A COMBINATORIAL CHARACTERIZATION OF $S^3 \times S^1$
AMONG CLOSED 4-MANIFOLDS

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Abstract. The topological product $S^3 \times S^1$ is proved to be the unique closed connected 4-manifold of regular genus one. As a consequence, the complex projective plane $\mathbb{C}P^2$ has regular genus two.

1. Crystallizations

Throughout the paper we work in the piecewise-linear (PL) category in the sense of [RS]. The prefix PL will always be omitted. Manifolds are assumed to be connected and compact. As general reference for graph theory see [H, and W]. We shall use the term graph instead of multigraph, hence loops are forbidden but multiple edges are allowed.

An edge-coloration $c$ on a graph $G = (V(G),E(G))$ is a map $c: E(G) \to C$ (where $C$ is a finite nonempty set, called the colour set) such that $c(e) \neq c(f)$ for any two adjacent edges $e, f \in E(G)$. An $(n+1)$-coloured graph is a pair $(G,c)$ where $G$ is a graph, regular of degree $n+1$, and $c: E(G) \to \Delta_n = \{0,1,\ldots,n\}$ is an edge-coloration on $G$. If $\Gamma \subseteq \Delta_n$, then we set $G_{\Gamma} = (V(G),c^{-1}(\Gamma))$. If $\bar{i}$ denotes the set $\Delta_n - \{i\}$, then $(G,c)$ is said to be contracted if $G_{\bar{i}}$ is connected for each $i \in \Delta_n$. If $\Gamma = \{i,j\} \subseteq \Delta_n$ (resp. $\Gamma = \{r,s,t\} \subseteq \Delta_n$), then $g_{ij}$ (resp. $g_{rst}$) represents the number of components of $G_{\Gamma}$. The $n$-dimensional pseudocomplex $K = K(G)$ (see [HW, p. 49], associated with $(G,c)$ is defined as follows: (1) take an $n$-simplex $A^n(v)$ for each vertex $v \in V(G)$ and label its vertices by $\Delta_n$; (2) if $v,w$ are joined in $G$ by an $i$-coloured edge, then identify the $(n-1)$-faces of $A^n(v)$ and of $A^n(w)$ opposite to the vertex labelled by $i$ so that equally labelled vertices coincide. By abuse of language we call simplexes the balls of $K$, as each $h$-ball of $K$ is actually isomorphic to a standard $h$-simplex. For each $\Gamma \subseteq \Delta_n$ with
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\[ \# \Gamma = h \leq n , \] there is a bijection between the set of components of $G_h$ and the set of $(n-h)$-simplexes of $K$ whose vertices are labelled by $\Delta_n - \Gamma$. Note that, if $(G,c)$ is contracted, then $K$ has exactly $n+1$ vertices.

Now let $A$ be a simplex of $K$. Then the **disjoined star** $\text{std}(A,K)$ is defined as the disjoint union of the $n$-simplexes of $K$ containing $A$ with re-identification of the $(n-1)$-faces containing $A$ and of their faces. The **disjoined link** is the subcomplex $\text{lk}_d(A,K) = \{ B \in \text{std}(A,K) | A \cap B = \emptyset \}$. The polyhedron $|K|$ is a closed $n$-manifold iff $\text{lk}_d(A^h,K)$ is a combinatorial $(n-h-1)$-sphere for each $h$-simplex $A^h \in K$. A **crystallization** of a closed $n$-manifold $M$ is a contracted $(n+1)$-coloured graph $(G,c)$ such that $|K(G)|$ is homeomorphic to $M$. Moreover we say that $K(G)$ is a **contracted triangulation** of $M$ and that $(G,c)$ **represents** $M$ and every homeomorphic space. Each closed $n$-manifold can be represented by a crystallization (see [P]). For a general survey on crystallizations see [FGG].

