ALGEBRAS OF HOLOMORPHIC FUNCTIONS BETWEEN $H^p$ AND $N$.  

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Abstract. For the algebra $N^p$, $p > 1$, introduced by Stoll with the notation $(\log^+ H)^*$ in [5], a characterization of the outer functions will be given, which can be used to derive results analogous to those of $N$. [4].

1. The algebra $N^p$

In this section, some introductory remarks will be made. Let $U$ and $T$ denote the unit disk in $\mathbb{C}$ and the unit circle. For $\phi \in L^1(T)$, a holomorphic function $H[\phi]$ is defined by

$$H[\phi](z) = (2\pi)^{-1} \int_0^{2\pi} H(z, e^{it}) \phi(e^{it}) dt \quad (z \in U),$$

where $H(z, e^{it}) = (e^{it} + z)(e^{it} - z)^{-1}$. Note that $H = P + iQ$, with $P$ the Poisson kernel. $P[\phi]$ will denote the Poisson integral. We denote by $N^p$, for $p > 1$, the class of functions $f$ holomorphic in $U$ and satisfying

$$\sup_{0 < r < 1} \int_0^{2\pi} (\log^+ |f(re^{it})|)^p dt < +\infty.$$

If $f \in N^p$, then $\log(1 + |f^*|) \in L^p(T)$ and

$$(1) \quad (\log(1 + |f(w)|))^p \leq P[(\log(1 + |f^*|))^p](w) \quad (w \in U),$$

where $f^*$ is the boundary function of $f$ on $T$. Under the metric $d_p$, defined for $f, g \in N^p$ by

$$d_p(f, g) = \left( (2\pi)^{-1} \int_0^{2\pi} \left( \log(1 + |f^*(e^{it}) - g^*(e^{it})|) \right)^p dt \right)^{1/p},$$

$N^p$ becomes an $F$-algebra. For $f \in N^p$, (1) implies that

$$(2) \quad \log(1 + |f(w)|) \leq 2^{1/p} d_p(f, 0)(1 - |w|)^{-1/p} \quad (w \in U).$$
It is known that

\[ N^q \subset N^p \ (q > p), \quad \bigcup_{p > 0} H^p \subset \bigcap_{p > 1} N^p, \quad \text{and} \quad \bigcup_{p > 1} N^p \subset N_*, \]

where the first containment is proper. To see that the second is proper, let \( \phi(e^{it}) = (\log t)^2 \) (\( t \in (0, 2\pi) \)). Then \( \phi \in L^p(T) \) for all \( p > 1 \) and \( e^\phi \notin L^p(T) \) for any \( p > 0 \). Define \( f \) by \( f(z) = \exp(H[\phi](z)) \) (\( z \in U \)). Since \( (\log^+ |f(z)|)^p \leq P[\phi](z) \), we have \( f \in N^p \) for all \( p > 1 \). On the other hand, \( |f^*| = e^\phi \) a.e. on \( T \) implies that \( f \notin H^p \), for \( p > 0 \). Next let \( \psi(e^{it}) = t^{-1}(1 + |\log t|)^{-2} \) (\( t \in (0, 2\pi) \)) and define \( f \) by \( f(z) = \exp(H[\psi](z)) \). Since \( \psi \in L^1(T) \) and \( \log^+ |f(z)| = P[\psi](z) \), the uniform integrability of the functions \( \{\log^+ |f_r(e^{it})| : 0 < r < 1\} \) follows, i.e., \( f \in N_* \), and \( \log^+ |f^*| = \psi \notin L^p(T), \ p > 1 \), implies \( f \notin N^p, \ p > 1 \). Thus the third containment is also proper.

