MEROMORPHIC FUNCTIONS ON A COMPACT RIEMANN SURFACE
AND ASSOCIATED COMPLETE MINIMAL SURFACES

KICHOON YANG

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Abstract. We prove that given any meromorphic function $f$ on a compact Riemann surface $M'$ there exists another meromorphic function $g$ on $M'$ such that \{df, g\} is the Weierstrass pair defining a complete conformal minimal immersion of finite total curvature into Euclidean 3-space defined on $M'$ punctured at a finite set of points. As corollaries we obtain i) any compact Riemann surface can be immersed in Euclidean 3-space as in the above with at most $4p + 1$ punctures, where $p$ is the genus of the Riemann surface; ii) any hyperelliptic Riemann surface of genus $p$ can be so immersed with at most $3p + 4$ punctures.

Introduction

Let $M$ be a (connected) Riemann surface and consider a conformal minimal immersion $\varphi : M \to \mathbb{R}^3$. It is a fundamental theorem due to Chern and Osserman [CO] that for a complete $\varphi$ (i.e., the induced metric on $M$ is complete) the total curvature is finite if and only if the Gauss map is algebraic. (In fact this result is true in any $\mathbb{R}^n$. However, our interest lies solely in the case $n = 3$.) For the sake of simplicity we shall call a complete conformal minimal immersion $\varphi : M \to \mathbb{R}^3$ of finite total curvature an algebraic minimal surface. In particular if $\varphi : M \to \mathbb{R}^3$ is an algebraic minimal surface then $M$ is, via a biholomorphism, identified with a compact Riemann surface $M'$ punctured at finitely many points and the Gauss map of $\varphi$ extends holomorphically to all of $M'$. Klotz and Sario [KS] proved that there exists an algebraic minimal surface of every genus. Hoffman and Meeks [HM] later exhibited an algebraic minimal surface of every genus with exactly three punctures that is actually embedded. On the other hand Gackstatter and Kunert [GK] proved that any compact Riemann surface can be immersed in $\mathbb{R}^3$ as an algebraic minimal surface with finitely many punctures.

In the present paper we prove that given any meromorphic function $f$ on a compact Riemann surface $M'$ there exists another meromorphic function $g$ on $M'$ so that \{df, g\} is the Weierstrass pair giving an algebraic minimal
surface defined on \( M' \) punctured at the supports of the polar divisors of \( f \) and \( g \). Since there always are an abundant supply of meromorphic functions on a Riemann surface our theorem implies the Gackstatter–Kunert theorem. As corollaries of our theorem we also obtain the following: i) any compact Riemann surface of genus \( p \) can be immersed in \( \mathbb{R}^3 \) as an algebraic minimal surface with at most \( 4p + 1 \) punctures, ii) any hyperelliptic Riemann surface of genus \( p \) can be immersed in \( \mathbb{R}^3 \) as an algebraic minimal surface with at most \( 3p + 4 \) punctures.

Our proof uses the Riemann–Roch theorem in an essential way and the technique is a variation on the ones used in [GK], [CG], and [BC].

§1. THE WEIERSTRASS REPRESENTATION FORMULA

Consider a conformal minimal immersion \( \varphi = (\varphi^a): M \to \mathbb{R}^3 \) from a Riemann surface \( M \). The Gauss map of \( \varphi \) is a map \( M \to \mathbb{C}P^2 \) given by

\[
\Phi: z \mapsto \left[ \frac{\partial \varphi^a}{\partial z} \right],
\]

where \( z \) is a local holomorphic coordinate in \( M \). The differential of \( \varphi \) gives globally defined holomorphic 1-forms \( (\zeta^a) \) on \( M \) given locally by

\[
\zeta^a = \eta^a \, dz,
\]

where \( \eta^a = \frac{\partial \varphi^a}{\partial z} \).

We then must have

1. \( \Sigma |\eta^a|^2 > 0 \);
2. \( \Sigma (\eta^a)^2 = 0 \);
3. the \( \zeta^a \)'s have no real periods.