2. **Regular genus**

Following [G2], we state the definition of the regular genus for a closed $n$-manifold. A **2-cell imbedding** (see [W]) $\alpha: |G| \to F$ of an $(n+1)$-coloured graph $(G,c)$ into a closed surface $F$ is said to be regular iff there is a cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ of $\Delta_n$ such that each region of $\alpha$ is bounded by the image of a cycle of $G$ with edges alternatively coloured by $\varepsilon_i, \varepsilon_{i+1}$ ($i$ being an integer mod $n+1$). The **regular genus** $g(G)$ of $G$ is the smallest integer $h$ such that $(G,c)$ regularly imbeds into a closed surface of genus $h$. The **regular genus** $g(M)$ of a closed $n$-manifold $M$ is defined as the nonnegative integer

\[
g(M) = \min\{g(G) | (G,c) \text{ is a crystallization of } M \}.
\]

Given an $(n+1)$-coloured graph $(G,c)$, call $p$ the order of $G$ divided by 2. If $G$ is bipartite (resp. nonbipartite), for each cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ of $\Delta_n$ there exists exactly one regular imbedding $\alpha: |G| \to F_\varepsilon$, where $F_\varepsilon$ is the orientable (resp. nonorientable) closed surface with Euler characteristic

\[ \chi_\varepsilon = \sum_{i \in \mathbb{Z}_{n+1}} q_{\varepsilon_i, \varepsilon_{i+1}} + (1 - n)p \]

(see [G1, Proposition 19]).

All closed $n$-manifolds or regular genus zero are proved to be homeomorphic to the $n$-sphere $S^n$ (see [FG, Main Theorem]). In the present paper, we are interested in closed 4-manifolds of regular genus one. Further results about closed 4-manifolds of regular genus greater than one will appear in a subsequent paper of the author.
3. Main results

Now we state our Main Theorem.

**Main Theorem.** Let $M^4$ be a closed 4-manifold. Then $g(M^4) = 1$ if and only if $M^4$ is homeomorphic to $S^3 \times S^1$.

**Corollary.** If $CP^2$ is the complex projective plane, then $g(CP^2) = 2$.

In order to prove these statements we need three lemmas and some preliminary constructions. Let $(G, c)$ be a crystallization of a closed 4-manifold $M$, $K = K(G)$ the associated contracted triangulation of $M$, and $\{v_i / i \in \Delta_4\}$ the vertex-set of $K$. We may assume that $v_i$ corresponds to $G_i$ for each $i \in \Delta_4$. If $\{i,j\} = \Delta_4 - \{r,s,t\}$, then $K(i,j)$ (resp. $K(r,s,t)$) denotes the subcomplex of $K$ generated by the vertices $v_i$ and $v_j$ (resp. $v_r$, $v_s$ and $v_t$). Obviously the number of edges (resp. triangles) of $K(i,j)$ (resp. $K(r,s,t)$) equals $g_{rst}$ (resp. $g_{ij}$). If $Sd K$ is the first barycentric subdivision of $K$, let $H(i,j)$ be the largest subcomplex of $Sd K$, disjoint from $Sd K(i,j) \cup Sd K(r,s,t)$. Then the polyhedron $H(i,j)$ is a closed 3-manifold $F(i,j)$ which splits $M$ into two complementary 4-manifolds $N(i,j)$ and $N(r,s,t)$ having $F(i,j)$ as common boundary. Furthermore $N(i,j)$, $N(r,s,t)$ are regular neighbourhoods in $M$ of $|Sd K(i,j)|$, $|Sd K(r,s,t)|$ respectively.

**Lemma 1.** Let $(G, c)$ be a crystallization of a closed 4-manifold $M$. For each triple $(r,s,t)$ of distinct elements of $\Delta_4$, we have

$$2g_{rst} = g_{rs} + g_{st} + g_{tr} - p. \tag{1}$$

**Proof.** If $\{i,j,k\} \subseteq \Delta_4$, then $q_h(i,j)$ (resp. $q_h(i,j,k)$) denotes the number of $h$-simplexes of $K$ containing $v_i$ and $v_j$ (resp. $v_i$, $v_j$ and $v_k$) as their vertices. If $\{i,j\} = \Delta_4 - \{r,s,t\}$, then it is easily proved that

$$q_1(i,j) = g_{rst},$$
$$q_2(i,j) = q_2(i,j,t) + q_2(i,j,r) + q_2(i,j,s) = g_{rs} + g_{st} + g_{tr},$$
$$q_3(i,j) = 3p \quad \text{and} \quad q_4(i,j) = 2p.$$

