STABLE ISOMORPHISM OF HEREDITARY $C^*$-SUBALGEBRAS
AND STABLE EQUIVALENCE OF OPEN PROJECTIONS

SHUANG ZHANG

(Communicated by John B. Conway)

Abstract. We relate the stable isomorphism of two hereditary $C^*$-subalgebras
to the stable equivalence of the corresponding open projections. We prove that
if $A$ is completely $\sigma$-unital, then her($p$) and her($q$) generate the same closed
ideal of $A$ iff $p \otimes 1 \sim q \otimes 1$ in $(A \otimes K)^{**}$ iff the central supports of $p$ and $q$
in $A^{**}$ are the same. If, in addition, $p \perp q$, then the above three equivalent
conditions are equivalent to the condition: $p \otimes 1$ and $q \otimes 1$ are in the same
path component of open projections in $(A \otimes K)^{**}$.

0. Introduction

Let $A$ be any $C^*$-algebra. We denote the multiplier algebra of $A$ by $M(A)$
and the Banach space double dual of $A$ by $A^{**}$. The Murray-von Neumann
equivalence of projections in $A^{**}$ is denoted by $\sim$. The set of hereditary
$C^*$-subalgebras of $A$ is denoted by $H(A)$ and the set of closed ideals of $A$
by $I(A)$. It is clear that $I(A) \subset H(A)$. Strong Morita equivalence is an
equivalence relation on $H(A)$ (see [10] and [4]). We denote the set of strong
Morita equivalence classes of $H(A)$ by $\tilde{H}(A)$ and the set of strong Morita
equivalence classes of $I(A)$ by $\tilde{I}(A)$. If $A_0$ is a hereditary $C^*$-subalgebra of
$A$, we denote the strong Morita equivalence class containing $A_0$ by $[A_0]$.

We say that a $C^*$-algebra $A$ is completely $\sigma$-unital if every hereditary $C^*$-
subalgebra of $A$ is $\sigma$-unital. It is clear that any separable $C^*$-algebra is complet-
ely $\sigma$-unital.

It was proved in [5] that two $\sigma$-unital $C^*$-algebras $A$ and $B$ are strongly
Morita equivalent iff $A$ and $B$ are stably isomorphic (i.e. $A \otimes K \simeq B \otimes K$).
Hence for a completely $\sigma$-unital $C^*$-algebra $A$ each class $[A_0]$ coincides with
the equivalence class $[A_0]_1$ in the sense of stable isomorphism. It follows easily
from the results of [5] and [3, 2.5] that if $A$ is $\sigma$-unital, then there is a bijection
between $\tilde{I}(A)$ and $\tilde{H}(A)$.

Received by the editors August 17, 1988.
Key words and phrases. Hereditary $C^*$-subalgebras, open projections, stable isomorphism, path
component of open projections.

©1989 American Mathematical Society
0002-9939/89 $1.00 + $.25 per page

677
It is well known that $A^{**} 
i p \leftrightarrow p A^{**} p \cap A \in H(A)$ is a bijection between the set of open projections and the set of all hereditary $C^*$-subalgebras of $A$. Moreover central open projections correspond to closed ideals under the bijection (see [2]). Here $\text{her}(p)$ denotes the hereditary $C^*$-subalgebra of $A$ corresponding to an open projection $p$ in $A^{**}$.

Two open projections in $A^{**}$ are said to be stably equivalent if there is a $v \in (A \otimes K)^{**}$ such that $vv^* = p \otimes 1$ and $v^*v = q \otimes 1$. It is clear that the stable equivalence of open projections is an equivalence relation on the set of open projections in $A^{**}$. Let $\tilde{O}(A)$ be the set of equivalence classes of open projections in $A^{**}$ under stable equivalence.

In this short note we shall show that there is a bijection $I(A) \leftrightarrow \tilde{O}(A)$ and describe the stable isomorphism of hereditary $C^*$-subalgebras of $A$ by the stable equivalence of the corresponding open projections. Moreover we shall prove that two mutually orthogonal hereditary $C^*$-subalgebras $\text{her}(p)$ and $\text{her}(q)$ are stably isomorphic iff $\text{her}(p) \otimes K$ can be continuously deformed through hereditary $C^*$-subalgebras of $A \otimes K$ to $\text{her}(q) \otimes K$. In particular, if two $\sigma$-unital $C^*$-algebras $A$ and $B$ are strongly Morita equivalent (or equivalently stably isomorphic) and

$$L = \begin{pmatrix} A & X \\ \bar{X} & B \end{pmatrix}$$

is the linking algebra constructed in [5], then $A \otimes K$ and $B \otimes K$ are path connected by mutually stably isomorphic hereditary $C^*$-subalgebras of $L \otimes K$. We think that this gives a more basic understanding of the stable isomorphism of hereditary $C^*$-subalgebras.

