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MEROMORPHIC FUNCTIONS ON A COMPACT RIEMANN SURFACE AND ASSOCIATED COMPLETE MINIMAL SURFACES

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ABSTRACT. We prove that given any meromorphic function f on a compact Riemann surface M' there exists another meromorphic function g on M' such that $\{df, g\}$ is the Weierstrass pair defining a complete conformal minimal immersion of finite total curvature into Euclidean 3-space defined on M' punctured at a finite set of points. As corollaries we obtain i) any compact Riemann surface can be immersed in Euclidean 3-space as in the above with at most $4p + 1$ punctures, where p is the genus of the Riemann surface; ii) any hyperelliptic Riemann surface of genus p can be so immersed with at most $3p + 4$ punctures.

INTRODUCTION

Let M be a (connected) Riemann surface and consider a conformal minimal immersion $\varphi: M \to \mathbf{R}^3$. It is a fundamental theorem due to Chern and Osserman [CO] that for a complete φ (i.e., the induced metric on M is complete) the total curvature is finite if and only if the Gauss map is algebraic. (In fact this result is true in any $\mathbf{R}^n$. However, our interest lies solely in the case $n = 3$.) For the sake of simplicity we shall call a complete conformal minimal immersion $\varphi: M \to \mathbf{R}^3$ of finite total curvature an *algebraic minimal surface.* In particular if $\varphi: M \to \mathbf{R}^3$ is an algebraic minimal surface then M is, via a biholomorphism, identified with a compact Riemann surface M' punctured at finitely many points and the Gauss map of φ extends holomorphically to all of M'. Klotz and Sario [KS] proved that there exists an algebraic minimal surface of every genus. Hoffman and Meeks [HM] later exhibited an algebraic minimal surface of every genus with exactly three punctures that is actually embedded. On the other hand Gackstatter and Kunert [GK] proved that any compact Riemann surface can be immersed in $\mathbf{R}^3$ as an algebraic minimal surface with finitely many punctures.

In the present paper we prove that given any meromorphic function f on a compact Riemann surface M' there exists another meromorphic function g on M' so that $\{df, g\}$ is the Weierstrass pair giving an algebraic minimal

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surface defined on M' punctured at the supports of the polar divisors of f and g. Since there always are an abundant supply of meromorphic functions on a Riemann surface our theorem implies the Gackstatter–Kunert theorem. As corollaries of our theorem we also obtain the following: i) any compact Riemann surface of genus p can be immersed in $\mathbf{R}^3$ as an algebraic minimal surface with at most $4p+1$ punctures, ii) any hyperelliptic Riemann surface of genus p can be immersed in $\mathbf{R}^3$ as an algebraic minimal surface with at most $3p+4$ punctures.

Our proof uses the Riemann–Roch theorem in an essential way and the technique is a variation on the ones used in [GK], [CG], and [BC].

§1. The Weierstrass representation formula

Consider a conformal minimal immersion $\varphi = (\varphi^\alpha): M \to \mathbf{R}^3$ from a Riemann surface M. The Gauss map of φ is a map $M \to \mathbf{C}P^2$ given by

$$\Phi: z \mapsto^t \left[\left(\frac{\partial\varphi^\alpha}{\partial z}\right)\right],$$

where z is a local holomorphic coordinate in M. The differential of φ gives globally defined holomorphic 1-forms (ζ^α) on M given locally by

$$\zeta^\alpha = \eta^\alpha\, dz, \qquad \text{where } \eta^\alpha = \frac{\partial\varphi^\alpha}{\partial z}.$$

We then must have

(1) $$\Sigma|\eta^\alpha|^2 > 0;$$

(2) $$\Sigma(\eta^\alpha)^2 = 0;$$

(3) the ζ^α's have no real periods.