Let now $e$ be an arbitrary edge of $K(i,j)$. Then the Euler characteristic $\chi(e)$ of $\lkd(e, K)$ is given by

$$2 = \chi(e) = q_2(e) - q_3(e) + q_4(e), \quad \text{where} \quad q_h(e) \quad \text{is the number of} \quad h\text{-simplexes of} \quad K \quad \text{containing} \quad e \quad \text{as their face.}$$

Summation over the edges of $K(i,j)$ gives

$$2g_{rst} = 2q_1(i,j) = q_2(i,j) - q_3(i,j) + q_4(i,j) = g_{rs} + g_{st} + g_{tr} - p \quad \text{as claimed.} \quad \Box$$

Now we assume that $(G, c)$ regularly imbeds into the closed orientable surface of genus $g = g(M)$ and of Euler characteristic

$$g_{01} + g_{12} + g_{23} + g_{34} + g_{40} - 3p = 2 - 2g. \tag{2}$$

Each subgraph $G_i$ ($i \in \Delta_4$) regularly imbeds into an orientable closed surface since $G_i$ represents the combinatorial 3-sphere $\lkd(v_i, K)$. Then we can define
the nonnegative integer \( g_i \) \((i \in \Delta_4)\) as follows (see §2)

\[
\begin{align*}
(3) & \quad g_{12} + g_{23} + g_{34} + g_{41} - 2p = 2 - 2g_0, \\
(4) & \quad g_{02} + g_{23} + g_{34} + g_{40} - 2p = 2 - 2g_1, \\
(5) & \quad g_{01} + g_{13} + g_{34} + g_{40} - 2p = 2 - 2g_2, \\
(6) & \quad g_{01} + g_{12} + g_{24} + g_{40} - 2p = 2 - 2g_3, \\
(7) & \quad g_{01} + g_{12} + g_{23} + g_{30} - 2p = 2 - 2g_4.
\end{align*}
\]

By substituting each relation \((k)\), \(3 \leq k \leq 7\), into (2) and by using (1), we get

\[
\begin{align*}
(8) & \quad g_{14} = g_{014} + g - g_0, \\
(9) & \quad g_{02} = g_{012} + g - g_1, \\
(10) & \quad g_{13} = g_{123} + g - g_2, \\
(11) & \quad g_{24} = g_{234} + g - g_3, \\
(12) & \quad g_{03} = g_{034} + g - g_4.
\end{align*}
\]

As a direct consequence, the inequalities \( g \geq g_i \geq g(G_i) \) hold for each color \( i \in \Delta_4 \).

**Lemma 2.** With the above notation, we have

\[
\begin{align*}
(13) & \quad g_{134} = 1 + g - g_0 - g_2, \\
(14) & \quad g_{124} = 1 + g - g_0 - g_3, \\
(15) & \quad g_{024} = 1 + g - g_1 - g_3, \\
(16) & \quad g_{023} = 1 + g - g_1 - g_4, \\
(17) & \quad g_{013} = 1 + g - g_2 - g_4, \\
(18) & \quad g_{012} + g_{014} + g_{034} + g_{123} + g_{234} = 4 + p + g - \sum g_i, \\
(19) & \quad g_{02} + g_{03} + g_{13} + g_{14} + g_{24} = 4 + 6g + p - 2\sum g_i, \\
(20) & \quad \chi(M) = 2 - 2g + \sum g_i.
\end{align*}
\]

**Proof.** For each pair \((i, j) \in \{(0, 2), (0, 3), (1, 3), (1, 4), (2, 4)\}\), we get the formula \((11 + i + j)\) of the statement by simply adding the relations \((3 + i), (3 + j)\) and by using (1). Summation directly gives

\[
\begin{align*}
(21) & \quad g_{134} + g_{124} + g_{024} + g_{023} + g_{013} = 5 + 5g - 2\sum g_i.
\end{align*}
\]

