PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 4, April 1989

ON KODAIRA VANISHING FOR SINGULAR VARIETIES

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(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. If X is a complex projective variety with an ample line bundle $\mathscr{L}$, we show that $H^i(X,\mathscr{L}^{-1}) = 0$ for any $i < \operatorname{codim}[\operatorname{Sing}(X)]$, provided that X satisfies Serre's condition S_{i+1}. We also give examples to show that these results are sharp. Finally, we prove a vanishing theorem (for H^1) for seminormal varieties

INTRODUCTION

Consider a complex projective variety X, together with an ample line bundle $\mathscr{L}$. Kodaira [4] has proved that if X is nonsingular, then $H^i(X,\mathscr{L}^{-1}) = 0$ for all $i < \dim(X)$. To what extent is this true when X is singular? If one is only concerned with the depth and the dimension of the singularities of X, then there is a simple answer: H^i vanishes provided that $i < \operatorname{codim}[\operatorname{Sing}(X)]$ and X satisfies property S_{i+1} of Serre. If X is Cohen-Macaulay, this was known: see [8, §7.80]. We reduce to this case easily. Our result generalizes a theorem of Mumford [7], who proved that if X is normal (of dimension at least two), then H^1 vanishes.

The above vanishing criterion cannot be improved in any naive way. Grothendieck knew that the depth condition was essential [3, XII 1.3]. The regularity condition is also essential: for any integers $0 < i < n$, there exists a projective Cohen-Macaulay variety X of dimension n, such that $\operatorname{codim}[\operatorname{Sing}(X)] = i$, together with an ample line bundle $\mathscr{L}$ such that $H^i(X,\mathscr{L}^{-1}) \neq 0$. If $i > 1$, these examples are trivial variants of an example of Sommese [9]. For $i = 1$, a different construction is needed.

One may search for more delicate local criteria for vanishing. For instance, a consequence of [2] is that H^i vanishes if X has rational singularities: see [8]. By definition, if X is rational, it is normal, so this criterion is not applicable to singularities which live in codimension one. We prove a theorem which applies

Received by the editors May 2, 1988 and, in revised form, July 18, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32L20, 14F99.
Key words and phrases. Vanishing theorem, seminormal.
The first author was partially supported by the National Science Foundation

to exactly this situation: if X is seminormal, S_2, and has dimension at least two, then $H^1(X, \mathscr{L}^{-1})$ vanishes.

In this paper, we shall assume that all varieties are defined over $\mathbf{C}$.

1. Vanishing based on regularity and depth

We prove:

Proposition 1.1. *Let X be a projective variety, and let $\mathscr{L}$ be an ample line bundle on X. Fix $k \geq 1$. Assume that $k <$ codim[Sing(X)] and that X is S_{k+1}. Then $H^k(X, \mathscr{L}^{-1}) = 0$.* (If X is smooth, we define codim[Sing(X)] = dim(X).)

Proof. The statement was known if X is Cohen-Macaulay. (See [8, §7.80].) Let $n = \dim(X)$. Let $H \subset X$ be a sufficiently general and sufficiently ample hyperplane. We have an exact sequence

$$0 \to \mathscr{L}^{-1}(-H) \to \mathscr{L}^{-1} \to \mathscr{L}^{-1}|_H \to 0,$$

and taking cohomology we obtain

$$H^k(\mathscr{L}^{-1}(-H)) \to H^k(\mathscr{L}^{-1}) \to H^k(\mathscr{L}^{-1}|_H).$$

We may assume that X is not Cohen-Macaulay and hence that $k + 1 < \dim(X)$. This, together with the appropriate Bertini theorems imply that $k <$ codim[Sing(H)] and that H is S_{k+1}. Since X is S_{k+1}, a result of Grothendieck [3, XII 1.3], implies that $H^k(\mathscr{L}^{-1}(-H)) = 0$ so long as H is sufficiently ample. By induction on the dimension $H^k(\mathscr{L}^{-1}|_H)$ is zero, hence the result. □