Condition (1) means that φ is an immersion. Condition (2) provides that φ is conformal. The holomorphy of (ζ^α) then reflects the fact that φ is minimal. Condition (3) says that the line integrals $\operatorname{Re}\int^z(\zeta^\alpha)$ are path independent. This is so since we must have

(4) $$\varphi^\alpha(z) = 2\operatorname{Re}\int^z \zeta^\alpha.$$

Conversely once we have holomorphic 1-forms (ζ^α) on M satisfying (1), (2), and, (3) then (4) defines a conformal minimal immersion $M \to \mathbf{R}^3$.

Assume now that $\varphi(M)$ does not lie in the xy-plane in $\mathbf{R}^3$. Introduce the holomorphic 1-form μ and the meromorphic function g by

(5) $$\mu = \zeta^1 - i\zeta^2, \; g = \zeta^3/\eta,$$

where $\mu = \eta\, dz$. Note that μ is a holomorphic 1-form on M and g is a meromorphic function on M such that whenever g has a pole of order m at a point then μ has a zero of order $2m$ at the same point. (See [L], p. 113.)

$\{\mu, g\}$ is called the *Weierstrass pair of* φ. Conversely given a pair $\{\mu, g\}$ on M whose zeros and poles are related as mentioned above we may put

$$(6) \qquad \zeta^1 = \frac{1}{2}(1 - g^2)\mu, \qquad \zeta^2 = \frac{i}{2}(1 + g^2)\mu, \qquad \zeta^3 = g\mu$$

giving rise to holomorphic 1-forms (ζ^α) on M satisfying (1) and (2). It follows that (ζ^α) defines a conformal minimal immersion at least on the universal cover of M. In order for (ζ^α) to define a conformal minimal immersion on M we must have the condition (3) met also.

Let $M = M' \backslash \Sigma$, where M' is a compact Riemann surface and Σ is a finite set. Take an exact meromorphic 1-form μ (μ is df for some meromorphic function f on M') on M' and a meromorphic function g on M' such that restricted to M μ and g are holomorphic. A sufficient condition (cf. [GK]) that (ζ^α) given by (6) have no real periods on M is

(7) $g\mu$ and $g^2\mu$ have no residues and no periods on M'.

Given that the condition (7) is met (4) defines a conformal minimal immersion

$$(8) \qquad \varphi\colon M' \backslash \Sigma \to \mathbf{R}^3.$$

The Gauss map of φ in (8) extends holomorphically to all of M' since the ζ^α's involved have at worst a pole at the points of Σ. (See [L], p. 134 for a proof of this fact.)

The induced metric on M is given by $h(z)\, dz \cdot d\bar{z}$ with $h(z) = 2\Sigma|\eta^\alpha|^2$ and the immersion φ is complete given that

$$(9) \qquad \Sigma|\eta^\alpha|^2 = c/|z|^{2m} + \text{ higher-order terms},$$

where $c \in \mathbf{C}$, z is a local holomorphic coordinate centered at one of the points in Σ, and $\eta^\alpha = \partial\varphi^\alpha/\partial z$. The expansion shows that any path approaching one of the punctures has infinite arc length.

§2. The main result

Theorem. *Let f be any nonconstant meromorphic function on a compact Riemann surface M' of genus $p > 0$. Then there exists another meromorphic function g on M' such that $\{df, g\}$ is the Weierstrass pair giving a complete conformal minimal immersion of finite total curvature*

$$\varphi\colon M = M' \backslash \Sigma \to \mathbf{R}^3,$$

where $\Sigma = \operatorname{supp}(f)_\infty \cup \operatorname{supp}(g)_\infty$.

