A THEOREM ON FUNCTION SPACES

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Abstract. Let X and Y be normal and first countable spaces, such that \( C_p(X) \) and \( C_p(Y) \) are linearly homeomorphic. Suppose \( X^{(\alpha)} \) is countably compact for some \( \alpha < \omega_1 \). We prove that if \( \alpha = 1 \) then \( Y^{(\alpha)} \) is also countably compact. The first countability condition in this result is essential. We also present examples that if \( \alpha \) is not a prime component, then \( Y^{(\alpha)} \) need not to be countably compact.

0. Introduction

Let X and Y be Tychonov spaces. By \( C(X) \) we denote the set of all realvalued continuous functions on X. We endow \( C(X) \) with a topological vectorspace-structure by considering it to be a subspace of \( \mathbb{R}^X \). With this topology we denote \( C(X) \) by \( C_p(X) \).

In [1] Arhangelskii proved that if \( C_p(X) \) is linearly homeomorphic to \( C_p(Y) \), and X is compact, then Y is compact. In addition, if X is pseudocompact then Y is pseudocompact. This means in particular that if X and Y are normal then X is countably compact if and only if Y is countably compact. In this note we prove that if X and Y are both normal and first countable such that \( C_p(X) \) is linearly homeomorphic to \( C_p(Y) \), then \( X^{(1)} \) is countably compact if and only if \( Y^{(1)} \) is countably compact (\( X^{(1)} \) is the set of accumulation points of X). Our technique is inspired by Arhangel'skii [1] and Baars, de Groot, van Mill and Pelant [3]. We give two examples showing that our result is "best possible". There exist a first countable normal space X and a normal space Y such that \( C_p(X) \) and \( C_p(Y) \) are linearly homeomorphic but \( X^{(1)} \) is not countably compact and \( Y^{(1)} \) is countably compact. In addition, there exist two metric spaces X and Y such that \( C_p(X) \) and \( C_p(Y) \) are linearly homeomorphic but \( X^{(2)} \) is compact while \( Y^{(2)} \) is not compact (\( X^{(2)} \) is the second derivative of X).

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1. Preliminaries

In this section we give some results from Baars and de Groot [2], and results and definitions from Arhangelskii [1], which we use in section 2.

Let $X$ be a topological space and $A$ a subset of $X$. Let $Y = Y_{X,A}$ be the quotientspace obtained from $X$ by identifying $A$ to one point, say $\infty$. Let $C_{p,A}(X)$ be the subspace of $C_p(X)$ consisting of those functions which vanish on $A$, and let $C_{p,0}(Y)$ be the subspace of $C_p(Y)$ consisting of those functions which are zero at $\infty$.

If two linear spaces $X$ and $Y$ are linearly homeomorphic then we denote that by $X \sim Y$.

1.1 Lemma [2]. Let $X$ be a space and $A$ a subset of $X$. Then $C_{p,A}(X) \sim C_{p,0}(Y)$.

1.2 Lemma. Let $\alpha$ be an ordinal. Then $C_{p,0}([1,\alpha]) \sim C'_{p}([1,\alpha])$.

1.3 Theorem. Let $\omega \leq \alpha$, $\beta < \omega_1$. Then $C_{p}([1,\alpha]) \sim C_{p}([1,\beta])$ iff $\alpha \leq \beta < \alpha^\omega$.

1.4 Proposition. ([1] Arhangelskii). Let $X$ and $Y$ be Tychonov spaces and $\phi: C(X) \rightarrow C(Y)$ a linear mapping. For every $y \in Y$, the support of $y$ in $X$ is defined to be the set $\text{supp}(y)$ of all $x \in X$ satisfying the condition that for every neighborhood $U$ of $x$, there is an $f \in C(X)$ such that $f(X \setminus U) = \{0\}$ and $\phi(f)(y) \neq 0$. For a subset $A$ of $Y$, we denote $\bigcup_{y \in A} \text{supp}(y)$ by $\text{supp} A$. Furthermore $\phi$ is said to be effective if for every $f$, $g \in C(X)$ and $y \in Y$, such that $f$ and $g$ coincide on a neighborhood of $\text{supp}(y)$, $\phi(f)(y) = \phi(g)(y)$.

A subset $A$ of $X$ is said to be bounded if for every $f \in C(X)$, $f(A)$ is bounded in $\mathbb{R}$.