Adding the following relations (see Lemma 1): \(2g_{013} = g_{01} + g_{13} + g_{30} - p\), \(2g_{024} = g_{02} + g_{24} + g_{40} - p\), \(2g_{023} = g_{02} + g_{23} + g_{30} - p\), \(2g_{134} = g_{13} + g_{34} + g_{41} - p\), \(2g_{124} = g_{12} + g_{24} + g_{41} - p\) and making use of (2) and (21), we obtain formula (19).
Substituting formula (1) into (2) easily gives

\[(22) \quad g_{123} + g_{034} - g_{013} + g_{01} - p = 1 - g,\]
\[(23) \quad g_{012} + g_{234} - g_{024} + g_{04} - p = 1 - g,\]
\[(24) \quad g_{014} + g_{234} - g_{124} + g_{12} - p = 1 - g,\]
\[(25) \quad g_{034} + g_{012} - g_{023} + g_{23} - p = 1 - g,\]
\[(26) \quad g_{014} + g_{123} - g_{134} + g_{34} - p = 1 - g.\]

Adding these relations and using (2), (21) gives formula (18). Now call \(q_h\) \((h \in \Delta_4)\) the number of \(h\)-simplexes of \(K = K(G)\). By construction, we have

\[q_0 = 5, \quad q_1 = \sum_{r,s,t} g_{rst}, \quad q_2 = \sum_{i,j} g_{ij}, \quad q_3 = 5p \quad \text{and} \quad q_4 = 2p.\]

Then the Euler characteristic \(\chi(M)\) of \(M = |K|\) is given by

\[
\chi(M) = \sum_h (-1)^h q_h = 5 - \sum_r g_{rst} + \sum_i g_{ij} - 3p
\]
\[
= 5 - \left(4 + p + g - \sum_i g_i + 5 + 5g - 2\sum_i g_i\right)
\]
\[
+ \left(3p + 2 - 2g + 4 + 6g + p - 2\sum_i g_i\right) - 3p
\]
\[
= 2 - 2g + \sum_i g_i. \quad \square
\]

**Lemma 3.** If \(g = 1\), then we have \(g_i = 0\) for each \(i \in \Delta_4\), \(\beta_0 = \beta_1 = \beta_3 = \beta_4 = 1\), and \(\beta_2 = 0\), where \(\beta_k\) is the \(k\)th Betti number of \(M\).

**Proof.** If \(g = 1\), then \(M\) is orientable since the regular genus of a closed nonorientable 4-manifold is an even positive number (see [G2, Corollary 8]). By Lemma 2, it suffices to prove that the sum \(R = g_{013} + g_{023} + g_{024} + g_{124} + g_{134}\) equals 10. By (21), Lemma 2 and \(g_i \leq 1\) \((i \in \Delta_4)\), the inequalities \(6 \leq R = 10 - 2\sum_i g_i \leq 10\) hold, whence \(R \in \{6, 8, 10\}\). Now we show that the cases \(R = 6\) or \(R = 8\) give a contradiction.

(1) If \(R = 6\), then we must have that one of the terms of \(R\) is 2 and the other four terms are 1 since \(g_{rst} \geq 1\).

Now we first suppose \(g_{134} = 2\), so that \(g_{013} = g_{023} = g_{024} = g_{124} = 1, g_0 = g_1 = g_2 = 0\) and \(g_3 = g_4 = 1\) by Lemma 2. Since \(g_{124} = 1\), \(K(0, 3)\) consists of exactly one edge, hence \(N(0, 3)\) is a 4-ball. Furthermore \(K(1, 4)\) and \(K(2, 4)\) are also formed by one edge each since \(g_{023} = g_{013} = 1\). Thus all triangles of \(K(1, 2, 4)\) have two edges in common. Since \(g_{034} - g_{03} = g_4 - g = 0\), \(K(1, 2, 4)\) consists of as many triangles as there are edges in \(K(1, 2)\). Therefore \(K(1, 2, 4)\) is a cone over the 1-pseudocomplex \(K(1, 2)\). Then the polyhedron \(|K(1, 2, 4)|\) is contractible, \(N(1, 2, 4)\) is a 4-ball and \(M\) is homeomorphic to \(S^4\). This contradicts the fact that \(g(M)\) is not zero and also proves that
$g_{024} = 2$ is impossible. In the other three possible cases, i.e., $g_{013} = 2$, $g_{124} = 2$, or $g_{023} = 2$, we repeat the above arguments replacing the pair $(K(0,3), K(1,2,4))$ with $(K(1,4), K(0,2,3))$, $(K(0,2), K(1,3,4))$, and $(K(2,4), K(0,1,3))$, respectively.

