PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 3, March 1989

SIMPLE NEAR-RINGS ASSOCIATED WITH MEROMORPHIC PRODUCTS

C. J. MAXSON AND K. C. SMITH

(Communicated by Donald S. Passman)

ABSTRACT. Let H be a subgroup of G^2 and let

$$M_0(G,2,H) = \{f \in M_0(G) | f(H) \subseteq H\}.$$

In this paper we characterize in terms of properties of H when $M_0(G,2,H)$ is a simple near-ring.

I. INTRODUCTION

Let G be a group written additively and k a positive integer, $k \geq 2$. R. Remak has pointed out in [5] that one can construct subgroups of the direct power G^k as follows. For $j \in \{1, 2, \ldots, k\}$, let B_j be a subgroup of G, $\overline{B}_j$ a normal subgroup of B_j such that $B_j/\overline{B}_j \cong B_{j+1}/\overline{B}_{j+1}$ with isomorphisms σ_j, $j \in \{1, \ldots k-1\}$. Let α be an ordinal, $\{b_{1\eta} | \eta < \alpha\}$ a set of coset representatives of $\overline{B}_1$ in B_1 where $b_{10} = 0$ and define a subset $H \subseteq G^k$ by $H = \bigcup_{\eta<\alpha}[(b_{1\eta} + \overline{B}_1) \times \prod_{j=1}^{k-1}(\sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_1(b_{1\eta} + \overline{B}_1))]$. Here H is called a *k-fold meromorphic product* and will be denoted by $H = B_1/\overline{B}_1 \overset{\times}{\sim}_{\sigma_1} B_2/\overline{B}_2 \overset{\times}{\sim}_{\sigma_2} \cdots \overset{\times}{\sim}_{\sigma_{k-1}} B_k/\overline{B}_k$. It is straightforward to verify that H is a subgroup of G^k. However, only for $k = 2$ can every subgroup of G^k be obtained as a meromorphic product.

Theorem I.1 (Klein–Fricke) [5]. *Every subgroup of $G \times G$ is a 2-fold meromorphic product.*

Let $M(G) = \{f : G \to G\}$ act on G^k componentwise, i.e. let $f(x_1, x_2, \ldots, x_k) = (f(x_1), \ldots, f(x_k))$. For any subgroup H of G^k we define $M(G,k,H) = \{f \in M(G) | f(H) \subseteq H\}$. In a similar manner, let $M_0(G) = \{f : G \to G | f(0) = 0\}$ and define $M_0(G,k,H) = \{f \in M_0(G) | f(H) \subseteq H\}$. These $M(G,k,H)$ and $M_0(G,k,H)$ are subnear-rings of $M(G)$ with identity id: $G \to G$, $\text{id}(x) = x$, $\forall x \in G$.

Received by the editors October 17, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 16A76; Secondary 20E07.

H. Wielandt [6], suggested relating the properties of $M(G,k,H)$ and $M_0(G,k,H)$ with those of the subgroup H. Because of the above theorem much more can be said for the case $k=2$ and henceforth we restrict ourselves to this case.

Recall, for any group G, with $|G|>2$, $M(G)$ and $M_0(G)$ are simple near-rings. (For this and other basic information about near-rings the reader is referred to the book by Meldrum [3] or the book by Pilz [4].) As one might expect, the situation for $M(G,2,H)$ and $M_0(G,2,H)$ is much different from that of $M(G)$ and $M_0(G)$. In fact, as we shall see, neither of these near-rings need be simple. Further when $M_0(G,2,H)$ is simple, it need not be the case that $M(G,2,H)$ is simple.

Example I.2. Let $G=\mathbb{Z}_2+\mathbb{Z}_2$, $H=G/A \overset{\times}{\sim}_\sigma G/B$ where $A=\mathbb{Z}_2+\{0\}$ and $B=\{0\}+\mathbb{Z}_2$. By calculations, or using Theorem III.12 we see that $M_0(G,2,H)$ is simple. However, if $c_a\colon G\to G$ denotes the constant function $c_a(x)=a$, $\forall x\in G$ then $M_c(G,2,H)=\{c_{(0,0)},c_{(1,1)}\}$ is an ideal of $M(G,2,H)$. Thus $M_0(G,2,H)$ is simple but $M(G,2,H)$ is not.

It is the purpose of this paper to characterize in terms of properties of the subgroup H when $M_0(G,2,H)$ is simple. In the next section we present some general results which reduce the problem to a special case. In the final section we focus on this particular case.

II. General results

Let $H=B_1/\overline{B}_1 \overset{\times}{\sim}_\sigma B_2/\overline{B}_2$. We first show that in some special situations, $M_0(G,2,H)$ and $M(G,2,H)$ are always simple. In fact if $\overline{B}_1\cap\overline{B}_2=G$ then $H=G/G \overset{\times}{\sim}_\sigma G/G$ so $H=G\times G$ and $M_0(G,2,H)=M_0(G)$ while $M(G,2,H)=M(G)$. Thus $M_0(G,2,H)$ is simple and $M(G,2,H)$ is simple when $|G|>2$. If $B_1\cup B_2=\{0\}$ then $H=\{(0,0)\}$ so $M_0(G,2,H)=M(G,2,H)=M_0(G)$. Thus we take $\overline{B}_1\cap\overline{B}_2\neq G$ and $B_1\cup B_2\neq\{0\}$. We now give an easy but basic result used throughout the paper.

Lemma II.1. *Let $H=B_1/\overline{B}_1 \overset{\times}{\sim}_\sigma B_2/\overline{B}_2$ and let $f\in M_0(G,2,H)$. Then $f(\overline{B}_i)\subseteq\overline{B}_i$, $i=1,2$.*

Proof. For $\overline{b}_1\in\overline{B}_1$, $(\overline{b}_1,0)\in H$. Therefore $(f(\overline{b}_1),0)\in H$ which implies $f(\overline{b}_1)\in\overline{B}_1$.

Another useful tool is given next. The proof is omitted since the result is well known.

Lemma II.2. *Let $N_0=M_0(G,2,H)$ and let U be a subset of G with the property $f(U)\subseteq U$, $\forall f\in N_0$. Define* $\mathrm{Ann}_{N_0}(U)=\{f\in N_0 | f(U)=\{0\}\}$. *Then* $\mathrm{Ann}_{N_0}(U)$ *is an ideal of* N_0.

