STABILITY OF SURFACES WITH CONSTANT MEAN CURVATURE

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Abstract. We estimate the Gaussian curvature of a conformal metric on a surface of constant mean curvature in space form $M^3(c)$. By use of the estimates, we study stability of surfaces with constant mean curvature in $M^3(c)$.

1. Introduction

Let $M^3(c)$ be the three-dimensional space form of constant sectional curvature $c$. Let $M$ be a surface with constant mean curvature $H$ in $M^3(c)$, $g$ be the induced metric, and $K$ be the Gaussian curvature of $g$. We get the following results:

Theorem 1. The Gaussian curvature $\bar{K}$ of the conformal metric $g = \sigma g$ satisfies $\bar{K} \leq 1$, where

$$\sigma = \begin{cases} 2H^2 - K + 2c, & \text{when } c \geq 0 \\ -K, & \text{when } c < 0 \text{ and } H^2 + c \leq 0 \end{cases}$$

and $\bar{K} \equiv 1$ if and only if $c = 0$ and $H = 0$, or $c < 0$ and $H^2 + c = 0$.

Corollary 1.1 (Proposition 2.2 of [1]). Let $M$ be a minimal surface of $M^3(c)$. Then the Gaussian curvature $\bar{K}$ of the conformal metric $g = \sigma g$ satisfies $\bar{K} \leq 1$, where $\sigma = 2c - K$, when $c > 0$, and $\sigma = -K$, when $c \leq 0$.

Let $X : M \to M^3(c)$ be an immersion with constant mean curvature $H$. Let $D \subset M$ be a domain in $M$ with compact closure $\overline{D}$ and piecewise smooth boundary $\partial D$. Following §5 of [4], we say that $D$ is strongly stable if

$$I(f) = \int_D [||\nabla f||^2 - 2(2c + 2H^2 - K)f^2]dA > 0$$

for all functions $f : D \to \mathbb{R}$ such that $f|_{\partial D} = 0$.
Making use of Theorem 1, we obtain:

**Theorem 2.** Let $X: \mathcal{M} \to \mathcal{M}^3(c)$ $(c \geq 0)$ be an immersion with constant mean curvature $H$. Assume that $D \subset \mathcal{M}$ is simply connected and that

\[(1.3) \int_D (2c - K + 2H^2) dA < 2\pi.\]

Then $D$ is strongly stable.

**Theorem 3.** Let $X: \mathcal{M} \to \mathcal{M}^3(c)$ $(c < 0)$ be an immersion with constant mean curvature $H$ and $H^2 + c \leq 0$. Assume that $D \subset \mathcal{M}$ is simply connected and that

\[(1.4) \int_D -K dA < 2\pi.\]

Then $D$ is strongly stable.

From definition of strongly stable, when $H = 0$, we easily see that strongly stable reduces to stable of minimal surfaces. We get from Theorem 2 and Theorem 3:

**Corollary 2.1** ([5], Theorem 1.2 of [1]). Let $X: \mathcal{M} \to \mathcal{M}^3(c)$ $(c \geq 0)$ be a minimal immersion. Assume that $D \subset \mathcal{M}$ is simply connected and $\int_D (2c - K) dA < 2\pi$. Then $D$ is stable.

**Corollary 3.1** (Theorem 1.3 of [1]). Let $X: \mathcal{M} \to \mathcal{M}^3(c)$ $(c < 0)$ be a minimal immersion. Assume that $D \subset \mathcal{M}$ is simply connected and $\int_D |K| dA < 2\pi$. Then $D$ is stable.

**Corollary 3.2** (Proposition 5.2 of [4]). Let $X: \mathcal{M} \to \mathcal{M}^3(-1)$ be an immersion with constant mean curvature one. Let $D \subset \mathcal{M}$ be a simply connected compact domain. If $\int_D -K dA < 2\pi$, then $D$ is strongly stable.

Let $\mathcal{M}$ is a minimal surface; it is a well known that the Gaussian curvature $\bar{K} \equiv 1$ of $\bar{g} = -\kappa g$. We now generalize the result to surfaces with constant mean curvature in $\mathcal{M}^3(c)$.

**Theorem 4.** Let $M$ be a surface with constant mean curvature $H$ in $\mathcal{M}^3(c)$ and $M$ is not totally umbilic. Then the Gaussian curvature $\bar{K}$ of $\bar{g} = \sigma g$ satisfies

\[(1.5) \bar{K} = 1 - \frac{H^2 + c}{H^2 - K + c}\]

where $\sigma = H^2 - K + c > 0$, and $\bar{K} \equiv 1$ if and only if $H^2 + c = 0$.

