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THE VARIETY OF PAIRS OF MATRICES WITH $\operatorname{rank}(AB - BA) \leq 1$

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ABSTRACT. We will show that the variety of pairs of $n \times n$ matrices over an algebraically closed field with rank one commutator consists of $n-1$ irreducible components each of dimension $n^2 + 2n - 1$.

Let F be an algebraically closed field, $M_n(F)$ the algebra of $n \times n$ matrices over F and

$$M_n^{(k)}(F) = \{(A, B) \in M_n(F) \times M_n(F) | \operatorname{rank}(AB - BA) \leq k\}.$$

It is well known [6 or 1] that the variety $M_n^{(0)}(F)$ is irreducible. Guralnick [2] showed that the variety $M_n^{(1)}(F)$ is not irreducible, while Hulek [3] showed that $M_n^{(k)}(C)$ is irreducible for $k \geq 2$. In this note, we will show that $M_n^{(1)}(F)$ is the union of $n - 1$ irreducible components of dimension $n^2 + 2n - 1$. First we introduce some notation. Set

$$\begin{aligned}
[A, B] &= AB - BA \\
H &= \{A \in M_n(F) \mid A \text{ is diagonal}\} \\
K &= \{A \in M_n(F) \mid A \text{ has distinct eigenvalues}\} \\
K' &= H \cap K \\
U &= \{(A, B) \in M_n^{(1)}(F) \mid A \text{ is nonderogatory}\} \\
V &= \{(A, B) \in M_n^{(1)}(F) \mid A \in K\}.
\end{aligned}$$

K is an open set in the Zariski topology of $M_n(F)$ and hence V is an open set of $M_n^{(1)}(F)$. It has also been shown in [2] that U is open in $M_n^{(1)}(F)$.

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We will also refer to the following sets:

$$N_i = \left\{A \in M_n(F) \mid A = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \text{ where } C \in M_{i\times(n-i)}(F)\right\} \quad \text{for } i = 0, \dots, n-1.$$

$$N_i^{(1)} = \{A \in N_i \mid \operatorname{rank} A \le 1\} \quad \text{for } i = 0, \dots, n-1.$$

$$W_i = \left\{(A, B) \in V \mid C, CA, \dots, CA^i \text{ are linearly dependent}\right\} \bigcap \left\{(A, B) \in V \mid C, AC, \dots, A^{n-i}C \text{ are linearly dependent}\right\}$$
$$\text{for } i = 1, \dots, n-1 \text{ where } C = [A, B].$$

If W is a set, let $\overline{W}$ denote the closure of W in the Zariski topology of the underlying space.

The first lemma summarizes some known results.

Lemma 1.

1. *If $(A, B) \in M_n^{(1)}(F)$ then A and B have the P-property i.e. they can be put in upper triangular form simultaneously.*
2. *U is dense in $M_n^{(1)}(F)$.*
3. *Let $p(x) \in F[x]$. Then $(A, B) \in M_n^{(1)}(F)$ (respectively $U, V, \overline{U}, \overline{V}$ or W_i) iff $(A, B - p(A)) \in M_n^{(1)}(F)$ (respectively $U, V, \overline{U}, \overline{V}$ or W_i).*
4. *Let $P \in GL_n(F)$. Then $(A, B) \in M_n^{(1)}(F)$ (respectively $U, V, \overline{U}, \overline{V}$ or W_i) iff $(PAP^{-1}, PBP^{-1}) \in M_n^{(1)}(F)$ (respectively $U, V, \overline{U}, \overline{V}$ or W_i).*
5. *If*
$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}$$
is nonderogatory, where $A_1 \in M_i(F)$ and $A_4 \in M_{n-i}(F)$, then A_1 and A_4 are also nonderogatory.
6. *$N_i^{(1)}$ is irreducible and* $\dim N_i^{(1)} = n - 1$ *for $i = 1, \dots, n-1$.*
7. *If $A \in M_i(F)$ and $B \in M_{(n-i)}(F)$ have no common eigenvalue and $C \in M_{i\times(n-i)}(F)$, then there exists a unique $X \in M_{i\times(n-i)}(F)$ such that $AX - XB = C$.*

Proof.