Lemma II.3. *If $B_1\cup B_2\neq G$ then $M_0(G,2,H)$ is not simple.*

Proof. Let $U = B_1 \cup B_2$ and note that $f(U) \subseteq U$, $\forall f \in M_0(G, 2, H)$ and so by the above lemma, $I = \mathrm{Ann}_{N_0}(U)$ is an ideal in $M_0(G, 2, H)$. Moreover, since id $\notin I$, $I \subsetneqq M_0(G, 2, H)$. Further, $I \neq \{0\}$ since the function $h: G \to G$ given by $h(x) = 0$ if $x \in U$ and $h(x) = x$ if $x \notin U$ is in I.

We have $B_1 \cup B_2 \neq \{0\}$. Suppose now one of B_1, B_2 is $\{0\}$, say $B_1 = \{0\}$, so $H = \{0\}/\{0\} \overset{\times}{\sim} B_2/B_2$. Thus $M_0(G, 2, H) = M(G, 2, H)$. If $M_0(G, 2, H)$ is simple, then from the above lemma, $B_2 = G$, hence $H = \{0\} \times G$ and $M(G, 2, H) = M_0(G, 2, H) = M_0(G)$. Conversely if $B_2 = G$, $M_0(G, 2, H) = M_0(G)$ is simple.

Corollary II.4. *Let* $H = \{0\}/\{0\} \overset{\times}{\sim}_\sigma B_2/B_2$. *Then* $M(G, 2, H) = M_0(G, 2, H)$ *and* $M_0(G, 2, H)$ *is simple* $\Leftrightarrow B_2 = G$.

Henceforth we also take $B_1 \neq \{0\} \neq B_2$. In the following result we collect some necessary conditions for $M_0(G, 2, H)$ to be simple.

Proposition II.5. *Let* $H = B_1/\overline{B}_1 \overset{\times}{\sim}_\sigma B_2/\overline{B}_2$ *and let* $N_0 = M_0(G, 2, H)$ *be simple. Then*

(i) $\overline{B}_1 \cap \overline{B}_2 = \{0\}$;

(ii) *if* $B_1 \neq G$, $H = B_1/\overline{B}_1 \overset{\times}{\sim}_\sigma G/\{0\}$;

(iii) *if* G *is finite,* $H = G/\overline{B}_1 \overset{\times}{\sim}_\sigma G/\overline{B}_2$.

Proof. (i) We take $U = \overline{B}_1 \cap \overline{B}_2$ and suppose $U \neq \{0\}$. From Lemma II.1, $f(U) \subseteq U$ and so $I = \mathrm{Ann}_{N_0}(U)$ is an ideal, $I \neq N_0$. Let $\overline{b}$ be a nonzero element in U and define $h: G \to G$ by $h(x) = 0$, $x \in U$, $h(x) = \overline{b}$, $x \notin U$. Then $h \neq 0$ and $h \in I$, a contradiction to the simplicity of N_0.

(ii) From Lemma II.3, $B_1 \cup B_2 = G$ and since $B_1 \neq G$ we must have $B_2 = G$. Assume $\overline{B}_2 \neq \{0\}$ and choose $0 \neq \overline{b}_2 \in \overline{B}_2$. Define $h: G \to G$ by $h(x) = 0$, $x \in B_1$, and $h(x) = \overline{b}_2$ if $x \notin B_1$. For $(u, v) \in H$,

$$(f(u), f(v)) = \begin{cases} (0, 0), & v \in B_1, \\ (0, \overline{b}_2), & v \notin B_1. \end{cases}$$

Thus $h \in N_0$ and for $U = B_1$, $\{0\} \subsetneqq \mathrm{Ann}_{N_0}(U) \subsetneqq N_0$, a contradiction.

(iii) From Lemma II.3, $B_1 = G$ or $B_2 = G$, say $B_1 = G$. If $B_2 \neq G$ then from (ii), $H = G/\{0\} \overset{\times}{\sim}_\sigma B_2/\overline{B}_2$ which is impossible when G is finite. Thus we must have $B_2 = G$ and so $H = G/\overline{B}_1 \overset{\times}{\sim}_\sigma G/\overline{B}_2$.

Convention. For the remainder of this paper we take G to be a *finite* group, written additively with identity 0. We let $S^* = S\backslash\{0\}$ for any subset S of G. We also let N_0 denote $M_0(G, 2, H)$.

From (iii) of the above proposition we need only consider subgroups H of the form $H = G/A \overset{\times}{\sim}_\sigma G/B$ where $A \cap B = \{0\}$. We handle first the case $A = \{0\}$.

Theorem II.6. *Let $H = G/A \overset{\times}{\sim}_\sigma G/B$, $A = \{0\}$. Then $B = \{0\}$ and considering σ as an automorphism of G we have $N_0 = \{f \in M_0(G) | f\sigma = \sigma f\}$. Moreover, N_0 is simple $\Leftrightarrow$ all $\mathscr{A}$-orbits of G^k have the same size where $\mathscr{A} = \langle\sigma\rangle$, the cyclic group of automorphisms of G generated by σ.*

Proof. Since G is finite, $A = \{0\}$ implies $B = \{0\}$ so $H = \{(a, \sigma a) | a \in G\}$ where we consider σ as an automorphism of G. But $f \in N_0 \Leftrightarrow (f(a), f\sigma(a)) \in H$. However $(f(a), u) \in H \Leftrightarrow u = \sigma f(a)$, so $N_0 = \{f \in M_0(G) | f\sigma = \sigma f\}$, the centralizer near-ring $\mathscr{C}(\mathscr{A}; G)$ where $\mathscr{A} = \langle\sigma\rangle$. The final statement now follows from Theorem 1.1 of [2].

We conclude this section with a general result about functions in $M(G, 2, H)$ and obtain as a corollary a result to be used in the remaining case.