**Corollary 4.1.** Let $M$ be a minimal surface in $\mathcal{R}^3$. Then the Gaussian curvature $\bar{K} \equiv 1$ of $\bar{g} = -K g$.

**Corollary 4.2** (Proposition 3 of [7]). Let $M$ be a surface with constant mean curvature one in $\mathcal{M}^3(-1)$. Then the Gaussian curvature $\bar{K} \equiv 1$ of $\bar{g} = -K g$. 

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2. Fundamental Formulas

Let $M$ be a surface in $M^3(c)$ and let $e_1, e_2, e_3$ be a local field of orthonormal frames in $M^3(c)$, such that, restricted to $M$, the vector field $e_3$ is normal to $M$. Then, the second fundamental form $B$ and the mean curvature $H$ for $M$ can be written as

\[
B = \sum_{i,j} h_{ij} \omega_i \omega_j e_3, \quad H = \frac{1}{2} \sum_i h_{ii}.
\]

The Gauss–Codazzi equations for $M$ are

\[
(2.2) \quad K = c + 2H^2 - |B|^2/2, \quad \text{where } |B|^2 = \sum_{i,j} h_{ij}^2
\]

\[
(2.3) \quad h_{ijk} = h_{ikj} \quad (1 \leq i, j, k, \cdots \leq 2).
\]

We denote by $\Delta$ the Laplacian relative to the induced metric on $M$. If $H =$ constant, then ([2])

\[
(2.4) \quad \frac{1}{2} \Delta |B|^2 = |\nabla B|^2 - |B|^4 + 2c|B|^2 - 4cH^2 + 2HW
\]

where

\[
(2.5) \quad |\nabla B|^2 = \sum_{i,j,k} (h_{ijk})^2, \quad W = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}.
\]

We get by a direct computation

\[
(2.6) \quad 2HW = 6H^2|B|^2 - 8H^4, \quad \text{and } |h|^2 \leq 2
\]

\[
\therefore \lambda_1 + \lambda_2 = 2H, \quad \therefore \lambda_1 \lambda_2 = 2H^2 - \sum
\]

\[
\therefore \Delta |B|^2 = 2H(\lambda_1^3 + \lambda_2^3)
\]

\[
\lambda_1 + \lambda_2 = 2H \lambda_1^2 + \lambda_2^2 - \lambda_1 \lambda_2
\]

\[
= 2H(\lambda_1 + \lambda_2)(\lambda_1^2 + \lambda_2^2 - \lambda_1 \lambda_2)
\]

\[
\]

From (2.4) and (2.6), we have

\[
(2.7) \quad \frac{1}{2} \Delta |B|^2 = |\nabla B|^2 - |B|^4 + 2c|B|^2 - 4cH^2 + 6H^2|B|^2 - 8H^4.
\]

Proposition 2.1. Let $M$ be a surface with $H =$ constant in $M^3(c)$, then

\[
(2.8) \quad |\nabla(|B|^2)|^2 = 2(|B|^2 - 2H^2) \cdot |\nabla B|^2 \leq 2|B|^2 \cdot |\nabla B|^2.
\]

Proof. At any point of $M$, let $h_{ij} = \lambda_i \delta_{ij}$, we have

\[
(2.9) \quad |\nabla(|B|^2)|^2 = 4 \sum_k \left( \sum_{i,j} h_{ij} h_{ijk} \right)^2 = 4 \sum_k \left( \sum_i \lambda_i h_{iik} \right)^2
\]

\[
= 4 \sum_k (\lambda_1 h_{11k} + \lambda_2 h_{22k})^2.
\]
But $H = \text{constant}$ implies

\begin{equation}
(2.10) \quad h_{11k} + h_{22k} = 0, \quad \lambda_1 + \lambda_2 = 2H.
\end{equation}

(2.9) and (2.10) yield

\begin{equation}
(2.11) \quad |\nabla(|B|^2)|^2 = 4(\lambda_1 - \lambda_2)^2 \sum_k h_{11k}^2 = 2(\lambda_1 - \lambda_2)^2 \sum_{i,k} h_{iik}^2
\end{equation}

\begin{equation}
= 4(|B|^2 - 2H^2) \sum_{i,k} h_{iik}^2.
\end{equation}

On the other hand, we easily establish by a direct computation:

\begin{equation}
(2.12) \quad |\nabla B|^2 = 3 \sum_{i \neq k} h_{iik}^2 + \sum_k h_{kkk}^2 = 2 \sum_{i,k} h_{iik}^2 + \sum_{i,k} h_{iik}^2
\end{equation}

\begin{equation}
= 2 \sum_{i,k} h_{iik}^2.
\end{equation}

Combining (2.11) with (2.12), we obtain (2.8). Q.E.D.

