A NOTE ON THE DIFFERENTIAL EQUATIONS OF GLEICK-LORENZ

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(Communicated by Kenneth R. Meyer)

Abstract. It is shown that for the Gleick-Lorenz equations, every solution in the positive octant blows up.

We consider the nonlinear system of differential equations

\[
\begin{align*}
\dot{x}_1 &= 10(x_2 - x_1) = F_1(x) \\
\dot{x}_2 &= x_1x_3 + 28x_1 - x_2 = F_2(x) \\
\dot{x}_3 &= x_1x_2 - (8/3)x_3 = F_3(x)
\end{align*}
\]

for \( x = (x_1, x_2, x_3) \) in the positive orthant \( \mathbb{R}^3_+ \), attributed to E. N. Lorenz by J. Gleick [Gleick 1987, p. 323]. Although Gleick describes their dynamics as chaotic [Gleick 1987, p. 30], in a simulation by C. Deno [Deno 1988] the forward orbit of any point other than the origin blows up. We rigorously verify this dynamic behavior.

For vectors \( u, f \) we write \( u > v \) in case \( u_i > v_i \) for all \( i \).

The system is cooperative in \( \mathbb{R}^3_+ \), i.e. \( \partial F_i/\partial x_j \geq 0 \) for \( i \neq j \). Therefore the Müller-Kamke theorem on differential inequalities implies that if \( x(t) \) and \( y(t) \) are solutions with \( x(0) > y(0) \geq 0 \) then \( x(t) > y(t) \) for all \( t \geq 0 \) at which both solutions are defined [Müller 1926, Kamke 1932; or see Coppel 1965].

It is easily verified that for any solution \( x(t) \) with \( x(0) > 0 \), there is a solution \( y(t) \) such that \( x(0) > y(0) > 0 \) and \( F(y(0)) > 0 \). It follows from the theory of cooperative systems that each \( y_i(t) \) is strictly increasing for \( t \geq 0 \) [Selgrade 1980]. Since there are no equilibria except the origin, \( y(t) \) cannot converge; therefore some \( y_i(t) \to \infty \) and \( \|y(t)\| \to \infty \). The Müller-Kamke theorem now implies \( \|x(t)\| \to \infty \).

For any solution \( z(t) \) with \( z(0) \geq 0, z \neq 0 \) it is easily seen that \( z(t) > 0 \) for all \( t > 0 \), and the preceding argument shows that \( \|z(t)\| \to \infty \).

It should be noted that these equations differ from those of [Lorenz 1963] in the sign of the term \( x_1x_3 \).

Received by the editors April 24, 1988.

1980 Mathematics Subject Classification (1985 Revision). Primary 34C11, 58F13.

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0002-9939/89 $1.00 + $.25 per page
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