Lemma II.7. *Let $H = B_1/\overline{B}_1 \overset{\times}{\sim}_\sigma B_2/\overline{B}_2$ and let $f \in M(G, 2, H)$. Then $\forall u \in B_1$, $f(u + \overline{B}_1) \subseteq f(u) + \overline{B}_1$ and $\forall v \in B_2$, $f(v + \overline{B}_2) \subseteq f(v) + \overline{B}_2$.*

Proof. Let $\sigma(u + \overline{B}_1) = x + \overline{B}_2$ and $\sigma(f(u) + \overline{B}_1) = y + \overline{B}_2$. Since $(u, x) \in H$, $(f(u), f(x)) \in H$ so $f(x) \in y + \overline{B}_2$. But then for every $b \in \overline{B}_1$, $(u + b, x) \in H$ implies $(f(u+b), f(x)) \in H$, and since $f(x) \in y + \overline{B}_2$, $f(u+b) \in f(u) + \overline{B}_1$. Similarly we get $f(v + \overline{B}_2) \subseteq f(v) + \overline{B}_2$.

Corollary II.8. *Let $H = G/A \overset{\times}{\sim}_\sigma G/B$ with $A \cap B = \{0\}$. If $f \in N_0$, $a \in A$, $b \in B$, $f(a + b) = f(a) + f(b)$.*

Proof. Since A and B are normal subgroups of G with $A \cap B = \{0\}$, $a + b = b + a$. From the above lemma, $f(a + b) = f(a) + \overline{b}$, $\overline{b} \in B$ and $f(b + a) = f(b) + \overline{a}$, $\overline{a} \in A$. Since $a + b = b + a$, $f(a) + \overline{b} = f(b) + \overline{a} = \overline{a} + f(b)$ and since $A \cap B = \{0\}$, $\overline{a} = f(a)$, $\overline{b} = f(b)$.

III. The case $H = G/A \overset{\times}{\sim}_\sigma G/B, A \neq \{0\}$

We start this section by defining two ascending chains of normal subgroups of G which play an important role in determining whether or not N_0 is simple. Let $A_0 = A$. Let A_1 be the subgroup of G such that $A_1/A = \sigma^{-1}((A + B)/B)$. Then A_1 is a normal subgroup of G and $A_0 \subseteq A_1$. Inductively we define A_j to be the subgroup of G such that $A_j/A = \sigma^{-1}((A_{j-1} + B)/B)$. These A_j are normal subgroups of G and $A_{j-1} \subseteq A_j$. Analogously we define $B_0 = B$, B_1 to be the subgroup of G such that $\sigma((B + A)/A) = B_1/B$ and again inductively B_j to be the subgroup of G such that $\sigma((B_{j-1} + A)/A) = B_j/B$. The B_j are normal subgroups of G with $B_{j-1} \subseteq B_j$. Since G is finite these ascending chains must eventually become stationary.

Let $H = G/A \overset{\times}{\sim}_\sigma G/B$, $A \neq \{0\}$. Since G is finite, $|A| = |B|$ and also $\exists$ integer $l \geq 0$ such that $A_l = A_{l+1} = \cdots = A_{l+t}$ for each integer t, $t \geq 0$. From $\sigma^{-1}((A_j + B)/B) = A_{j+1}/A$ and the fact that $|A| = |B|$ we obtain

$|A_j+B| = |A_{j+1}|$ and by symmetry $|B_j+A| = |B_{j+1}|$. Since $|A_l| = |A_{l+1}| = |A_l+B| = |A_l|\,|B|/|A_l \cap B|$ we get $|A_l \cap B| = |B|$, so $B_0 = B \subseteq A_l$. Suppose $B_k \subseteq A_l$. By definition, $\sigma^{-1}(B_{k+1}/B) = (B_k+A)/A \subseteq A_l/A = A_{l+1}/A = \sigma^{-1}((A_l+B)/B)$. Hence $B_{k+1}/B \subseteq (A_l+B)/B$ which implies $B_{k+1} \subseteq A_l$. Thus, if for some m, $B_m = G$ then for some n, $A_n = G$. Since the above argument is symmetric in A and B we have the following.

Lemma III.1. $\exists n,\ A_n = G \Leftrightarrow \exists m,\ B_m = G$.

If there exists an integer n such that $A_n = G$ then there is a least such integer n. In this case we say G *has A-length* n and denote this by $n = Al(G)$. In a symmetric manner we define the B-length of G. The previous lemma says that G has an A-length $\Leftrightarrow G$ has a B-length.

We show now that the subgroups A_i and B_j are N_0-invariant. By Lemma II.7 $A_0 = A$ and $B_0 = B$ are N_0-invariant. If $a_1 \in A_1$ then there exists an $a \in A_0$ such that $\sigma(a_1+A) = a+B$, so $(a_1, a) \in H$. If $f \in N_0$ then $f(a_1, a) = (f(a_1), f(a)) \in H$ and since $f(a) \in A_0$ then $f(a_1) \in A_1$. Continuing we have $f(A_i) \subseteq A_i$ by induction. Similarly for the B_j's.

Suppose G does not have an A-length and let l be the least non-negative integer such that $A_l = A_{l+1} = \cdots$ and $A_l \neq G$. Define $f: G \to G$ by $f(x) = 0$, $x \in A_l$ and $f(x) = x$, $x \notin A_l$. We show $f \in N_0$. Let $(u, v) \in H$. If $u \in A_l$ then $\exists w \in A_{l-1}$, $b \in B$ such that $v = w + b$. But then $v \in A_{l-1} + B \subseteq A_l$ since we showed above $A_l = A_{l+1}$ implies $B \subseteq A_l$. Thus $(f(u), f(v)) = (0, 0) \in H$. If $u \notin A_l$ then $v \notin A_l$, hence $(f(u), f(v)) = (u, v) \in H$. Thus $f \in \mathrm{Ann}_{N_0}(U)$ where $U = A_l$ and $\{0\} \subsetneqq N_0$. This establishes the next result.

Lemma III.2. *If N_0 is simple, $\exists n$ such that $n = Al(G)$.*

Lemma III.3. $A \cap B = A_1 \cap B = \cdots = A_l \cap B = \{0\} \Leftrightarrow A \cap B = A \cap B_1 = \cdots = A \cap B_l = \{0\}$.