We call a hereditary $C^*$-subalgebra $\text{her}(p)$ of $A$ essential if there is no nonzero hereditary $C^*$-subalgebra $\text{her}(c)$ such that $xy = 0$ for all $x$ in $\text{her}(p)$, $y$ in $\text{her}(c)$. It is equivalent that there be no nonzero open projection $q$ such that $q \perp p$. Here $\perp$ means orthogonal.

1. Main result

1. **Theorem.** If $A$ is a completely $\sigma$-unital $C^*$-algebra and $p, q$ are two open projections in $A^{**}$, then the following are equivalent:

(a) $\text{her}(p)$ and $\text{her}(q)$ generate the same closed ideal of $A$.

(b) The central supports of $p$ and $q$ in $A^{**}$ are the same.

(c) $p$ and $q$ are stably equivalent.

If, in addition, $p \perp q$, then the above three conditions are equivalent to:

(d) $p \otimes 1$ and $q \otimes 1$ are in the same path component of open projections in $(A \otimes K)^{**}$.

Consequently any one of the conditions (a)-(d) implies that $\text{her}(p) \otimes K \simeq \text{her}(q) \otimes K$.

**Proof.** (a) $\leftrightarrow$ (b): Let $I(p)$ be the ideal of $A$ generated by $\text{her}(p)$ for any open projection $p$. It suffices to show that the central open projection corresponding to $I(p)$ is the central cover $c(p)$ of $p$ in $A^{**}$.
Let $q_0$ be the central open projection corresponding to $I(p)$. First of all, $p \leq q_0$ is clear since $\text{her}(p) \subset I(p)$. We need only show that $q_0 \leq c(p)$. Let $A_1 = c(p)A^{**} \cap A$. Then $A_1$ is a closed ideal of $A$. Let $q_1$ be the central open projection corresponding to $A_1$. Then $q_1 \leq c(p)$. Since $pA^{**}p \subset c(p)A^{**}c(p)$, $\text{her}(p) \subset A_1$. It follows that $I(p) \subset A_1$ and so $q_0 \leq q_1$. Thus $q_0 \leq c(p)$.

(b) $\Rightarrow$ (c): We show that $p$ and $c(p)$ are stably equivalent. Then similarly $q$ and $c(p)$ are stably equivalent, and so $p$ and $q$ are stably equivalent.

Let $I(p)$ be the closed ideal generated by $\text{her}(p)$. As above $I(p) = c(p)A^{**} \cap A$. By hypothesis $\text{her}(p)$ and $I(p)$ are both $\sigma$-unital. By Lemma (6.2) in [9], there is a sequence $\{a_i\} \subset I(p)$ such that $a_i a_i^* \in \text{her}(p)$ for all $i \geq 1$ and $\sum_{i=1}^{\infty} a_i^* a_i = c(p)$ with convergence in the strict topology in $M(I(p))$.

Define $u = \sum_{i=1}^{\infty} a_i \otimes e_{11}$, then it is routine to check that the sum converges in the strict topology in $M(I(p) \otimes K)$ and $u^* u = c(p) \otimes 1$ and $u u^* \leq p \otimes 1$ (see the proof of Lemma (2.4) of [3]), where we need the fact that $a_i a_i^* \in \text{her}(p)$ implies $a_i^* p a_i = a_i^* a_i$ for any $i \geq 1$. Since $M(I(p) \otimes K) \subset (I(p) \otimes K)^{**} \subset (A \otimes K)^{**}$ and the strict topology of $M(I(p) \otimes K)$ is stronger than the relative $w^*$-topology induced by the one of $(I(p) \otimes K)^{**}$, $u = \sum_{i=1}^{\infty} a_i \otimes e_{11}$ converges in the $w^*$-topology of $(I(p) \otimes K)^{**}$ and so it converges in the $w^*$-topology of $(A \otimes K)^{**}$. Now the proof of Lemma (2.5) of [3] can be repeated to obtain a partial isometry $v \in (A \otimes K)^{**}$ such that $v^* v = c(p) \otimes 1$ and $vv^* = p \otimes 1$.