If \( f' \in H^p, \ 0 < p < 1 \), then \( f \in H^q \) with \( q = p(1-p)^{-1} \) (Hardy-Littlewood, [1]). On the other hand, \( f' \in N \) does not imply \( f \in N \) (Hayman, [3]). Further, \( f' \in N_* \) does not imply \( f \in N \) (Yanagihara, [6]). In contrast to \( H^p, \ N_* \), and \( N \), the class \( N^p \) has the following property: If \( f' \in N^p \), then \( f \in N^p \). If \( q > p \), then there exists \( f \) such that \( f' \in N^p \), yet \( f \notin N^q \). The former is easily seen by a maximal function argument [3, p. 183]. To see the latter, let \( f(z) = \exp((1-z)^{-\alpha}) \) (\( z \in U \)) with \( q^{-1} < \alpha < p^{-1} \). Since \( (1-z)^{-\alpha} \in H^p \), we have \( f \in N^p \) and hence \( f'(z) = \alpha f(z)(1-z)^{-\alpha-1} \in N^p \). Let \( M_\infty(f; r) = \max\{|f(z)| : |z| = r\} \). Then \( \log^+ M_\infty(f; r) = (1-r)^{-\alpha} \) (\( 0 < r < 1 \)), and hence \( (1-r)^{1/q} \log^+ M_\infty(f; r) \to +\infty \) as \( r \to 1 \). It follows from (2) that \( f \notin N^q \).

2. Algebra homomorphisms

By the same argument as in [4], we can prove that if \( \gamma \) is a nontrivial multiplicative linear functional on \( N^p \), then there exists \( \lambda \in U \) such that \( \gamma(f) = f(\lambda) \) (\( f \in N^p \)) and \( \gamma \) is continuous, by (2). This fact will be used to see part (4) of the following Theorem 1.

Let \( \Psi: U \to U \) be a holomorphic map. For \( f \) holomorphic on \( U \), we define \( C_{\Psi}f \) by

\[ (C_{\Psi}f)(z) = (f \circ \Psi)(z) \quad (z \in U). \]

**Theorem 1.** (3) Let \( \Psi: U \to U \) be holomorphic. Then, for \( q \geq p \), \( C_{\Psi}: N^q \to N^p \) is a continuous algebra homomorphism.

(4) Suppose \( \Gamma: N^q \to N^p \) is a nontrivial algebra homomorphism. Then there exists \( \Psi: U \to U \), holomorphic, such that \( \Gamma f = C_{\Psi}f \) (\( f \in N^q \)). Hence, if \( q \geq p \), then \( \Gamma \) is continuous.

(5) Suppose \( \Gamma: N^q \to N^p \) is an algebra homomorphism onto \( N^p \). Then \( p = q \) and \( \Gamma \) is an isomorphism. The map \( \Psi: U \to U \), determined by \( \Gamma \), is a conformal map onto \( U \) and \( \Gamma^{-1} = C_{\Psi^{-1}} \).
Proof. (3) Let \( f \in N^q \). Then from (1) with \( w = \Psi(z) \), (2.5) in [4], and Hölder's inequality we have, for \( 0 < r < 1 \),

\[
\begin{align*}
(2\pi)^{-1} \int_0^{2\pi} (\log(1 + |(f \circ \Psi)(re^{i\theta})|))^{p} \, d\theta \\
\leq \frac{1 + |\Psi(0)|}{1 - |\Psi(0)|} \left( (2\pi)^{-1} \int_0^{2\pi} (\log(1 + |f^*(e^{i\theta})|))^{q} \, d\theta \right)^{p/q}.
\end{align*}
\]

This shows that \( f \circ \Psi \in N^p \) and, at the same time, that \( d_p(C_\Psi f, 0) \leq K d_q(f, 0) \) with a constant \( K \) independent of \( f \). Thus \( C_\Psi \) is continuous. (4) This part is the same as in [4]. (5) \( \Gamma \) is written in the form \( \Gamma = C_\Phi \), by (4). \( \Psi(U) \) is a nonempty open subset of \( U \), so \( C_\Psi \) is one-to-one and \( \Gamma^{-1} = C_\Phi \) with a holomorphic map \( \Phi: U \to U \). From \( \Psi \circ \Phi = \Phi \circ \Psi = \text{identity} \), we see that \( \Psi \) is a conformal map of \( U \) onto \( U \). Finally, suppose \( q < p \) and let \( f(z) = \exp((1 - z)^{-\alpha}) \) with \( p^{-1} < \alpha < q^{-1} \). Then \( f \notin N^p \) and \( f \in N^q \), so \( C_\Psi f \in N^p \) by assumption. But we can conclude from (3) that \( f = C_\Phi(C_\Psi f) \) belongs to \( N^p \), a contradiction. From \( C_\Phi: N^p \to N^q \) we see that \( p \geq q \), as well.