Proof. Suppose $A \cap B = A_1 \cap B = \cdots = A_l \cap B = \{0\}$. We show $A \cap B_j = \{0\}$, $0 \leq j \leq l$. To this end suppose $\exists x_j \in A \cap B_j$, $x_j \neq 0$. Then $\exists x_{j-1} \neq 0$ in $A_1 \cap B_{j-1}$ such that $(x_{j-1}, x_j) \in H$. For if $x_{j-1} = 0$ then $x_j \in A \cap B = \{0\}$, a contradiction. Continuing one obtains $x_1 \neq 0$ in $A_{j-1} \cap B_1$. But then $\exists x_0 \neq 0$, $x_0 \in A_j \cap B$, a contradiction. The converse follows in the same manner.

Lemma III.4. *Let $A \cap B = A_1 \cap B = \cdots = A_{k-1} \cap B = \{0\}$ and $A_k \cap B \neq \{0\}$. Then $\forall j$, $1 \leq j \leq k$, $A_j \cap B_{k-j} \neq \{0\}$ and $\exists$ isomorphisms $\sigma_j: A_j \cap B_{k-j} \to A_{j-1} \cap B_{k-(j-1)}$.*

Proof. As in the above lemma, $A_k \cap B \neq \{0\}$ implies $A_j \cap B_{k-j} \neq \{0\}$ $1 \leq j \leq k$. Now let $0 \neq x \in A_j \cap B_{k-j}$, $j \geq 1$. Then $\exists y \neq 0$ in $A_{j-1} \cap B_{k-(j-1)}$ such that $(x, y) \in H$. For if $y = 0$, then $x \in A \cap B_{k-j} = \{0\}$, a contradiction to the above lemma. Also y is unique since $(x, y), (x, \overline{y}) \in H$ implies $y - \overline{y} \in B$ and since $y - \overline{y} \in A_{j-1}$ we get $y = \overline{y}$. Thus we have a function $\sigma_j: A_j \cap B_{k-j} \to A_{j-1} \cap B_{k-(j-1)}$ defined by $\sigma_j(x) = y$. Moreover σ_j is one-one since

$(x, y), (\overline{x}, y)$ in H implies $x - \overline{x} \in A \cap B_{k-j}$. Since $j \geq 1$ this gives $x = \overline{x}$. Also, σ_j is onto for if $y \in A_{j-1} \cap B_{k-(j-1)}$ then $\exists x \in A_j \cap B_{k-j}$ such that $(x, y) \in H$. But this means $\sigma_j(x) = y$. To show σ_j is an isomorphism let $x_1, x_2 \in A_j \cap B_{k-j}$. Then there exist unique $y_1, y_2 \in A_{j-1} \cap B_{k-(j-1)}$ such that $(x_1, y_1), (x_2, y_2) \in H$. Since H, $A_j \cap B_{k-j}$, and $A_{j-1} \cap B_{k-(j-1)}$ are groups we have $(x_1 + x_2, y_1 + y_2) \in H$ with $\sigma_j(x_1 + x_2) = y_1 + y_2 = \sigma_j(x_1) + \sigma_j(x_2)$.

Theorem III.5. *If N_0 is simple then $\exists n$, $n = Al(G)$ and $\forall k$, $0 \leq k < n$, $A_k \cap B = \{0\}$.*

Proof. From Lemma III.2, $\exists n$, $n = Al(G)$. Suppose $\exists k < n$ such that $A \cap B = \cdots = A_{k-1} \cap B = \{0\}$ but $A_k \cap B \neq \{0\}$. We show there exists a nonzero function $f \in N_0$ such that $f(A) = \{0\}$. But then for $U = A$ in Lemma II.2, N_0 is not simple, contrary to hypothesis.

To construct the desired function, we use the above lemma to obtain nonzero $\overline{b}_j$ in $A_{k-j} \cap B_j$, $0 \leq j \leq k$ such that

$$(+) \qquad (\overline{b}_0, \overline{b}_1), (\overline{b}_1, \overline{b}_2), \dots, (\overline{b}_{k-1}, \overline{b}_k) \in H.$$

Let F denote the $(k+1)$-tuple $(\overline{b}_0, \overline{b}_1, \dots, \overline{b}_{k-1}, \overline{b}_k)$. Since $n = Al(G)$, $\exists m$, $m = Bl(G)$ and since $A_{k-1} \cap B = \{0\}$ implies $A \cap B_{k-1} = \{0\}$ we have $m \geq k$. Let $x \in G^*$. Thus, there is a unique j, $0 \leq j \leq m$ such that $x \in B_j \backslash B_{j-1}$ where we take $B_{-1} = \{0\}$. Let $x = b_j$. Then there exist elements $b_{j-1} \in B_{j-1} \backslash B_{j-2}, \dots, b_{j-k} \in B_{j-k} \backslash B_{j-(k+1)}$ such that

$$(++) \qquad (b_{j-k}, b_{j-k+1}), \dots, (b_{j-1}, b_j) \in H.$$

We make the convention that if any of the subscripts i in $(++)$ are negative then $B_i = \{0\}$. In the $(k+1)$-tuple, $(b_{j-k}, b_{j-k+1}, \dots, b_j)$, each element b_i is in A_k or is not. In this way we get a $(k+1)$-tuples of 0's and 1's associated to $(b_{j-k}, \dots, b_j)$, i.e. define ε_{j-i} by

$$\varepsilon_{j-i} = \begin{cases} 0 & \text{if } b_{j-i} \in A_k. \\ 1 & \text{if } b_{j-i} \notin A_k. \end{cases}$$

We call $(\varepsilon_{j-k}, \dots, \varepsilon_j)$ the *signature of* $x = b_j$, denoted by $E(x)$. We must show $E(x)$ is well defined. To this end let $(b'_{j-k}, \dots, b'_j)$ be another $(k+1)$-tuple associated with $x = b_j$, that is

$$(b'_{j-k}, b'_{j-k+1}), \dots, (b'_{j-1}, b_j) \in H \qquad \text{where } b'_{j-i} \in B_{j-i} \backslash B_{j-i-1}.$$

Note that $b'_{j-1} - b_{j-1} \in A_0$ so $b'_{j-1} = b_{j-1} + a$ and $b'_{j-1} \in A_k \Leftrightarrow b_{j-1} \in A_k$. Then from $(b'_{j-2}, b_{j-1} + a), (b_{j-2}, b_{j-1}) \in H$ we get $(b'_{j-2} - b_{j-2}, a) \in H$ so $b'_{j-2}, b_{j-2} \in A_1$, thus $b'_{j-2} \in A_k \Leftrightarrow b_{j-2} \in A_k$. In general, $b'_{j-i} - b_{j-i} \in A_{i-1}$, $1 \leq i \leq k$, so $b'_{j-i} \in A_k \Leftrightarrow b_{j-i} \in A_k$. This shows that $E(x)$ is well defined.