(c) $\Rightarrow$ (b): Let $c(p \otimes 1)$ and $c(q \otimes 1)$ be the central supports of $p \otimes 1$ and $q \otimes 1$ in $(A \otimes K)^{**}$, respectively. Since $p \otimes 1 \sim q \otimes 1$, $c(p \otimes 1) = c(q \otimes 1)$ by [7, 6.2.8]. It is easy to check that $c(p \otimes 1) = c(p) \otimes 1$ and $c(q \otimes 1) = c(q) \otimes 1$. It follows that $c(p) = c(q)$.

(a) implies that $\text{her}(p) \otimes K \simeq \text{her}(q) \otimes K$ since $\text{her}(p)$ and $\text{her}(q)$ are $\sigma$-unital and strongly Morita equivalent ([3, 2.5]).

(d) $\Rightarrow$ (c) since $p \otimes 1$ and $q \otimes 1$ are unitarily equivalent. It is a well known fact that two projections in a $C^*$-algebra are unitarily equivalent if the distance (in norm) between them is less than one.

(a) $\Rightarrow$ (d): Since $\text{her}(p)$ and $\text{her}(q)$ generate the same closed ideal $I$, $\text{her}(p)$ and $\text{her}(q)$ are strongly Morita equivalent. Let $X = [\text{her}(p)I \text{her}(q)]^{-} = [\text{her}(p)A \text{her}(q)]^{-}$. Then $X \subset A$ and $X$ is a her(p) -- her(q)-imprimitivity bimodule. Let

$$L = \begin{pmatrix} \text{her}(p) & X \\ \bar{X} & \text{her}(q) \end{pmatrix}$$

be the linking algebra of [5]. Since $p \perp q$, $L$ can be identified with a subalgebra of $A$ and so $L \otimes K$ can be identified with a subalgebra of $A \otimes K$. Consequently $M(L \otimes K)$ can be identified with a subalgebra of $(A \otimes K)^{**}$. We assume that $L \otimes K \subset M(L \otimes K) \subset (A \otimes K)^{**}$ from now on. By [3, 2.5], there is $v \in M(L \otimes K)$ such that $v v^* = p \otimes 1$ and $v^* v = q \otimes 1$. Let $u = v + v^*$. Then $u = u^*$, $u^2 = (p + q) \otimes 1$ and $u(p \otimes 1)u = q \otimes 1$. Define a path of unitaries in
\[(p + q) \otimes 1]((p + q) \otimes 1)\) by
\[u(t) = \frac{1}{2}(1 + e^{it\pi})(p + q) \otimes 1 + \frac{1}{2}(1 - e^{it\pi})u : \quad 0 \leq t \leq 1.\]

Then define a path of projections in \((A \otimes K)^{**}\) by
\[p(t) = u(t)^* (p \otimes 1) u(t) : \quad 0 \leq t \leq 1.\]

It is easy to check that \(p(0) = p \otimes 1\) and \(p(1) = q \otimes 1\). It remains to show that \(p(t)\) is an open projection for each \(t \in [0, 1]\). Since \(p\) is open, there is a net of positive elements in \(A\) such that \(a_x \not\to p\) (\([2]\) and \([8]\)). Let \(f_n = \sum_{i=1}^{n} e^{it}\) for each \(n\). It is obvious that \(u(t)^*[a_x \otimes f_n] u(t) \not\to p(t)\) in the \(w^*\)-topology of \((A \otimes K)^{**}\), where the directed set \(\Lambda \times N\) has the dictionary order.

It is sufficient to show \(u(t)^*[a_x \otimes f_n] u(t) \in A \otimes K\) for each \(t \in [0, 1]\). By construction it suffices to show that \(u[a_x \otimes f_n], [a_x \otimes f_n] u,\) and \(u[a_x \otimes f_n] u\) are all in \(A \otimes K\). In fact, since \(a_x \otimes f_n \in (p \otimes 1)(A \otimes K)(p \otimes 1)\) and \(u \in M(L \otimes K)\), we obtain that
\[u[a_x \otimes f_n] = v^*[a_x \otimes f_n], [a_x \otimes f_n] u = [a_x \otimes f_n] v,\]
and \(u[a_x \otimes f_n] u = v^*[a_x \otimes f_n] v\) are all in \(L \otimes K \subset A \otimes K\).

2. Remarks. (1) In Theorem 1 \((a) \Rightarrow (d)\), if \(p\) and \(q\) are in \(M(A)\), then the path of projections between \(p \otimes 1\) and \(q \otimes 1\) can be chosen in \(M(A \otimes K)\) by the same proof.

(2) An easy consequence of Theorem 1 \((a) \Leftrightarrow (b)\) is that a central projection \(r\) in \(A^{**}\) is open iff \(r = c(p)\) for some open projection \(p\) in \(A^{**}\).