$(//)$ If $R = 8$, then $\sum_i g_i = 1$ implies $\chi(M) = 1$ (see (20)). Since at least one of the $g_{ijk}$'s in $R$ equals 1, the fundamental group $\prod_1(M)$ of $M$ is null (see [G3]). Thus we have $\beta_0 = \beta_4 = 1$, $\beta_1 = \beta_3 = 0$, hence $\chi(M) = 2 + \beta_2 \geq 2$, which is a contradiction.

Now the result $R = 10$ (hence $\sum_i g_i = 0$) directly implies $\chi(M) = 0$ and $g_i = 0$ for each $i \in \Delta_4$. Furthermore the equalities $\chi(M) = 0$ and $g_{013} = 2$ give $\prod_1(M) \simeq \mathbb{Z}$ (compare again [G3]), $\beta_0 = \beta_1 = \beta_3 = \beta_4 = 1$, and $\beta_2 = 0$. This concludes the proof. $\clubsuit$

4. Proof of main theorem

If $M^4$ is homeomorphic to $S^3 \times S^1$, then $g(M) = 1$ as proved in [FG, Corollary 1]. Now we prove the converse implication. Since $g_{024} = 2$ (by Lemma 3 and (15)), $K(1,3)$ consists of exactly two edges, hence $N(1,3)$ is homeomorphic to $S^1 \times B^3$, $B^3$ being a 3-ball. Since $g_{134} = g_{013} = 2$, $K(0,2)$, and $K(2,4)$ are also formed by exactly two edges each. Furthermore $K(0,2,4)$ has one more triangle than there are edges in $K(0,4)$ since $g_{13} = 0$ (by Lemma 3 and (10)).

Call $A_1$, $A_2$ the two triangles of $K(0,2,4)$ which have a common edge $e \in K(0,4)$ as their face. Then $K(0,2,4)$ collapses to the subcomplex $\overline{K} = K(0,2) \cup K(2,4) \cup \{A_1, A_2\}$. If $\partial A_1 \neq \partial A_2$, it is very easy to see that $\overline{K}$ collapses to a 1-sphere $S^1$, hence $N(0,2,4)$ is homeomorphic to $S^1 \times B^3$. If $\partial A_1 = \partial A_2$, then $\overline{K}$ is homotopy equivalent to $S^1 \vee S_1 \vee S_2$ (where $X \vee Y$ means the one-point union of $X$ and $Y$), $S^1_i$, $i = 1,2$, (resp. $S^2_2$) being a 1-sphere (resp. 2-sphere). In the latter case, we would have $\beta_2 \geq 1$ which contradicts $\beta_2 = 0$ (see Lemma 3). Therefore we have $N(1,3) \simeq N(0,2,4) \simeq S^1 \times B^3$, $\partial N(1,3) = \partial N(0,2,4) \simeq S^1 \times S^2$, whence $M = N(1,3) \cup N(0,2,4) \simeq S^3 \times S^1$ by Theorem 2 of [M].

Proof of Corollary. The genus of $CP^2$ is proved to be $\leq 2$ (see [FGG, p. 134]) by constructing a simple crystallization of $CP^2$ with regular genus two. Thus the statement follows from our Main Theorem. $\square$

Note. It is easily proved that the regular genus is subadditive, by direct construction. Thus the connected sum $(S^3 \times S^1) \# (S^3 \times S^1)$ also has regular genus two from our Main Theorem.
REFERENCES


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