We define $f: G \to G$ as follows. For $x \in G$, $x \in B_j \backslash B_{j-1}$, say $x = b_j$, define $f(x) = \varepsilon_{j-k}\overline{b}_k + \varepsilon_{j-(k-1)}\overline{b}_{k-1} + \cdots + \varepsilon_j \overline{b}_0 = E(x) \cdot F$, where $E(x) \cdot F$

denotes the scalar product of the $(k+1)$-tuples $E(x)$ and F. From the previous paragraph f is well defined.

Claim 1. $f(b_j + a_0) = f(b_j)$ $\forall a_0 \in A$.

Since $a_0 \in A$, for $i = 1, 2, \ldots, k \exists a_i \in A_i \backslash A_{i-1}$ such that $(a_k, a_{k-1}), \ldots, (a_1, a_0)$ are in H. Using (++) we obtain

$$(b_{j-k} + a_k, b_{j-k+1} + a_{k-1}), \ldots, (b_{j-1} + a_1, b_j + a_0) \in H.$$

Since $a_0, a_1, \ldots, a_k \in A_k$ we have $b_{j-i} + a_i \in A_k \Leftrightarrow b_{j-i} \in A_k$, i.e. $E(b_j) = E(b_j + a_0)$. Hence $f(b_j + a_0) = f(b_j)$.

Claim 2. $f \in N_0$.

Every element of H has the form $(b_{j-1} + a, b_j)$ for some j where $b_j \in B_j \backslash B_{j-1}$, $b_{j-1} \in B_{j-1} \backslash B_{j-2}$ and $a \in A$. From Claim 1, $f(b_{j-1} + a) = f(b_{j-1})$ so to show $f \in N_0$ it suffices to show $(f(b_{j-1}), f(b_j)) \in H$. Let $F_1 = (b_{j-k-1}, b_{j-k}, \ldots, b_{j-1})$ be a $(k+1)$-tuple determined by b_{j-1} as in (++). But then $F_2 = (b_{j-k}, \ldots, b_{j-1}, b_j)$ is a $(k+1)$-tuple determined by b_j so $f(b_j) = E(b_j) \cdot F_1$ and $f(b_{j-1}) = E(b_{j-1}) \cdot F_2$. From (+), $(\varepsilon_{j-k}\overline{b}_{k-1}, \varepsilon_{j-k}\overline{b}_k), \ldots, (\varepsilon_{j-2}\overline{b}_1, \varepsilon_{j-2}\overline{b}_2), (\varepsilon_{j-1}\overline{b}_0, \varepsilon_{j-1}\overline{b}_1) \in H$ and since $\overline{b}_0 \in B$, $\overline{b}_k \in A$ we have $(0, \overline{b}_0)$, $(\overline{b}_k, 0) \in H$. Adding gives $(E(b_{j-1}) \cdot F_1, E(b_j) \cdot F_2) \in H$. Thus $f \in N_0$.

Claim 3. $f(A) = \{0\}$.

For $a \in A$, $E(a) = (0, \ldots, 0)$ so $f(a) = 0$.

From Claims 2 and 3, $f \in \mathrm{Ann}_{N_0}(A)$. Moreover $B \nsubseteq A_k$ (for if $B \subseteq A_k$ then $|A_{k+1}| = |A_k + B| = |A_k|$ and since $A_k \subseteq A_{k+1}$ then $A_k = A_{k+1} = A_{k+2} = \cdots$, contradicting $n = Al(G)$) so for $b \in B \backslash A_k$, $f(b) = \overline{b}_0 \neq 0$ hence we have the desired function.

In a sequence of lemmas we establish the converse. For convenience we say $H = G/A \overset{\times}{\sim}_\sigma G/B$ *has property* Σ if $\exists n$, $n = Al(G)$ and $\forall k$, $0 \leq k < n$, $A \cap B_k = \{0\}$.

Lemma III.6. *Let H satisfy property Σ. Then for each i, $0 \leq i \leq n$, $|A_i| = |B_i| = |A|^{i+1}$. Therefore $|G| = |A|^{n+1}$ and $Bl(G) = n$.*

Proof. We know $|A_j| = |A_{j-1} + B|$ for all j, $1 \leq j \leq n$. Since $A_{j-1} \cap B = \{0\}$, $|A_j| = |A_{j-1}|\,|B|$. From $|A_1| = |A|\,|B| = |A|^2$ we get $|A_j| = |A|^{j+1}$, hence $|G| = |A_n| = |A|^{n+1}$. From Lemma II.3, $B \cap A_k = \{0\}$ $\forall k$, $0 \leq k < n$ so in a symmetric manner we get $|B_n| = |A|^{n+1}$, hence $B_n = G$. But this means $Bl(G) = n$.

Let $\psi \colon N_0 \to M_0(A)$ be the restriction map, $\psi(f) = f/A$. It is clear that ψ is a near ring homomorphism. When H satisfies property Σ we show ψ is an isomorphism. Since $M_0(A)$ is simple we will have the converse of Theorem III.5. We first show ψ is one to one.

Lemma III.7. *Let H satisfy property Σ.*

(i) $\forall j$, $0 \le j \le n$, $A_j \cap B_{n-j-1} = \{0\}$.

(ii) $(A_1 \cap B_{n-1}) + (A_2 \cap B_{n-2}) + \cdots + (A_{n-1} \cap B_1)$ *is a direct sum.*

(iii) $A_{n-1} \cap B_{n-1} = \sum_{i=1}^{n-1} \oplus (A_i \cap B_{n-i})$.