Remark 1.2. Let Irr(X) denote the *irrational locus* of X, given by

$$\operatorname{Irr}(X) = \bigcup_{i>0} \operatorname{Supp}(R^i \pi_* \mathscr{O}_{\widetilde{X}}),$$

where $\pi: \widetilde{X} \to X$ is a resolution of singularities. In the proposition, one can weaken the hypothesis on the singular locus to the condition that $k <$ codim[Irr(X)]. The same proof works, allowing one to reduce to the case where X is Cohen-Macaulay, which is well known. One needs to known the following Bertini-type lemma: if $H \subset X$ is a general hyperplane, then $\operatorname{Irr}(X \cap H) \subset \operatorname{Irr}(X) \cap H$. The proof of this lemma is left to the reader.

2. Counterexamples

The result of this section is: *Fix integers $0 < k < n$. Then there exists a projective Cohen-Macaulay variety X of dimension n, with* codim[Sing(X)] = *k, and an ample line bundle $\mathscr{L}$ on X such that $H^k(X, \mathscr{L}^{-1}) \neq 0$.*

Actually, except for the case $k = 1$, this is obtained by a trivial variant of a known construction [9, 0.2.4]. The author has kindly informed us of a critical typographical error, so we summarize the construction for the convenience of the reader.

We first assume that $k > 1$. (The case $(n = 3, k = 2)$ occurs in [9].) Let

$$P = \mathbf{P}(\mathscr{O}_{\mathbf{P}^{n-k}} \oplus [\mathscr{O}_{\mathbf{P}^{n-k}}(1)]^{\oplus(k+1)}).$$

Let $\mathscr{T}$ denote the tautological bundle on P. Let X be a general member of the linear system associated to $[\mathscr{T} \otimes \mathscr{O}_{\mathbf{P}^{n-k}}(-1)]^{\otimes(k+2)}$. Let $\mathscr{L} = \mathscr{T} \otimes \mathscr{O}_{\mathbf{P}^{n-k}}(1)$. Then $H^k(X, \mathscr{L}^{-1}) \neq 0$. The fibers of $X \to \mathbf{P}^{n-k}$ are cones, each having a unique singular point. Some details may be found in the original source.

Now we deal with the case $k = 1$. We describe the subcase $n = 2$. The construction is easily modified for higher values of n.

Let Y be the ruled surface $\mathbf{P}(\mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}(-2))$. Let C_0 and C_∞ be the sections of $\pi: Y \to \mathbf{P}^1$ with self-intersection -2 and 2, respectively. (So $C_\infty \sim C_0 + 2f$.) Let f be a fiber of π. Let X be the cyclic double cover of Y, branched along the divisor $3C_0 + C_\infty$. (The divisor $3C_0 + C_\infty$ is uniquely divisible by 2 in $\mathrm{Pic}(Y)$: $3C_0 + C_\infty \sim 2(2C_0 + f)$.) Let $\mathscr{L}$ be the line bundle on X which is the pullback to X of $\mathscr{O}_Y(C_0 + 3f)$. The singularities of X are analytically of the form $\mathbf{C}[[x, y, t]]/(x^2 + y^3)$. Then X is Cohen-Macaulay (but not normal), $\mathscr{L}$ is ample, and $H^1(X, \mathscr{L}^{-1}) \neq 0$. (One may calculate that

$$H^1(X, \mathscr{L}^{-1}) \cong H^1(Y, \mathscr{O}(-C_0 - 3f)) \oplus H^1(Y, \mathscr{O}(-3C_0 - 4f))$$

and the second summand is nonzero.) The details are left to the reader.

For higher values of n, replace $\mathbf{P}^1$ by $\mathbf{P}^{n-1}$, and replace $\mathscr{L}$ by $\mathscr{O}_Y(C_0 + (n+1)f)$.

3. Vanishing of H^1 for seminormal varieties

The result of this section is:

Theorem 3.1. *Let S be a projective seminormal S_2 variety of dimension at least two. Let $\mathscr{L}$ be an ample line bundle on S. Then $H^1(S, \mathscr{L}^{-1}) = 0$.*

Remark 3.2. It is known [5, 3.9] that seminormal S_2 varieties have a particularly simple geometry: they are those S_2 varieties, which (outside a subset of codimension two) look locally like $N \times \mathbf{C}^r$, where N is the union of the coordinate axes in $\mathbf{C}^k$. (But k can vary.)