3. Outer functions in \( N^p \)

It is well known that if \( f \in N_\ast \), then \( \log|f^*| \in L^1(T) \). \( f \in N^p \) does not imply, however, that \( \log|f^*| \in L^p(T) \), while \( \log^+|f^*| \in L^p(T) \). Indeed, \( f(z) = \exp(H[\psi]z)(z \in U) \) with \( \psi(e^{i\theta}) = -t^{-1/p} \) \( (t \in (0,2\pi]) \) belongs to \( H^\infty \), but \( \log|f^*| \notin L^p(T) \). Now let

\[
f(z) = a \exp(H[\log \phi](z)) \quad (z \in U),
\]

where \( \phi(e^{i\theta}) \geq 0 \), \( \log \phi \in L^1(T) \), \( \log^+ \phi \in L^p(T) \), and \( a \in \mathbb{C} \) with \( |a| = 1 \). We shall call \( f \) an outer function in \( N^p \). If \( f \in N^p \), \( f \neq 0 \), then \( f \) admits the factorization: \( f = BSF \), as a function in \( N_\ast \), where \( B \) is the Blaschke product with respect to the zeros of \( f \), \( S \) is a singular inner function, and \( F \) is an outer function in \( N_\ast \). Here, since \( F = a \exp(H[\log|f^*|]) \), \( F \) becomes an outer function in \( N^p \). In \( N_\ast \), \( f \) is outer if and only if \( f^{-1} \notin N_\ast \). But an outer function in \( N^p \) is not necessarily invertible in \( N^p \), as is seen from the example \( f \) such that \( \log|f^*| \notin L^p(T) \).

Let \( f \in N^p \). If there is a sequence \( \{f_k\} \subset N^p \) such that \( f_k f \to 1 \) in \( N^p \) as \( k \to \infty \), we shall call \( \{f_k\} \) an approximate inverse of \( f \). This concept characterizes the outer functions in \( N^p \), as follows.

**Theorem 2.** Let \( f \in N^p \). Then \( f \) is outer if and only if \( f \) has an approximate inverse. When this is the case, \( f \) is approximated by invertible functions in \( N^p \).

**Proof.** Suppose first that \( f \) is outer in \( N^p \), with \( a = 1: f(z) = \exp(H[\log \phi](z))(z \in U) \). Let \( E_k = \{t \in [0,2\pi]|\phi(e^{it}) \geq k^{-1}\} \) and \( G_k = \{t|\phi(e^{it}) < k^{-1}\} \). Put \( \phi_k(e^{it}) = \phi(e^{it})^{-1} \) for \( t \in E_k \) and \( \phi_k(e^{it}) = 1 \) for \( t \in G_k \) \( (k = 1,2,\ldots) \).
Then $\log \phi_k \in L^1(T)$ and $\log^+ \phi_k \in L^p(T)$, hence $f_k := \exp(H[\log \phi_k])$ belongs to $N^p$. Put $\psi_k(t) = 1$ for $t \in E_k$ and $\psi_k(e^{it}) = \phi(e^{it})$ for $t \in G_k$. Then $f_k(z)f(z) = \exp(H[\log \psi_k](z)) = \exp(P[\log \psi_k](z) + iv_k(z))$, where $v_k = Q[\log \psi_k]$. As $r \to 1$, with $z = re^{i\theta}$, we have $P[\log \psi_k]^*(e^{i\theta}) = \log \psi_k(e^{i\theta})$ for a.e. $\theta \in [0, 2\pi]$, and $v_k(e^{i\theta})$ also exists for a.e. $\theta$ [2, p. 103]. Thus $f_k^*(e^{i\theta})f^*(e^{i\theta}) = \psi_k(e^{i\theta})\exp(i\theta v_k(e^{i\theta}))$. Take $q$, $0 < q < 1$. By Theorem 4.2 in [1], we see that $M_q(\psi_k ; r) \leq C_q M_1(P[\log \psi_k] ; r) \leq C_q \|\log \psi_k\|_1$ ($0 < r < 1$), where $C_q$ is a constant depending only on $q$, and hence $\|v_k\|_q \leq C_q \|\log \psi_k\|_1$, by Fatou's lemma. Since the right side tends to 0 as $k \to \infty$, by the dominated convergence theorem, a subsequence of $\{v_k\}$, denoted by the same symbol again, tends to 0 for a.e. $\theta \in [0, 2\pi]$. Hence $f_k^*(e^{i\theta})f^*(e^{i\theta}) \to 1$ as $k \to \infty$, for a.e. $\theta$. Now from $\log(1 + |f_kf^* - 1|) \leq \log 3$, we conclude that $d_p(f_kf^*, 1) \to 0$.