(iv) $G = A \oplus B \oplus (A_{n-1} \cap B_{n-1})$.

Proof. (i) Let $x \in A_j \cap B_{n-j-1}$. Then $\exists a_{j-1}, \ldots, a_0$ such that (x, a_{j-1}), $(a_{j-1}, a_{j-2}), \ldots, (a_1, a_0) \in H$ where $a_i \in A_i \cap B_{n-i-1}$. Thus $a_0 = 0$ and from this $a_1 = a_2 = \cdots = a_{j-1} = x = 0$.

(ii) It suffices to show $(A_1 \cap B_{n-1}) + \cdots + (A_{i-1} \cap B_{n-(i-1)}) \cap (A_i \cap B_{n-i}) = \{0\}$, for all i, $2 \le i \le n-1$. But this is immediate since $x \in (A_1 \cap B_{n-1}) + \cdots + (A_{i-1} \cap B_{n-(i-1)}) \cap (A_i \cap B_{n-i})$ implies $x \in A_{i-1} \cap B_{n-i} = \{0\}$ by part (i).

(iii) Consider $A_i \cap B_{n-i}$ for any i, $1 \le i \le n-1$. From part (i) $A_{i-1} \cap (A_i \cap B_{n-i}) = \{0\}$ so $A_{i-1} \oplus (A_i \cap B_{n-i})$ is a subgroup of A_i with order $|A|^i |A_i \cap B_{n-i}|$. By Lemma III.4, $|A_i \cap B_{n-i}| = |A|$ and therefore $|A_{i-1} \oplus A_i \cap B_{n-i}| = |A|^i |A| = |A|^{i+1}$. This in turn implies $|\sum_{i=1}^{n-1} \oplus (A_i \cap B_{n-i})| = |A|^{n-1}$. On the other hand, from $|A|^{n+1} = |G| \ge |A_{n-1} + B_{n-1}| = |A|^{2n}/|A_{n-1} \cap B_{n-1}|$ we get $|A_{n-1} \cap B_{n-1}| \ge |A|^{n-1}$ and since $A \cap (A_{n-1} \cap B_{n-1}) = \{0\}$, $|A \oplus (A_{n-1} \cap B_{n-1})| = |A|\,|A_{n-1} \cap B_{n-1}| \le |A|^n$ or $|A_{n-1} \cap B_{n-1}| = |A|^{n-1}$. Since $\sum_{i=1}^{n-1} \oplus (A_i \cap B_{n-i}) \subseteq A_{n-1} \cap B_{n-1}$ the result follows.

(iv) It is clear that $A \oplus B \cap (A_{n-1} \cap B_{n-1}) = \{0\}$. Hence $A \oplus B \oplus (A_{n-1} \cap B_{n-1})$ is a subgroup of G of order $|A|\,|A|\,|A|^{n-1} = |A|^{n+1}$, i.e. $G = A \oplus B \oplus A_{n-1} \cap B_{n-1}$.

Corollary III.8. *Let H satisfy property Σ and let $w \in G$. Then $w = a + b + c$, $a \in A$, $b \in B$, $c \in A_{n-1} \cap B_{n-1}$ and $\forall f \in N_0$, $f(w) = f(a+b+c) = \hat{a} + \hat{b} + f(c)$, $\hat{a} \in A$, $\hat{b} \in B$.*

Proof. From part (iv) of the above lemma, $w = a + b + c$, $a \in A$, $b \in B$, $c \in A_{n-1} \cap B_{n-1}$. From Lemma II.7, $f(a+b+c) = f(b+c+a) = f(b+c) + \hat{a}$, $\hat{a} \in A$ and $f(b+c) = f(c+b) = f(c) + \hat{b}$, $\hat{b} \in B$.

Lemma III.9. *Let H satisfy property Σ and let $f \in \mathrm{Ann}_{N_0}(A)$.*

(i) $f(A_j) \subseteq A_{j-1}$ $\forall_j$, $1 \le j \le n$.

(ii) $f(B) = \{0\}$.

(iii) $f(B_j) \subseteq B_{j-1}$, $\forall_j$, $1 \le j \le n$.

(iv) $f(A \oplus B) = \{0\}$.

(v) $f(G) \subseteq A_{n-1} \cap B_{n-1}$.

Proof. (i) Let $a_1 \in A_1$. Thus $\exists a_0 \in A$ such that $(a_1, a_0) \in H$. But then $(f(a_1), f(a_0)) = (f(a_1), 0) \in H$ implies $f(a_1) \in A = A_0$. Continuing, let $a_j \in A_j$ and suppose $f(A_{j-1}) \subseteq A_{j-2}$. Then for some $a_{j-1} \in A_{j-1}$, $(a_j, a_{j-1}) \in H$ so $(f(a_j), f(a_{j-1})) \in H$. Since $f(a_{j-1}) \in A_{j-2}$, $f(a_j) \in A_{j-1}$.

(ii) From part (i), $f(G) = f(A_n) \subseteq A_{n-1}$ and so $f(B) \subseteq A_{n-1}$. But from Lemma II.1, $f(B) \subseteq B$ so we have $f(B) \subseteq A_{n-1} \cap B = \{0\}$.

(iii) Follows from (ii) using the same arguments as in (i).

(iv) Follows from (ii) and Corollary II.8.

(v) From (i) and (iii), $f(G) = f(A_n) \subseteq A_{n-1}$ and $f(G) = f(B_n) \subseteq B_{n-1}$.

Lemma III.10. *If* $f \in N_0$ *and* $f(A) = \{0\}$ *then* $f(G) = \{0\}$.

Proof. Let $f \in N_0$ such that $f \in \operatorname{Ker}\psi$, i.e. $f(A) = \{0\}$. For $x \in G$, from Corollary III.8, $x = a + b + c$, $a \in A$, $b \in B$, $c \in A_{n-1} \cap B_{n-1}$ and $f(x) = \hat{a} + \hat{b} + f(c)$, $\hat{a} \in A$, $\hat{b} \in B$. From the above lemma, $f(x) \in A_{n-1} \cap B_{n-1}$. Thus if $f(c) = 0$ then $f(x) \in (A_{n-1} \cap B_{n-1}) \cap A \oplus B = \{0\}$ from Lemma III.7, (iv). Therefore $f(A) = \{0\}$ implies f is the zero map which means $\operatorname{Ker}\psi = \{0\}$. To complete the proof we show $f(A_{n-1} \cap B_{n-1}) = \{0\}$.