We need the following well-known lemma:

Lemma 3.3. *Let X be a reduced projective scheme over $\mathbf{C}$ (having no zero dimensional components). Let $\mathscr{L}$ be an ample line bundle on X. Then $H^0(X, \mathscr{L}^{-1}) = 0$.*

Proof (of Theorem 3.1*).* We first prove the theorem in the case where S is a surface. Let $\pi: \widetilde{S} \to S$ be the normalization map. let $\mathscr{I}$ be the conductor of $\mathscr{O}_{\widetilde{S}}$ into $\mathscr{O}_S$. It is a sheaf of ideals in both $\mathscr{O}_S$ and in $\mathscr{O}_{\widetilde{S}}$. Let Δ be the closed subscheme of S determined by $\mathscr{I}$, and let $\widetilde{\Delta}$ be the closed subscheme of $\widetilde{S}$ determined by $\mathscr{I}$. By [6, 1.5] or [11, 1.3] Δ and $\widetilde{\Delta}$ are reduced schemes. (This is the only place where we use the hypothesis that S is seminormal.)

We shall make various calculations with $\mathcal{O}_S$-modules, and we shall write for instance $\mathcal{O}_{\widetilde{S}}$ instead of $\pi_*\mathcal{O}_{\widetilde{S}}$, to avoid cumbersome notation.

There is an exact sequence of $\mathcal{O}_S$-modules

$$0 \to \mathcal{O}_S \xrightarrow{p} \mathcal{O}_{\widetilde{S}} \oplus \mathcal{O}_\Delta \xrightarrow{q} \mathcal{O}_{\widetilde{\Delta}} \to 0$$

which is obtained from the canonical maps of $\mathcal{O}_S$-modules:

$$\begin{aligned} p_1 &: \mathcal{O}_S \to \mathcal{O}_{\widetilde{S}} \\ p_2 &: \mathcal{O}_S \to \mathcal{O}_\Delta \\ q_1 &: \mathcal{O}_{\widetilde{S}} \to \mathcal{O}_{\widetilde{\Delta}} \\ q_2 &: \mathcal{O}_\Delta \to \mathcal{O}_{\widetilde{\Delta}} \end{aligned}$$

by $p = p_1 + p_2$ and $q = q_1 - q_2$. It is easy to verify that the sequence is in fact exact. (This exact sequence has been used by Steenbrink [10], proof of theorem 3.)

Tensor the given exact sequence by $\mathcal{L}^{-1}$ and compute the long exact sequence of cohomology on S. By (3.3), $H^0(\mathcal{L}^{-1} \otimes \mathcal{O}_{\widetilde{\Delta}}) = 0$. Hence we have an exact sequence:

$$0 \to H^1(\mathcal{L}^{-1}) \to H^1(\mathcal{L}^{-1} \otimes \mathcal{O}_{\widetilde{S}}) \oplus H^1(\mathcal{L}^{-1} \otimes \mathcal{O}_\Delta) \xrightarrow{\phi} H^1(\mathcal{L}^{-1} \otimes \mathcal{O}_{\widetilde{\Delta}}).$$

We must show that ϕ is injective. Observe that $H^1(\mathcal{L}^{-1} \otimes \mathcal{O}_{\widetilde{S}})$ is isomorphic to $H^1(\widetilde{S}, \pi^*\mathcal{L}^{-1})$. This is H^1 of the dual of an ample invertible sheaf on a normal surface, which by a theorem of Mumford [7] is zero. (This also follows from 1.1.)

