mC_1 \nleq mC_2$, we have that $k = k_\alpha$ for some $\alpha < \omega_1$. But then $x_\alpha \in C_1 \cap X = B_1 \cap X$. However, $k_\alpha(x_\alpha) \notin \mathscr{O}_\beta(x_\beta, y_\beta)$ for any $\beta < \alpha$. Also $k_\alpha(x_\alpha) \notin \mathscr{O}_\alpha(x_\alpha)$. Suppose that $k_\alpha(x_\alpha) \in \mathscr{O}_\beta(x_\beta)$ for $\beta > \alpha$. Then $x_\beta \in \mathscr{O}_\beta(k_\alpha(x_\alpha))$ is a contradiction. Finally, suppose that $k_\alpha(x_\alpha) \in \mathscr{O}_\beta(y_\beta)$ for $\beta > \alpha$. Then $y_\beta \in \mathscr{O}_\beta(k_\alpha(x_\alpha))$, another contradiction.

We have shown that $k_\alpha(x_\alpha) \notin X$, which contradiction establishes the claim.

Claims 5 and 6 prove the first part of the theorem.

Suppose now that m satisfies condition (M). If $t(B_1 \cap X) \leq t(B_2 \cap X)$, then either $B_1 \cap X$ is countable, in which case $mB_1 = 0$, or else $B_1 \cap X$ is Borel-isomorphic with $C \cap X$, where C is some measurable subset of B_2. So $mB_1 = mC \leq mB_2$. On the other hand, suppose that $mB_1 \leq mB_2$. Condition (M) implies that there is some Borel set $C \subseteq B_2$ with $mC = mB_1$. Then

$$t(B_1 \cap X) = t(C \cap X) \leq t(B_2 \cap X),$$

as desired. Q.E.D.

Corollary (CH). *Let m be a continuous, completely homogeneous measure on $\mathscr{B}(\mathbf{R})$ taking values in a commutative divisibility monoid $\mathscr{M}$. Then the range of m is a generalized cardinal algebra (in the sense of Tarski* [6]).

Indication. Choose X as in the theorem. As noted previously, $\{t(B \cap X): B \subseteq \mathbf{R} \text{ Borel}\}$ is a generalized cardinal algebra. The mapping $t(B \cap X) \to mB$ is well defined and determines an isomorphism onto range(m).

The following result generalizes Proposition 6.6 in [5].

Corollary (CH). *Given $I = [0, 1]$ and $n = 1, 2, \ldots, \infty$, consider the cube I^n under coordinate-wise addition and partial order. There is a set $X \subset \mathbf{R}$ with $\{t(Y): Y \in \mathscr{B}(X)\}$ and I^n isomorphic* (*with respect to both addition and order*).

Indication. Let $\mu_1 \mu_2 \cdots \mu_n$ be the restrictions of Lebesgue measure to the respective intervals $(0, 1), (1, 2), \ldots, (n-1, n)$. (For $n = \omega$, use an infinite sequence.) Put $mB = (\mu_1 B, \mu_2 B, \ldots, \mu_n B)$ and apply the theorem.

Note. If $m(B) = \lambda(B \cap (0, 1))$, where λ is Lebesgue measure, then any set $X \subset \mathbf{R}$ as in the theorem is a Sierpiński subset of $(0, 1)$. It follows that the assumption of CH cannot be dropped from the theorem.

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