Condition (1) means that \( \varphi \) is an immersion. Condition (2) provides that \( \varphi \) is conformal. The holomorphy of \( (\zeta^a) \) then reflects the fact that \( \varphi \) is minimal. Condition (3) says that the line integrals \( \text{Re} \int z (\zeta^a) \) are path independent. This is so since we must have

\[
\varphi^a(z) = 2 \text{Re} \int z \zeta^a.
\]

Conversely once we have holomorphic 1-forms \( (\zeta^a) \) on \( M \) satisfying (1), (2), and, (3) then (4) defines a conformal minimal immersion \( M \to \mathbb{R}^3 \).

Assume now that \( \varphi(M) \) does not lie in the \( xy \)-plane in \( \mathbb{R}^3 \). Introduce the holomorphic 1-form \( \mu \) and the meromorphic function \( g \) by

\[
\mu = \zeta^1 - i\zeta^2, \quad g = \zeta^3 / \eta,
\]

where \( \mu = \eta \, dz \). Note that \( \mu \) is a holomorphic 1-form on \( M \) and \( g \) is a meromorphic function on \( M \) such that whenever \( g \) has a pole of order \( m \) at a point then \( \mu \) has a zero of order \( 2m \) at the same point. (See [L], p. 113.)
\{\mu, g\} is called the \textit{Weierstrass pair of \varphi}. Conversely given a pair \{\mu, g\} on \(M\) whose zeros and poles are related as mentioned above we may put
\begin{align}
\zeta^1 &= \frac{1}{2}(1 - g^2)\mu, \\
\zeta^2 &= \frac{i}{2}(1 + g^2)\mu, \\
\zeta^3 &= g\mu
\end{align}
giving rise to holomorphic 1-forms \((\zeta^\alpha)\) on \(M\) satisfying (1) and (2). It follows that \((\zeta^\alpha)\) defines a conformal minimal immersion at least on the universal cover of \(M\). In order for \((\zeta^\alpha)\) to define a conformal minimal immersion on \(M\) we must have the condition (3) met also.

Let \(M = M' \setminus \Sigma\), where \(M'\) is a compact Riemann surface and \(\Sigma\) is a finite set. Take an exact meromorphic 1-form \(\mu\) (\(\mu\) is \(df\) for some meromorphic function \(f\) on \(M'\)) on \(M'\) and a meromorphic function \(g\) on \(M'\) such that restricted to \(M\) \(\mu\) and \(g\) are holomorphic. A sufficient condition (cf. [GK]) that \((\zeta^\alpha)\) given by (6) have no real periods on \(M\) is
\begin{align}
g\mu \text{ and } g^2\mu \text{ have no residues and no periods on } M'.
\end{align}
Given that the condition (7) is met (4) defines a conformal minimal immersion
\begin{align}
\varphi: M' \setminus \Sigma \rightarrow \mathbb{R}^3.
\end{align}
The Gauss map of \(\varphi\) in (8) extends holomorphically to all of \(M'\) since the \(\zeta^\alpha\)'s involved have at worst a pole at the points of \(\Sigma\). (See [L], p. 134 for a proof of this fact.)

The induced metric on \(M\) is given by \(h(z)dz\cdot d\overline{z}\) with \(h(z) = 2|\eta^\alpha|^2\) and the immersion \(\varphi\) is complete given that
\begin{align}
\Sigma|\eta^\alpha|^2 = c/|z|^{2m} + \text{higher-order terms},
\end{align}
where \(c \in \mathbb{C}\), \(z\) is a local holomorphic coordinate centered at one of the points in \(\Sigma\), and \(\eta^\alpha = \partial \varphi^\alpha/\partial z\). The expansion shows that any path approaching one of the punctures has infinite arc length.

\section*{§2. The main result}

\textbf{Theorem.} \textit{Let \(f\) be any nonconstant meromorphic function on a compact Riemann surface \(M'\) of genus \(p > 0\). Then there exists another meromorphic function \(g\) on \(M'\) such that \(\{df, g\}\) is the Weierstrass pair giving a complete conformal minimal immersion of finite total curvature}
\begin{align}
\varphi: M = M' \setminus \Sigma \rightarrow \mathbb{R}^3,
\end{align}
\textit{where} \(\Sigma = \text{supp}(f)_\infty \cup \text{supp}(g)_\infty\).