We have exact sequences:

$$0 \to \mathcal{O}_\Delta \to \mathcal{O}_{\Delta_{\mathrm{nor}}} \to \mathcal{M} \to 0$$

and

$$0 \to \mathcal{O}_{\widetilde{\Delta}} \to \mathcal{O}_{\widetilde{\Delta}_{\mathrm{nor}}} \to \widetilde{\mathcal{M}} \to 0.$$

Then $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ have finite support, so $\mathcal{M} \cong \mathcal{M} \otimes \mathcal{L}^{-1}$ and $\widetilde{\mathcal{M}} \cong \widetilde{\mathcal{M}} \otimes \mathcal{L}^{-1}$, noncanonically. We obtain:

$$\begin{array}{ccccccccc} 0 & \to & H^0(\mathcal{M}) & \to & H^1(\mathcal{L}^{-1} \otimes \mathcal{O}_\Delta) & \to & H^1(\mathcal{L}^{-1} \otimes \mathcal{O}_{\Delta_{\mathrm{nor}}}) & \to & 0 \\ & & \downarrow \rho_1(\mathcal{L}) & & \downarrow \rho_2(\mathcal{L}) & & \downarrow \rho_3(\mathcal{L}) & & \\ 0 & \to & H^0(\widetilde{\mathcal{M}}) & \to & H^1(\mathcal{L}^{-1} \otimes \mathcal{O}_{\widetilde{\Delta}}) & \to & H^1(\mathcal{L}^{-1} \otimes \mathcal{O}_{\widetilde{\Delta}_{\mathrm{nor}}}) & \to & 0. \end{array}$$

We want to prove that $\rho_2(\mathcal{L})$ is injective. It suffices to show that $\rho_3(\mathcal{L})$ and $\rho_1(\mathcal{L})$ are injective. Now $\rho_3(\mathcal{L})$ is injective because $\mathcal{O}_{\Delta_{\mathrm{nor}}}$ is a smooth curve (perhaps disconnected) and hence the map

$$\mathcal{O}_{\Delta_{\mathrm{nor}}} \to \mathcal{O}_{\widetilde{\Delta}_{\mathrm{nor}}}$$

of $\mathcal{O}_{\Delta_{\mathrm{nor}}}$-modules is split injective (via trace). Because S is Cohen-Macaulay, the Serre duality and Serre vanishing theorems imply that $H^1(\mathcal{L}^{-n}) = 0$ for $n \gg 0$. Hence $\rho_2(\mathcal{L}^n)$ is injective for $n \gg 0$. Hence $\rho_1(\mathcal{L}^n)$ is injective for $n \gg 0$. But $\rho_1(\mathscr{L})$ is independent of $\mathscr{L}$, so in fact $\rho_1(\mathscr{L})$ is injective. Hence $\rho_2(\mathscr{L}) = \phi$ is injective. This proves the theorem in the case where S is a surface.

Now we prove the theorem in the general case, by induction on the dimension of S. If $\dim(S) = 2$ we are done. Otherwise, embed S in some projective space $\mathbf{P}$. Since S is S_2, we know by the lemma of Enriques-Severi-Zariski that $H^1(S, \mathscr{L}^{-1}(-n)) = 0$ for $n \gg 0$. Changing the embedding, we may assume that $H^1(S, \mathscr{L}^{-1}(-1)) = 0$. By the Bertini theorem for seminormality [1, 3.9 or 12], one knows that there exists a hyperplane $H \subset \mathbf{P}$ such that $S \cap H$ is seminormal. Also, we may choose H so that $S \cap H$ is S_2. Tensor the exact sequence:

$$0 \to \mathscr{O}_S(-1) \to \mathscr{O}_S \to \mathscr{O}_{S\cap H} \to 0$$

by $\mathscr{L}^{-1}$ and compute H^1. The theorem follows immediately. □

Remark 3.4. If the ground field has positive characteristic, by adding suitable hypotheses one can still make the proof work: let S be a projective seminormal Cohen-Macaulay surface. Assume that the Picard scheme $\mathbf{Pic}(S_{\text{nor}})$ is reduced. Let $\mathscr{L}$ be an ample line bundle on S. Then $H^1(S, \mathscr{L}^{-1}) = 0$.

Acknowledgments

Craig Huneke has kindly pointed out some simplifications to the proof of (3.1). Also, the referee has provided a shorter proof of (1.1).

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