1. See Theorem 1 in [2 or 4].
2. See Lemma 3 in [2].
3. For a fixed $p \in F[x]$, the map $(A, B) \mapsto (A, B - p(A))$ is an isomorphism of $M_n^{(1)}(F)$ which leaves $U, V, \overline{U}, \overline{V}$ and W_i invariant.
4. For a fixed $P \in GL_n(F)$, the map $(A, B) \mapsto (PAP^{-1}, PBP^{-1})$ is an isomorphism of $M_n^{(1)}(F)$ which leaves $U, V, \overline{U}, \overline{V}$ and W_i invariant.
5. If $m_{A_1}(x)$ and $m_{A_4}(x)$ are minimal polynomials of A_1 and A_4, then $(m_{A_1}m_{A_4})(A) = m_{A_1}(A)m_{A_4}(A) = 0$. Hence $\deg m_{A_1} = i$ and $\deg m_{A_4} = n - i$.

6. Clearly $N_i^{(1)}$ and $R = \{C \in M_{i\times(n-i)} \mid \operatorname{rank} C \leq 1\}$ are isomorphic as varieties. Define

$$\pi\colon M_{i\times 1}(F) \times M_{1\times(n-i)}(F) \to R$$
$$(v, w) \mapsto vw.$$

The mapping π is regular and onto. Since $M_{i\times 1}(F) \times M_{1\times(n-i)}(F)$ is irreducible, this implies that R is irreducible. Furthermore if $C \neq 0$, then $\dim\ \pi^{-1}(C) = 1$. Hence $\dim\ N_i^{(1)} = \dim\ R = \dim\ M_{i\times 1}(F) \times M_{1\times(n-i)}(F) - 1 = i + (n-i) - 1 = n - 1$.
7. See Theorems 2 and 3 on p. 422 in [5]. □

Our first goal is to prove that V is dense in $M_n^{(1)}(F)$. The following lemma establishes this for a subset of $M_n^{(1)}(F)$.

Lemma 2. *Let*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_4 \end{pmatrix}$$

such that $(A, B) \in U$; $A_1, B_1 \in M_i(F))$; $A_2, B_2 \in M_{i\times(n-i)}(F)$; $A_4, B_4 \in M_{n-i}(F)$; $[A_1, B_1] = 0$ *and* $[A_4, B_4] = 0$. *If* A_1 *and* A_4 *have no common eigenvalue, then* $(A, B) \in \overline{V}$.

Proof. Since A_1 and A_4 have no common eigenvalue there exists

$$P_2 \in M_{i\times(n-i)}(F)$$

such that $A_1P_2 - P_2A_4 = A_2$ by part 7 of Lemma 1. If we set

$$P = \begin{pmatrix} I_i & P_2 \\ 0 & I_{n-i} \end{pmatrix},$$

then

$$PAP^{-1} = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix} \quad \text{and} \quad PBP^{-1} = \begin{pmatrix} B_1 & B_2 + P_2B_4 - B_1P_2 \\ 0 & B_4 \end{pmatrix}.$$

Hence using part 4 of Lemma 1 we may assume that $A_2 = 0$.

Since $[A, \operatorname{diag}(B_1, B_4)] = 0$ and A is nonderogatory, there exists a polynomial $p(x) \in F[x]$ with $p(A) = \operatorname{diag}(B_1, B_4)$. Part 3 of Lemma 1 implies that it is enough to consider (A, B) with $A_2 = 0, B_1 = 0$ and $B_4 = 0$. Define

$$W = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} \mid X_1 \text{ and } X_4 \text{ have no common eigenvalue} \right\},$$

where $X_1 \in M_i(F)$ and $X_4 \in M_{n-i}(F)$, and define

$$W' = \{X \in W \mid X \text{ has distinct eigenvalues}\}.$$

Note that W' is dense in W and $A \in W$. For every $X \in W$ there exists a unique $S_X \in M_{i\times(n-i)}(F)$ by part 7 of Lemma 1 such that

$$X_1S_X - S_XX_4 = B_2(X_4 - A_4) - (X_1 - A_1)B_2.$$

Furthermore the entries of S_X are regular functions in the coordinates of X. Therefore we have a regular mapping

$$\begin{aligned} \phi\colon W &\to M_n^{(1)}(F) \\ X &\mapsto \left(X, B + \begin{pmatrix} 0 & S_X \\ 0 & 0 \end{pmatrix}\right). \end{aligned}$$