Let $w \in A_{n-1} \cap B_{n-1}$, $w = x_1 + x_2 + \cdots + x_{n-1}$ where $x_i \in A_i \cap B_{n-i}$, $i = 1, 2, \ldots, n-1$. If $w = 0$ we are finished so we take $w \neq 0$. Let l be the largest integer such that $x_l \neq 0$. If $l = 1$, $w = x_1$ and $f(w) \in f(A_1 \cap B_{n-1}) \subseteq A \cap B_{n-2} = \{0\}$. If $l = 2$, $w = x_1 + x_2$. Since $x_1 \in A_1 \cap B_{n-1}$, $\exists a_0 \in A \cap B_n$ such that $(x_1, a_0) \in H$. Also, since $x_2 \in A_2 \cap B_{n-2}$, $\exists a_1 \in A_1 \cap B_{n-1}$ with $(x_2, a_1) \in H$. Therefore $(x_1 + x_2, a_0 + a_1) \in H$ and thus $(f(x_1 + x_2), f(a_0 + a_1)) \in H$. However, from Lemma II.7, $f(a_0 + a_1) = \hat{a}_0 + f(a_1)$, $\hat{a}_0 \in A$ and by the previous case $f(a_1) = 0$. Now $(f(x_1 + x_2), \hat{a}_0) \in H$ implies $f(x_1 + x_2) \in A_1$. But $x_1, x_2 \in B_{n-1}$ imply $f(x_1 + x_2) \in B_{n-2}$. Thus $f(x_1 + x_2) \in A_1 \cap B_{n-2} = \{0\}$ from Lemma III.7, (i). Assume the results for all elements of the form $x_1 + \cdots + x_{l-1}$ and let $u = y_1 + \cdots + y_l$. As above, $\exists a_i \in A_i \cap B_{n-i}$ with $(y_{i+1}, a_i) \in H$, $0 \leq i \leq l-1$ and we obtain $(f(u), f(a_0 + \cdots + a_{l-1})) = (f(u), \hat{a}_0 + f(a_1 + \cdots + a_{l-1})) = (f(u), \hat{a}_0)$ by the induction hypothesis. Hence $f(u) \in A_1$ and $u \in B_{n-1}$ implies $f(u) \in B_{n-2}$, so we have $f(u) = 0$. The result follows by induction.

We now have that $\psi\colon N_0 \to M_0(A)$ defined by $\psi(f) = f/A$ is a monomorphism when H satisfies property Σ. It remains to show that every function f on A with $f(0) = 0$ can be extended to a function in N_0.

From Lemma III.4 we have isomorphisms σ_j,

$$B = A_n \cap B_0 \xrightarrow{\sigma_n} A_{n-1} \cap B_1 \xrightarrow{\sigma_{n-1}} \cdots \to A_j \cap B_{n-j} \xrightarrow{\sigma_j} A_{j-1} \cap B_{n-(j-1)} \to \cdots \to$$
$$A_1 \cap B_{n-1} \xrightarrow{\sigma_1} A_0 \cap B_n = A.$$

Let $f_0 \in M_0(A)$. We extend f_0 to a function $f \in N_0$. First let $f = f_0$ on $A = A_0 \cap B_n$. We extend to $A_1 \cap B_{n-1}$. Let $x \in A_1 \cap B_{n-1}$. From the definition of σ_1, $(x, \sigma_1(x)) \in H$ and $\sigma_1(x) \in A_0$. But then $f\sigma_1(x) \in A$. We define $f(x)$ to be the unique element in $A_1 \cap B_{n-1}$ such that $\sigma_1 f(x) = f\sigma_1(x)$. Assume now we have defined f on $A_{j-1} \cap B_{n-(j-1)}$ and let $x \in A_j \cap B_{n-j}$. Then $(x, \sigma_j(x)) \in H$, $\sigma_j(x) \in A_{j-1} \cap B_{n-(j-1)}$ and $f\sigma_j(x) \in A_{j-1} \cap B_{n-(j-1)}$. Define $f(x)$ to be the unique element in $A_j \cap B_{n-j}$ such that $\sigma_j f(x) = f\sigma_j(x)$.

Therefore $\forall j$, $0 \leq j \leq n$ we have f defined on $A_j \cap B_{n-j}$ and $f\sigma_j(x) = \sigma_j f(x)$ $\forall x \in A_j \cap B_{n-j}$. Recall from Lemma III.7 (iv), $G = (A \cap B_n) \oplus (A_1 \cap B_{n-1}) \oplus \cdots \oplus (A_{n-1} \cap B_1) \oplus (A_n \cap B)$. We extend f linearly to G, i.e. for $x \in G$, $x = \sum_{i=0}^n x_i$ where $x_i \in A_i \cap B_{n-i}$ define $f(x) = \sum_{i=0}^n f(x_i)$.

We next show that $f \in N_0$. Let $(u, v) \in H$, $u = \sum_{i=0}^n u_i$, $u_i \in A_i \cap B_{n-i}$ and $v = \sum_{i=0}^n v_i$, $v_i \in A_i \cap B_{n-i}$. For $i = 1, 2, \ldots, n$, $(u_i, \sigma_i(u_i)) \in H$ and since $(u_0, 0) \in H$, we have $(u, \sigma_1(u_1) + \cdots + \sigma_n(u_n)) \in H$. From $(u, v) \in H$ and $(0, v_n) \in H$ we get $(u, v_0 + v_1 + \cdots + v_{n-1}) \in H$. Thus $(0, \sigma_1(u_1) - v_0 + \sigma_2(u_2) - v_1 + \cdots + \sigma_n(u_n) - v_{n-1})$ is in H so $\sigma_1(u_1) - v_0 + \cdots + \sigma_n(u_n) - v_{n-1} \in B$. But also $\sigma_1(u_1) - v_0 + \cdots + \sigma_n(u_n) - v_{n-1} \in A_0 + \cdots + A_{n-1} = A_{n-1}$, hence $\sigma_1(u_1) - v_0 + \cdots + \sigma_n(u_n) - v_{n-1} = 0$ which in turn gives $v = \sigma_1(u_1) + \cdots + \sigma_n(u_n) + v_n$. Therefore $f(v) = f\sigma_1(u_1) + \cdots + f\sigma_n(u_n) + f(v_n) = \sigma_1 f(u_1) + \cdots + \sigma_n f(u_n) + f(v_n)$ and $f(u) = \sum_{i=0}^n f(u_i)$. But, $(f(u_0), 0) \in H$, $(f(u_i), \sigma_i f(u_i)) \in H$ for $i = 1, 2, \ldots, n$ and $(0, f(v_n)) \in H$ together imply that $(f(u), f(v)) \in H$, consequently $f \in N_0$. Combining this with the previous lemma gives the following.