\textbf{Proof.} Let \(f\) be a nonconstant meromorphic function on \(M'\) with polar divisor
\begin{align}
(f)_\infty = \Sigma b_ip_i; \quad 1 \leq i \leq n, \quad p_i \in M'.
\end{align}
Also put \(d = \Sigma b_i\). Then \(d\) is the degree of the polar divisor of \(f\). And \(df\), a meromorphic 1-form on \(M'\), has poles of order \(b_i + 1\) at \(p_i\) and no other poles. Put
\begin{align}
(df)_0 = \Sigma a_jq_j; \quad 1 \leq j \leq m, \quad q_j \in M'.
\end{align}
We then have $2p - 2 = \deg(df)_0 - \deg(df)_\infty$ since $(df) = (df)_0 - (df)_\infty$ is a canonical divisor. Thus

$$\Sigma a_j = (2p - 2) + d + n.$$ 

Define a divisor $D$ on $M'$ by

$$D = \Sigma a_j q_j - \Sigma c_i p_i,$$

where $\Sigma c_i = 3p - 2 + d + n$ and $c_i \geq b_i + 1$. It follows that $\deg D = -p$. The Riemann-Roch theorem then tells us that

$$\dim L(-D) = \deg(-D) - p + 1 + \dim L((df) + D) \geq 1,$$

where $L(-D) = \{G, \text{ meromorphic function on } M' : (G) \geq D\} \cup \{0\}$. Given $G \in L(-D)$ set

$$(G)_0 = \Sigma \hat{a}_j q_j + \Sigma \hat{a}_m q_m q_k + \Sigma \hat{c}_i p_i; \quad 1 \leq j \leq m, 1 \leq k \leq l,$$

$$(G)_\infty = \Sigma \hat{c}_i p_i; \quad 1 \leq i \leq n.$$ 

Note that we must have

$$\hat{c}_i \leq c_i; \quad \hat{a}_j \geq a_j; \quad \Sigma \hat{a}_j + \Sigma \hat{a}_m = \Sigma c_i.$$

The last condition reflects the fact that $(G)$ is a principal divisor and the first two conditions say that $G \in L(-D)$.

Define a meromorphic function $g$ on $M'$ by

$$g = \sum_{\alpha=1}^\lambda c_\alpha \frac{1}{G^\alpha},$$

where $\lambda = 2(n + m + l - 1) + 4p + 1$. The $c_\alpha$'s are complex constants to be chosen suitably later. Since $\text{supp}(g)_\infty = \text{supp}(G)_0$ we get

$$\text{supp}(g)_\infty = \{q_1, \ldots, q_{m+1}\}.$$ 

Consider the meromorphic 1-forms $g df$ and $g^2 df$ on $M'$. Observe that

$$\{q_{m+1}, \ldots, q_{m+1}\} \subset \text{supp}(g df)_\infty \subset \{q_1, \ldots, q_{m+1} ; p_1, \ldots, p_n\},$$

$$\{q_1, \ldots, q_{m+1}\} \subset \text{supp}(g^2 df)_\infty \subset \{q_1, \ldots, q_{m+1} ; p_1, \ldots, p_n\}.$$ 

We claim that we can choose $(c_\alpha)$, not all zero, such that $g df$ and $g^2 df$ have no residues and no periods on $M'$. Put

$$R_{i\alpha} = \text{the residue of } \frac{df}{G^\alpha} \text{ at } p_i,$$

$$R_{j\alpha} = \text{the residue of } \frac{df}{G^\alpha} \text{ at } q_j,$$

$$R_{k\alpha} = \text{the residue of } \frac{df}{G^\alpha} \text{ at } q_{m+k}.$$
So the residue of $g df$ at $p_i$ is $\sum_\alpha c_\alpha R_{i\alpha}$, etc. Thus $g df$ on $M'$ has no residues if and only if

$$\sum_\alpha c_\alpha R_{i\alpha} = 0; \quad \sum_\alpha c_\alpha R_{j\alpha} = 0; \quad \sum_\alpha c_\alpha R_{k\alpha} = 0.$$  

Now the total residue of any meromorphic 1-form must vanish. Hence

$$\sum_\alpha c_\alpha R_{i\alpha} + \sum_\alpha c_\alpha R_{j\alpha} + \sum_\alpha c_\alpha R_{k\alpha} = 0.$$  

It follows that (A) represents a homogeneous linear system in $(c_\alpha)$ containing at most $(n + m + l - 1)$ independent equations. Let $(e_1, \ldots, e_{2p})$ be 1-cycles representing a (canonical) homology basis of $M'$ and put

$$P_{\alpha a} = \int_{e_\alpha} \frac{df}{G^a}; \quad 1 \leq a \leq 2p, \quad 1 \leq \alpha \leq \lambda.$$  