Clearly $(A, B) \in \phi(W) \subseteq \overline{\phi(W')} \subseteq \overline{V}$. $\square$

We are now able to prove:

Proposition 1. *V is dense in $M_n^1(F)$.*

Proof. Since U is dense in $M_n^{(1)}(F)$ by Lemma 1 it suffices to show that V is dense in U. Let $(A, B) \in U$. Using parts 1 and 4 of Lemma 1 we may assume A and B are upper triangular. Therefore

$$[A, B] = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \in N_i^{(1)} \quad \text{for some } i = 0, \dots, n-1,$$

and

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_4 \end{pmatrix}$$

where $A_1, B_1 \in M_i(F)$ and $A_4, B_4 \in M_{n-i}(F)$ are upper triangular and A_2, $B_2 \in M_{i\times(n-i)}(F)$. Since $[A_1, B_1] = 0$ and A_1 is nonderogatory, $B_1 = p(A_1)$ for some $p(x) \in F[x]$. Considering $(A, B - p(A))$, we may also assume $B_1 = 0$ by part 3 of Lemma 2. We proceed by induction on i. The case $i = 0$ was proved in [6]. So assume $i > 0$ and let $\alpha_1, \dots, \alpha_i$ be the eigenvalues of A_1. Define $j(A) =$ order of $\{k \mid \alpha_k$ is not an eigenvalue of $A_4\}$. If $j(A) = i$, then A_1 and A_4 have no common eigenvalue. Therefore we can apply Lemma 2 and conclude that $(A, B) \in \overline{V}$.

Now use reverse induction on $j(A)$. If $j(A) < i$, then we can assume that α_i is an eigenvalue for A_4. Let r_k denote the kth row of C. If $r_i = 0$, then $[A, B] \in N_{i-1}^{(1)}$ and so are done by induction on i. If $r_i \neq 0$, then there exists $P_1 \in GL_i(F)$, upper triangular, such that

$$C' = P_1 C = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ r_i \end{pmatrix}.$$

After conjugating by

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & I_{n-i} \end{pmatrix}$$

we may assume

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & B_2 \\ 0 & B_4 \end{pmatrix} \quad \text{and} \quad [A, B] = \begin{pmatrix} 0 & C' \\ 0 & 0 \end{pmatrix}.$$

Let

$$L = \{(A + tE_{ii}, B) \mid t \in F\}.$$

As L is a line in $M_n^{(1)}(F)$ it is irreducible and since $B_1 = 0$, L is contained in $M_n^{(1)}(F)$. Except for finitely many $t \in F$, $j(A + tE_{ii}) > j(A)$. Hence, by induction on $j(A)$, $L \cap \overline{V}$ is cofinite in the line L. Thus $(A, B) \in L \subseteq \overline{V}$, as desired. This finishes the proof by induction on i. □

If $(A, B) \in V$, then there exists $P \in GL_n(F)$ such that $A_0 = PAP^{-1} \in K'$ and $B_0 = PBP^{-1}$ is upper triangular by Lemma 1. This implies $[A_0, B_0] \in N_i^{(1)}$ for some $i = 1, \dots, n-1$. An explicit computation of $[A_0, B_0]$ shows that in this case $B_0 = p(A_0) + N$ for some $N \in N_i$ and some $p(x) \in F[x]$. On the other hand, for a given $A \in K'$ and $C \in N_i^{(1)}$ there exists a unique $N_{A,C} \in N_i$ with $[A, N_{A,C}] = C$. Furthermore, the entries of $N_{A,C}$ are regular functions in the coordinates of A and C. Thus