Next suppose that $f \in N^p$ and $\{f_k\}$ is an approximate inverse of $f$. Then we have $f_k(z)f(z) \to 1$ ($z \in U$) as $k \to \infty$, so $f(z) \neq 0$ ($z \in U$). Thus the factorization of $f$ is of the form $f = SF$, with $S$ a singular inner function and $F$ outer in $N^p$. It is enough to see that $S^{-1} \in N^p$, since this implies that $S$ is a constant. Now we have $f_k f S^{-1} = f_k F \in N^p$ and $f_k(z)f(z) S^{-1}(z) \to S^{-1}(z)$ ($z \in U$) as $k \to \infty$. Since $|(S^{-1})^*| = 1$ a.e. on $T$, we see that $d_p(f_j f S^{-1}, f_k f S^{-1}) = d_p(f_j f, f_k f) \to 0$ as $j, k \to \infty$. Thus $\{f_k f S^{-1}\}$ converges to some $h \in N^p$, so $f_k(z)f(z) S^{-1}(z) \to h(z)$ ($z \in U$).

Finally, let $f$ be outer in $N^p$ and define $f_k$ as above. Put $g_k = f_k^{-1}$. Then, since $\log^+ \phi_k^{-1} = \log^+ \phi \in L^p(T)$, we see that $g_k \in N^p$, i.e., $g_k$ is invertible in $N^p$. Moreover, $|g_k^*(e^{i\theta})| = |f^*(e^{i\theta})|$ for $\theta \in E_k$ and $|g_k^*(e^{i\theta})| = 1$ for $\theta \in G_k$. Therefore, we have $|g_k^* - f^*| = |g_k^*||f_k^* - 1| \leq (|f^*| + 1)|f_k^* - 1|$, the right side tending to 0 as $k \to \infty$, a.e. on $T$. From $\log(1 + |g_k^* - f^*|) \leq \log(2 + 2|f^*|)$, we see that $d_p(g_k^*, f) \to 0$ as $k \to \infty$.

Remark. Let $S$ be a singular inner function. Then $S_r$ ($0 < r < 1$) is invertible in $N^p$, and $S_r \to S$ as $r \to 1$ (Theorem 4.2, [5]). This means that the converse of the second statement of Theorem 2 is not valid.

Corollary. Let $f \in N^p$. Then $f N^p$, the ideal generated by $f$, is dense in $N^p$ if and only if $f$ is outer.

4. Some ideals in $N^p$

Theorem 2 above enables us to deduce the following, which are analogues of Theorems 1 and 2 in [4].

Theorem 3. Let $M$ be a nonzero prime ideal in $N^p$ which is not dense in $N^p$. Then $M = M_\lambda := \{f \in N^p|f(\lambda) = 0\}$ for some $\lambda \in U$. Every closed maximal ideal is of the form $M_\lambda$.
Theorem 4. Let $M$ be a nonzero closed ideal in $N^p$. Then there exists a unique (modulo constants) inner function $I$ such that $M = IN^p$.

Proof. For the proof of Theorem 3, let $f \in M$, $f \neq 0$. Then $f = BSF$, where $F \notin M$ by the above corollary, so we have $BS \in M$. The remainder of the argument is completely analogous to that of [4]. For the proof of Theorem 4, let $f = BSF$, $f \in M$, $f \neq 0$. Take an approximate inverse \{\{f_k\}\} of $F$. Then $f_k f = BS(f_k F) \to BS$ as $k \to \infty$, so we have $BS \in M$ and hence $BS \in M \cap H^1$. The rest is the same as that of [4].

REFERENCES


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