(3) Note that the assumption \(p \perp q\) is not needed in proving \((d) \Rightarrow (c)\), but for \((a) \Rightarrow (d)\) this assumption cannot be removed in general. In fact, if \(p\) corresponds to a full hereditary \(C^*\)-subalgebra of \(A\), then by Theorem (2.5) of [3] \(\text{her}(p)\) generates \(A\) as a closed ideal, but \(p \otimes 1\) and \(1 \otimes 1\) cannot be connected by a path of open projections in \((A \otimes K)^{**}\).

2. Corollaries

3. Corollary. If \(A\) is a completely \(\sigma\)-unital simple \(C^*\)-algebra, and \(\text{her}(p), \text{her}(q)\) are nonessential hereditary \(C^*\)-subalgebras of \(A\), then \(p \otimes 1\) and \(q \otimes 1\) are in the same path component of open projections in \((A \otimes K)^{**}\) whenever \(pq = qp\).

Proof. By Theorem 1, we may assume \(r = pq \neq 0\). Since \(pq = qp = r\), \(r\) is the open projection corresponding to \(\text{her}(p) \cap \text{her}(q)\) by [1]. Since \(p\) and \(q\) are both nonessential, we can choose nonzero open projections \(p_1 \perp p\) and \(q_1 \perp q\). Then we have \(p_1 \perp r\) and \(q_1 \perp r\). Now Theorem 1 applies. \(p \otimes 1\) is path connected to \(p_1 \otimes 1\), \(p_1 \otimes 1\) is path connected to \(r \otimes 1\), \(r \otimes 1\) is path connected to \(q_1 \otimes 1\), \(q_1 \otimes 1\) is path connected to \(q \otimes 1\). Here all paths are in the set of open projections in \((A \otimes K)^{**}\). Therefore \(p \otimes 1\) and \(q \otimes 1\) are in the same path component of open projections in \((A \otimes K)^{**}\). \(\square\)

In [6], it was proved that all proper projections in \(M(A \otimes K)\) are in the same path component of projections if \(A\) is a \(\sigma\)-unital \(C^*\)-algebra, where a
projection $P$ in $M(A \otimes K)$ is proper iff $P \sim 1 \sim 1 - P$. With the aid of their result we prove the following:

4. **Corollary.** If $A$ is a completely $\sigma$-unital simple $C^*$-algebra such that every hereditary $C^*$-subalgebra contains a nonzero projection then $p \otimes 1$ and $q \otimes 1$ are in the same path component of open projections in $(A \otimes K)^{**}$ whenever $p$ and $q$ are nonzero nonessential open projections in $A^{**}$.

**Proof.** Since $p$ and $q$ are nonessential open projections in $A^{**}$, there are nonzero open projections $p_0 \perp p$ and $q_0 \perp q$. By hypotheses, there are nonzero projections $p_1$ in her($p$) and $q_1$ in her($q$). By Theorem 1 $p \otimes 1$ and $p_0 \otimes 1$ can be joined by a path of open projections in $(A \otimes K)^{**}$ and also $q \otimes 1$ and $q_0 \otimes 1$ can be joined by a path of open projections in $(A \otimes K)^{**}$. Similarly, $p_0 \otimes 1$ and $p_1 \otimes 1$ are in the same path component of open projections of $(A \otimes K)^{**}$, and so are $q_0 \otimes 1$ and $q_1 \otimes 1$. Since $A$ is simple, $A \otimes K$ is simple. Thus $p_1 \otimes 1$ and $q_1 \otimes 1$ are both proper by Theorem (2.5) of [3]. By [6, Lemma 1] $p_1 \otimes 1$ and $q_1 \otimes 1$ can be joined by a path of projections in $M(A \otimes K)$. Therefore $p \otimes 1$ and $q \otimes 1$ are in the same path component of open projections in $(A \otimes K)^{**}$. □

5. **Question.** If $A$ is a completely $\sigma$-unital $C^*$-algebra and her($p$) and her($q$) generate the same closed ideal of $A$, and $\|pq\| < 1$, does it follow that $p \otimes 1$ and $q \otimes 1$ are in the same path component of open projections in $(A \otimes K)^{**}$?

**ACKNOWLEDGMENT**

This is one of four independent parts of the author's Ph. D. thesis. The author sincerely thanks Professor Lawrence G. Brown for his supervision.

**REFERENCES**


Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

Current address: Department of Mathematics, University of Kansas, Lawrence, Kansas 66045-2142 U.S.A.