$P_{\alpha a}$ is the $e_\alpha$-period of $df/G^a$. So the $e_\alpha$-period of the meromorphic 1-form $g df$ is $\sum_\alpha c_\alpha P_{\alpha a}$. Thus $g df$ has no periods if and only if

$$\sum_\alpha c_\alpha P_{\alpha a} = 0.$$  

This gives a homogeneous linear system in $(c_\alpha)$ containing $2p$ equations. We now consider the meromorphic 1-form $g^2 df$. The residue at $p_i$ of $g^2 df$ is

$$R_i(c_\alpha) = R_{i2}c_1^2 + R_{i4}c_2^2 + \cdots + R_{i2\lambda}c_\lambda^2 + 2R_{i3}c_1c_2 + \cdots + 2R_{i2\lambda}c_{\lambda-1}c_\lambda,$$

where $R_{ij\lambda}$ denotes the residue at $p_i$ of $df/G^{2\lambda}$, etc. Thus $g^2 df$ has no residues if and only if

$$R_i(c_\alpha) = 0; \quad R_j(c_\alpha) = 0; \quad R_k(c_\alpha) = 0.$$  

Again we can eliminate one of the equations from (C) using the fact that the total residue of $g^2 df$ must vanish. Hence (C) represents a homogeneous quadratic system $(R_i, R_j, R_k$ are all homogeneous polynomials in $(c_\alpha)$ of degree 2) in $(c_\alpha)$ containing $(n + m + l - 1)$ equations. Requiring $g^2 df$ to have no periods we obtain another homogeneous quadratic system (D) containing $2p$ equations. The total number of equations in (A–D) is $2(n + m + l - 1) + 4p = \lambda - 1$ and the claim follows. (Observe that in solving the system (A–D) we are intersecting a set of hyperplanes and homogeneous hyperquadrics in $C^\lambda$.) Equation (7) now tells us that $\{df, g\}$ is the Weierstrass pair representing a conformal minimal immersion $\varphi: M' \setminus \Sigma \to \mathbb{R}^3$, where $\Sigma = \text{supp}(f)_\infty \cup \text{supp}(g)_\infty = \{p_1, \ldots, p_n; q_1, \ldots, q_{m+1}\}$. The Gauss map of $\varphi$ extends holomorphically to all of $M'$ since the $\zeta^\alpha$'s given by (6) with $\mu = df$ have at worst a pole at the points of $\Sigma$. Condition (9) is also routinely verified. For example, $df$ has a pole of order $b_i + 1$ at $p_i$ and condition (9) is met with $m \geq 2$. \(\square\)

Note that in the above proof

$$n \leq d; \quad m + l \leq 3p + d + n - 2.$$
Let $\#$ denote the total number of punctures of $\varphi$, i.e., $\#$ is the cardinality of $\Sigma$. Then we obtain

$$\# = n + m + l \leq 3p + 3d - 2.$$ 

**Corollary.** Let $M'$ be any compact Riemann surface of genus $p$. Then there exists a complete conformal minimal immersion of finite total curvature

$$\varphi: M' \backslash \Sigma \rightarrow \mathbb{R}^3 \quad \text{with} \quad |\Sigma| \leq 4p + 1.$$ 

**Proof.** Let $p_1 \in M'$ be a non-Weierstrass point. Then there exists a meromorphic function $f$ on $M'$ with $(f)_\infty = (p + 1)p_1$. So $n = 1$. Also

$$m + l \leq 3p + d + n - 2 = 4p$$

and the result follows. \hfill \Box

**Corollary.** Let $M'$ be any hyperelliptic Riemann surface of genus $p$. Then there exists a complete conformal minimal immersion of finite total curvature

$$\varphi: M' \backslash \Sigma \rightarrow \mathbb{R}^3 \quad \text{with} \quad |\Sigma| \leq 3p + 4.$$ 

**Proof.** On a hyperelliptic Riemann surface there exists a meromorphic function whose polar divisor has degree two. So we can take $d = 2$. \hfill \Box

**References**


Department of Mathematics, Arkansas State University, State University, Arkansas 72467