1.5 Proposition. ([1] Arhangelskii). Let $X$ and $Y$ be Tychonov spaces and $\phi: C_p(X) \rightarrow C_p(Y)$ a linear homeomorphism. Then

(a) $\phi$ is effective,
(b) if $A$ is a bounded subset of $Y$, then $\text{supp} A$ is bounded in $X$.

For details about ordinals we refer to [5] and [6].

2. Function spaces

In this section we prove the results, announced in the Introduction.

2.1 Lemma. Let $X$ and $Y$ be Tychonov spaces and $\phi: C_p(X) \to C_p(Y)$ a homeomorphism. Suppose that $(f_n)_{n \in \mathbb{N}}$ is a sequence in $C_p(X)$ such that $f_n$ converges pointwise to a discontinuous function $f \in \mathbb{R}^X$. Suppose $g: Y \to \mathbb{R}$ is an accumulation point of the set $\{\phi(f_n) | n \in \mathbb{N}\}$. Then $g$ is not continuous.

Proof. Since $\{f_n | n \in \mathbb{N}\}$ is closed and discrete in $C_p(X)$ we have $\{\phi(f_n) | n \in \mathbb{N}\}$ is closed and discrete in $C_p(Y)$. \hfill \Box

2.2 Theorem. Let $X$ and $Y$ be topological spaces which are both normal and first countable and let $C_p(X)$ and $C_p(Y)$ be linearly homeomorphic. Then $X^{(1)}$ is countably compact if and only if $Y^{(1)}$ is countably compact.

Proof. Suppose $X^{(1)}$ is not countably compact and $Y^{(1)}$ is countably compact. Since $X^{(1)}$ is not sequentially compact, there exists a closed discrete set $F = \{x_n | n \in \mathbb{N}\}$ in $X^{(1)}$. For every $n \in \mathbb{N}$ let $\{U^n_j | j \in \mathbb{N}\}$ be a decreasing open base at $x_n$ and $f^n_j$ a Urysohn function such that $f^n_j(x_n) = 1$ and $f^n_j(X \setminus U^n_j) = 0$. Then $f^n_j \to \chi_{x_n}$ pointwise, where $\chi_{x_n}$ is the characteristic function of $x_n$. Notice that $\chi_{x_n}$ is discontinuous. Furthermore let $\phi: C_p(X) \to C_p(Y)$ be a linear homeomorphism and let $g^n_j = \phi(f^n_j)$.

Claim. For every $y \in Y$ and $n \in \mathbb{N}$, the set $\{g^n_j(y) | j \in \mathbb{N}\}$ is bounded in $\mathbb{R}$.

Suppose not. Then there are $y \in Y$ and $n \in \mathbb{N}$, such that without loss of generality for every $k \in \mathbb{N}$ there is $j_k \in \mathbb{N}$, with $g^n_{j_k}(y) \geq 2^k$. The function $f = \sum_{k=1}^{\infty} 2^{-k} f^n_{j_k} \in C_p(X)$, so $\phi(f) = \sum_{k=1}^{\infty} 2^{-k} g^n_{j_k} \in C_p(Y)$. But then we have a contradiction since $\phi(f)(y) = \sum_{k=1}^{\infty} 2^{-k} g^n_{j_k}(y) = \infty$.

For every $y \in Y$, let $A_y$ be compact in $\mathbb{R}$ such that $\{g^n_j(y) | j \in \mathbb{N}\} \subseteq A_y$. Then $\prod_{y \in Y} A_y$ is a compact subset of $\mathbb{R}^Y$. Since $\{g^n_j | j \in \mathbb{N}\}$ has an accumulation point $\sigma_n$. By Lemma 2.1, $\sigma_n$ is discontinuous, say at $y_n$. Notice that $y_n \in Y^{(1)}$. Since $Y^{(1)}$ is sequentially compact, without loss of generality we may assume that there is $y \in Y$ such that $y_n \to y$. Let $\{V_n | n \in \mathbb{N}\}$ be a decreasing open base at $y$. Without loss of generality $y_n \in V_n$.

Since $Y$ is first countable, for every $n \in \mathbb{N}$ there is a sequence $(y^n_k)_k$ in $V_n$ such that $y^n_k \to y_n$ and

\[ \sigma_n(y^n_k) \to \sigma_n(y_n). \]

Let $K = \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \{y_n, y^n_k\} \cup \{y\}$. Then $K$ is compact. Indeed, let $\mathcal{U}$ be an
open cover of $K$. There is $V \in \mathcal{V}$ with $y \in V$. There is $n_0 \in \mathbb{N}$ such that $y \in V_{n_0} \subset V$. Then $\bigcup_{n \geq n_0} \bigcup_{k \in \mathbb{N}} \{y_n, y_k^n\} \cup \{y\} \subset V$. Since $\bigcup_{n < n_0} \bigcup_{k \in \mathbb{N}} \{y_n, y_k^n\}$ is compact, we are done.