$$\phi_i \colon G_i = GL_n(F) \times K' \times H \times N_i^{(1)} \to V$$
$$(P, A, B, C) \mapsto (PAP^{-1}, P(B + N_{A,C})P^{-1})$$

is a regular mapping for all $i = 1, \dots, n-1$. $V_i = \phi(G_i)$ is irreducible since it is the image of the irreducible variety G_i under the regular mapping ϕ_i. The argument above also shows that $V = V_1 \cup \dots \cup V_{n-1}$.

To compute dim V_i we note that $\{P \in M_n(F) \mid AP = PA\}$ is isomorphic to each of the $n!$ components of $\phi^{-1}((A, 0))$. Since A has distinct eigenvalues this shows that dim $\phi^{-1}((A, 0)) = n$. Hence $\dim V_i \geq \dim G_i - n = n^2 + 2n - 1$. We also have a regular mapping

$$\pi \colon V \to K \qquad (A, B) \mapsto A.$$

If $A \in K'$, then $\pi^{-1}(A) \supseteq \{(A, B) \mid B = D + N_{A,C}, D \in H, C \in N_i^{(1)}\}$. Hence $\dim \pi^{-1}(A) \geq \dim H + \dim N_i^{(1)} = 2n - 1$, which implies $\dim V_i = n^2 + 2n - 1$.

We now state the main result.

Theorem 1.

1. $V_i = W_i$.
2. *The irreducible components of* $M_n^{(1)}(F)$ *are* $\overline{V}_1, \dots, \overline{V_{n-1}}$. *In particular* $M_2^{(1)}(F)$ *is irreducible and if* $n > 2$, *then* $M_n^{(1)}(F)$ *is not irreducible.*
3. $\dim \overline{V}_i = \dim V_i = n^2 + 2n - 1$ *for all* $i = 1, \dots, n-1$.

Proof.

1. If $(A, B) \in V_i$, we can assume $[A, B] \in N_i^{(1)}$. Hence $(A, B) \in W_i$. If $(A, B) \in W_i$, we can assume $A \in K'$, $B = p(A) + N$ where $p(x) \in F[x]$ and $N \subset N_j$ for some $j = 1, \dots, n-1$. Hence $[A, B] = C \in N_j^{(1)}$ for some $i = 1, \dots, n-1$. If $j < i$ then the condition $C, CA, \dots, CA^{n-i}$ linearly dependent implies that C, and hence N,

has at most $n - i$ nonzero columns. However, this implies that there exists $T \in GL_n(F)$ with $TAT^{-1} \in K'$ and $TNT^{-1} \in N_i$. Therefore $(A, B) \in V_i$. Similarly, if $j > i$, then $(A, B) \in V_i$.

2. It was shown above that V_i is irreducible. Hence $\overline{V}_i$ is irreducible. From part 1 we know that if, $(A, B) \in \overline{V}_i$ then $[A, B] = C, CA, \dots, CA^i$ are linearly dependent and $C, AC, \dots, A^{n-i} \subset$ are linearly dependent. If we let $A = \operatorname{diag}(a_1, \dots, a_n) \in K'$ and $B_i = (b^i_{lk}) \in N_i$, where
$$b^i_{lk} = \begin{cases} (a_l - a_k)^{-1} & \text{if } 1 \le l \le i < k \le n \\ 0 & \text{otherwise}, \end{cases}$$
then $(A, B_i) \in \overline{V}_i$ but $(A, B_i) \in \overline{V}_j^c$ for $j \neq i$. Together with the fact $M_n^{(1)} = \overline{V}_1 \cup \cdots \cup \overline{V_{n-1}}$, this shows that the $\overline{V}_i'$s are the irreducible components of $M_n^{(1)}(F)$.

3. This was shown above. As a consequence the formula for the dimension of $M_n^{(k)}(F)$ given in [3] is also valid in the case $k = 1$. □

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