Proof. Let f be a nonconstant meromorphic function on M' with polar divisor

$$(f)_\infty = \Sigma b_i p_i; \qquad 1 \le i \le n, \quad p_i \in M'.$$

Also put $d = \Sigma b_i$. Then d is the degree of the polar divisor of f. And df, a meromorphic 1-form on M', has poles of order $b_i + 1$ at p_i and no other poles. Put

$$(df)_0 = \Sigma a_j q_j; \qquad 1 \le j \le m; \quad q_j \in M'.$$

We then have $2p-2=\deg(df)_0-\deg(df)_\infty$ since $(df)=(df)_0-(df)_\infty$ is a canonical divisor. Thus

$$\Sigma a_j=(2p-2)+d+n.$$

Define a divisor D on M' by

$$D=\Sigma a_j q_j-\Sigma c_i p_i\,,$$

where $\Sigma c_i=3p-2+d+n$ and $c_i\geq b_i+1$. It follows that $\deg D=-p$. The Riemann–Roch theorem then tells us that

$$\dim L(-D)=\deg(-D)-p+1+\dim L((df)+D)\geq 1\,,$$

where $L(-D)=\{G,$ meromorphic function on $M'\colon (G)\geq D\}\cup\{0\}$. Given $G\in L(-D)$ set

$$(G)_0=\Sigma\tilde{a}_j q_j+\Sigma\tilde{a}_{m+k}q_{m+k}\,;\qquad 1\leq j\leq m\,,1\leq k\leq l;$$
$$(G)_\infty=\Sigma\tilde{c}_i p_i\,;\qquad 1\leq i\leq n.$$

Note that we must have

$$\tilde{c}_i\leq c_i\,;\quad \tilde{a}_j\geq a_j\,;\qquad \Sigma\tilde{a}_j+\Sigma\tilde{a}_{m+k}=\Sigma\tilde{c}_i.$$

The last condition reflects the fact that (G) is a principal divisor and the first two conditions say that $G\in L(-D)$.

Define a meromorphic function g on M' by

$$g=\sum_{\alpha=1}^{\lambda}\frac{c_\alpha}{G^\alpha}\,,$$

where $\lambda=2(n+m+l-1)+4p+1$. The c_α's are complex constants to be chosen suitably later. Since $\operatorname{supp}(g)_\infty=\operatorname{supp}(G)_0$ we get

$$\operatorname{supp}(g)_\infty=\{q_1\,,\,\dots\,,q_{m+l}\}.$$

Consider the meromorphic 1-forms $g\,df$ and $g^2\,df$ on M'. Observe that

$$\{q_{m+1}\,,\,\dots\,,q_{m+l}\}\subset\operatorname{supp}(g\,df)_\infty\subset\{q_1\,,\,\dots\,,q_{m+l}\,;\;p_1\,,\,\dots\,,p_n\}\,,$$
$$\{q_1\,,\,\dots\,,q_{m+l}\}\subset\operatorname{supp}(g^2df)_\infty\subset\{q_1\,,\,\dots\,,q_{m+l}\,;\;p_1\,,\,\dots\,,p_n\}.$$

We *claim* that we can choose (c_α), not all zero, such that $g\,df$ and $g^2\,df$ have no residues and no periods on M'. Put

$$R_{i\alpha}=\text{ the residue of }\frac{df}{G^\alpha}\text{ at }p_i\,,$$
$$R_{j\alpha}=\text{ the residue of }\frac{df}{G^\alpha}\text{ at }q_j\,,$$
$$R_{k\alpha}=\text{ the residue of }\frac{df}{G^\alpha}\text{ at }q_{m+k}.$$

So the residue of $g\,df$ at p_i is $\sum_\alpha c_\alpha R_{i\alpha}$, etc. Thus $g\,df$ on M' has no residues if and only if

(A) $$\sum_\alpha c_\alpha R_{i\alpha} = 0; \qquad \sum_\alpha c_\alpha R_{j\alpha} = 0; \qquad \sum_\alpha c_\alpha R_{k\alpha} = 0.$$

Now the total residue of any meromorphic 1-form must vanish. Hence

$$\sum_{i,\alpha} c_\alpha R_{i\alpha} + \sum_{j,\alpha} c_\alpha R_{j\alpha} + \sum_{k,\alpha} c_\alpha R_{k\alpha} = 0.$$