Since $K$ is compact, it is bounded in $Y$. So by Proposition 1.4, $\text{supp} K$ is bounded in $X$. Since $F$ is closed and discrete and $X$ is normal, $F$ is not bounded. This implies that there is $n \in \mathbb{N}$ such that $x_n \notin \text{supp} K$. Since $X$ is regular there is $j_0 \in \mathbb{N}$ and a neighborhood $V$ of $\text{supp} K$ such that $U_{j_0}^{n_0} \cap V = \emptyset$. So for every $z \in K$ and $j \geq j_0$, $f_j^n$ and the zero function on $X$ are equal on $V$, which is a neighborhood of $\text{supp}(z)$. Since $\phi$ is linear and effective, this implies that $g_j^n(z) = 0$ for every $j \geq j_0$ and $z \in K$. But then $\sigma_n(y_k^n) = 0$ and $\sigma_n(y_n) = 0$, which gives a contradiction with ($\ast$). \(\square\)

By $X \oplus Y$ or $\oplus_{i=1}^{\infty} X_i$, we denote the topological sum of the topological spaces $X$ and $Y$ or $X_i (i \in \mathbb{N})$, respectively.

2.3 Example. In this example we show that the first countability condition in Theorem 2.2 is essential.

Let $X = \bigoplus_{i=1}^{\infty} [1, \omega]$. Let $A = X^{(1)}$ and $Y = Y_{X \times A}$ the quotient space obtained from $X$ by identifying $A$ to single point, say $\infty$. Then $X$ is clearly first countable and normal, and $Y$ is normal but not first countable. By Lemma 1.1 we have $C_p(A(X)) \sim C_p(A(Y))$. Furthermore we have

$$C_p(A(X)) \sim \prod_{i=1}^{\infty} C_p([1, \omega])_i$$

$$\sim \prod_{i=1}^{\infty} C_p([1, \omega]) \quad \text{(Lemma 1.2a)}$$

$$\sim C_p(X).$$

Notice that for every Tychonoff space $Z$ and for every $z \in Z$, $C_p(Z) \sim C_p([1, \omega]) \times \mathbb{R}$, where $C_p([1, \omega])$ consists of those functions in $C_p(Z)$ which vanish at $z$. So by Lemma 1.2, $C_p([1, \omega]) \sim C_p([1, \omega]) \times \mathbb{R}$. This implies $C_p(X) \sim C_p(X) \times \mathbb{R}$. So

$$C_p(X) \sim C_p(X) \times \mathbb{R}$$

$$\sim C_p(A(X)) \times \mathbb{R}$$

$$\sim C_p(A(Y)) \times \mathbb{R}$$

$$\sim C_p(Y).$$

However $X^{(1)} = A$ is not countably compact, and $Y^{(1)} = \{\infty\}$ is countably compact.

From Theorem 2.2 and the result in [1] for normal spaces, that if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic and $X$ is countably compact, then $Y$ is countably compact, one could conjecture the following: Let $\alpha$ be an arbitrary ordinal. If $X$ and $Y$ are both normal and first countable spaces such that
$C_p(X)$ and $C_p(Y)$ are linearly homeomorphic and $X^{(\alpha)}$ is countably compact, then $Y^{(\alpha)}$ is countably compact.

In the next example we show that if $\alpha$ is not a prime component, then the conjecture is false.

2.4 Example. Let $\alpha < \omega_1$ be an ordinal which is not a prime component. Observe that in this situation $1 \leq \alpha' < \alpha$.

Let $X = \bigoplus_{i=1}^{\infty} [1, \omega^\alpha]$ and $Y = \bigoplus_{i=1}^{\infty} [1, \omega^\alpha]$. By Theorem 1.3, $C_p[1, \omega^\alpha] \sim C_p[1, \omega^\alpha]$, so that $C_p(X) \sim C_p(Y)$. But $Y^{(\alpha)} \approx \mathbb{N}$ (see [2] or [6] p. 155) which is not countably compact, and $X^{(\alpha)} = \emptyset$ which is countably compact.

Questions. (1) Is the above conjecture true for prime components?
(2) Does Theorem 2.2 still hold if normal is replaced by Tychonov?

REFERENCES