Theorem III.11. *If* $\exists n$, $n \in Al(G)$ *and* $\forall k < n$ $A \cap B_k = \{0\}$ *then* N_0 *is simple.*

Combining Theorems III.11 and III.5 we have our major result.

Theorem III.12. *Let* $H = G/A \overset{\times}{\sim}_\sigma G/B$, $A \neq \{0\} \neq B$, $A \cap B = \{0\}$. $M_0(G, 2, H)$ *is simple* $\Leftrightarrow \exists n, n = Al(G)$ *and* $\forall k$, $0 \leq k < n$, $A \cap B_k = \{0\}$.

We conclude this section and the paper by showing that for the near-ring $M_0(G, 2, H)$ the concepts of simplicity and 2-semisimplicity coincide. This is identical with the situation for centralizer near-rings $\mathscr{C}(\mathscr{A}; G)$ when $\mathscr{A}$ is a cyclic group of automorphisms.

Theorem III.13. *Let* $H = G/A \overset{\times}{\sim}_\sigma G/B$, $A \cap B = \{0\}$, N_0 *is simple* $\Leftrightarrow N_0$ *is 2-semisimple.*

Proof. If $A = \{0\}$ then as in Theorem II.6, N_0 is the centralizer near-ring $\mathscr{C}(\mathscr{A}; G)$ where $\mathscr{A} = \langle \sigma \rangle$. Consequently this case follows from [2].

Now take $A \neq \{0\}$ and suppose N_0 is not simple. If $\forall n$, $A_n \neq G$ we let l be the least index such that $A_l = A_{l+1}$. Let $I = \{f \in N_0 | f(A_l) = \{0\}$ and $f(G \backslash A_l) \subseteq A_l\}$. Then I is a nilpotent N_0-subgroup. Suppose $\exists a_l \in A_l$, $a_l \neq 0$ such that $(a_l, a_l) \in H$. Define $f: G \to G$ by $f(x) = 0$ if $x \in A_l$ and $f(x) = a_l$ if $x \notin A_l$. Since $A_l = A_{l+1} = \cdots$, $B \subseteq A_l$, consequently, for $(x, y) \in H, x \in A_l \Leftrightarrow y \in A_l$. If $x \notin A_l, (f(x), f(y)) = (a_l, a_l) \in H$ while if $x \in A_l, (f(x), f(y)) = (0, 0) \in H$. Thus $f \in I$ and $I \neq \{0\}$. It remains to show that such an a_l exists. Since l is least with $A_l = A_{l+1} = \cdots$, $\exists b \in B$, $b \neq 0$, $b \in A_l \backslash A_{l-1}$. Further, since $b \in A_l$, $\sigma(b + A) = a_{l-1} + B$, $a_{l-1} \in A_{l-1}$. But then $\sigma(a_{l-1} + A) = a_{l-2} + B$, $a_{l-2} \in A_{l-2}$ and continuing we obtain $\sigma(a_{l-2} + A) = a_{l-3} + B, \ldots, \sigma(a_1 + A) = a_0 + B$, $\sigma(a_0 + A) = \sigma(A) = B$. Adding gives $\sigma(b + a_{l-1} + a_{l-2} + \cdots + a_1 + A) = a_{l-1} + a_{l-2} + \cdots + a_0 + B$. Now let

$a_l = b + a_{l-1} + a_{l-2} + \cdots + a_1 + a_0$. Then $a_l \in (b + a_{l-1} + \cdots + a_1 + A) \cap (a_{l-1} + \cdots + a_0 + B)$ so $(a_l, a_l) \in H$ and $a_l \neq 0$ since $b \notin A_{l-1}$ while $a_{l-1} + a_{l-2} + \cdots + a_0 \in A_{l-1}$.

If $\exists n$, $n = Al(G)$ and N_0 is not simple, then as in the proof of Theorem III.5 we get that $\mathrm{Ann}_{N_0}(A)$ is a nonzero N_0-subgroup of N_0. We note that if $f \in \mathrm{Ann}_{N_0}(N)$ then $f(A_1) \subseteq A_0$, $f(A_2) \subseteq A_1, \ldots$, and $f(G) = f(A_n) \subseteq A_{n-1}$. From this we have the product of any n elements in $\mathrm{Ann}_{N_0}(A)$ is 0, so $\mathrm{Ann}_{N_0}(A)$ is nilpotent and again N_0 is not 2-semisimple.

References

1. P. Fuchs and C. J. Maxson, *Near-fields associated with invariant linear k-relations*, Proc. Amer. Math. Soc. 103 (1988), 729–736.
2. C. J. Maxson, and K. C. Smith, *The centralizer of a group automorphism*, J. Algebra **54** (1978), 27–41.
3. J. D. P. Meldrum, *Near-rings and their links with groups*, Research Notes in Math. **134**, Pitman Publ. Co., London, 1986.
4. G. F. Pilz, *Near-rings*, 2nd Ed., North Holland, Amsterdam, 1983.
5. R. Remak, *Über die Darstellung der endlichen Gruppen als Untergruppen direkter Produkte*, J. Reine Angew. Math. **163** (1930), 1–44.
6. H. Wielandt, *How to single out function near-rings*, Oberwolfach Abstracts, 1972.

Department of Mathematics, Texas A&M University, College Station, Texas 77843