It follows that (A) represents a homogeneous linear system in (c_α) containing at most $(n+m+l-1)$ independent equations. Let $(e_1, \dots, e_{2p})$ be 1-cycles representing a (canonical) homology basis of M' and put

$$P_{a\alpha} = \int_{e_a} \frac{df}{G^\alpha}; \qquad 1 \le a \le 2p, \quad 1 \le \alpha \le \lambda.$$

$P_{a\alpha}$ is the e_a-period of df/G^α. So the e_a-period of the meromorphic 1-form $g\,df$ is $\sum_\alpha c_\alpha P_{a\alpha}$. Thus $g\,df$ has no periods if and only if

(B) $$\sum_\alpha c_\alpha P_{a\alpha} = 0.$$

This gives a homogeneous linear system in (c_α) containing $2p$ equations. We now consider the meromorphic 1-form $g^2\,df$. The residue at p_i of $g^2\,df$ is

$$R_i(c_\alpha) = R_{i2}c_1^2 + R_{i4}c_2^2 + \cdots + R_{i,2\lambda}c_\lambda^2 + 2R_{i3}c_1c_2 + \cdots + 2R_{i,2\lambda}c_{\lambda-1}c_\lambda,$$

where $R_{i,2\lambda}$ denotes the residue at p_i of $df/G^{2\lambda}$, etc. Thus $g^2\,df$ has no residues if and only if

(C) $$R_i(c_\alpha) = 0; \qquad R_j(c_\alpha) = 0; \qquad R_k(c_\alpha) = 0.$$

Again we can eliminate one of the equations from (C) using the fact that the total residue of $g^2\,df$ must vanish. Hence (C) represents a homogeneous quadratic system (R_i, R_j, R_k are all homogeneous polynomials in (c_α) of degree 2) in (c_α) containing $(n+m+l-1)$ equations. Requiring $g^2\,df$ to have no periods we obtain another homogeneous quadratic system (D) containing $2p$ equations. The total number of equations in (A–D) is $2(n+m+l-1)+4p = \lambda - 1$ and the claim follows. (Observe that in solving the system (A–D) we are intersecting a set of hyperplanes and homogeneous hyperquadrics in $\mathbf{C}^\lambda$.) Equation (7) now tells us that $\{df, g\}$ is the Weierstrass pair representing a conformal minimal immersion $\varphi: M'\backslash\Sigma \to \mathbf{R}^3$, where $\Sigma = \operatorname{supp}(f)_\infty \cup \operatorname{supp}(g)_\infty = \{p_1, \dots, p_n; q_1, \dots, q_{m+l}\}$. The Gauss map of φ extends holomorphically to all of M' since the ζ^α's given by (6) with $\mu = df$ have at worst a pole at the points of Σ. Condition (9) is also routinely verified. For example, df has a pole of order $b_i + 1$ at p_i and condition (9) is met with $m \ge 2$. □

Note that in the above proof

$$n \le d; \qquad m + l \le 3p + d + n - 2.$$

Let $\sharp$ denote the total number of punctures of φ, i.e., $\sharp$ is the cardinality of Σ. Then we obtain

$$\sharp = n + m + l \leq 3p + 3d - 2.$$

Corollary. *Let M' be any compact Riemann surface of genus p. Then there exists a complete conformal minimal immersion of finite total curvature*

$$\varphi : M' \backslash \Sigma \to \mathbf{R}^3 \qquad \textit{with } |\Sigma| \leq 4p + 1.$$

Proof. Let $p_1 \in M'$ be a non-Weierstrass point. Then there exists a meromorphic function f on M' with $(f)_\infty = (p+1)p_1$. So $n = 1$. Also

$$m + l \leq 3p + d + n - 2 = 4p$$

and the result follows. □

Corollary. *Let M' be any hyperelliptic Riemann surface of genus p. Then there exists a complete conformal minimal immersion of finite total curvature*

$$\varphi : M' \backslash \Sigma \to \mathbf{R}^3 \qquad \textit{with } |\Sigma| \leq 3p + 4.$$

Proof. On a hyperelliptic Riemann surface there exists a meromorphic function whose polar divisor has degree two. So we can take $d = 2$. □

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