3. Proofs of Theorem 1, 2, 3 and 4

(3.1) Proof of Theorem 1. Case $c > 0$: (1.1) and (2.2) yield

\begin{equation}
(3.2) \quad \sigma = 2H^2 - K + 2c = c + |B|^2/2 > 0,
\end{equation}

where $c = 0$; we assume that $M$ is not totally geodesic. Thus we can define a conformal metric $\bar{g} = \sigma g$ on $M$. As well known, the Gaussian curvature $\bar{K}$ of $\bar{g}$ satisfies ([3]):

\begin{equation}
(3.3) \quad \bar{K} = K - \frac{1}{2} \Delta \log \sigma.
\end{equation}

By (3.2) and (3.3), we have

\begin{equation}
(3.4) \quad -\sigma \bar{K} = \sigma - 2(c + H^2) + \frac{1}{2} \frac{\Delta \sigma}{\sigma} - \frac{1}{2\sigma} |\nabla \sigma|^2.
\end{equation}

From (2.7), (3.2) and Proposition 2.1, we get

\begin{equation}
(3.5) \quad \frac{1}{2} \Delta \sigma = \frac{1}{4} \Delta(|B|^2) = \frac{1}{2} |\nabla B|^2 - \frac{1}{2} |B|^4 + c |B|^2 - 2cH^2 + 3H^2 |B|^2 - 4H^4
\end{equation}

\begin{equation}
\geq \frac{1}{2} |\nabla \sigma|^2 / \sigma - 2\sigma^2 + 6\sigma c + 6H^2 \sigma - 4c^2 - 8H^2 c - 4H^4
\end{equation}

and equality holds if and only if $c = 0$ and $H = 0$.

Noting $\frac{1}{4} (\lambda_1 - \lambda_2)^2 = |B|^2/2 - H^2 \geq 0$, then

\begin{equation}
(3.6) \quad \sigma = c + |B|^2/2 \geq c + H^2.
\end{equation}

From (3.5) and (3.6), we have

\begin{equation}
(3.7) \quad \frac{1}{2} \Delta \sigma \geq \frac{1}{2} |\nabla \sigma|^2 / \sigma - 2\sigma^2 + 2\sigma c + 2H^2 \sigma + 4(c + H^2) c + 4H^2 (c + H^2)
\end{equation}

\begin{equation}
- 4c^2 - 8H^2 c - 4H^4
\end{equation}

\begin{equation}
= \frac{1}{2} |\nabla \sigma|^2 / \sigma - 2\sigma^2 + 2\sigma c + 2H^2 \sigma.
\end{equation}
Combining (3.4) with (3.7), we get $\bar{K} \leq 1$ and $\bar{K} \equiv 1$ if and only if $c = 0$ and $H = 0$.

Case $c < 0$: Assume $H^2 + c \leq 0$ and $M$ is not totally umbilic. Then \[
\frac{1}{4}(\lambda_1 - \lambda_2)^2 = \frac{|B|^2}{2} - H^2 > 0,
\]
and
\[
(3.8) \quad \sigma = -K = -c - 2H^2 + \frac{|B|^2}{2} > 0.
\]
Thus we can define a conformal metric $\bar{g} = \sigma g$ on $M$.

(3.3) and (3.8) yield
\[
(3.9) \quad -\sigma \bar{K} = \sigma + \frac{1}{2} \frac{\Delta \sigma}{\sigma} - \frac{1}{2\sigma^2} |\nabla \sigma|^2.
\]

From (2.7), (3.8), and Proposition 2.1, we get
\[
(3.10) \quad \frac{1}{2} \Delta \sigma = \frac{1}{4} \Delta(|B|^2)
\]
\[
= \frac{1}{2} \frac{|\nabla(|B|^2)|^2}{(|B|^2 - 2H^2)} - \frac{1}{2} |B|^4 + c|B|^2 - 2cH^2 + 3H^2|B|^2 - 4H^4
\]
\[
\geq \frac{1}{2} \frac{|\nabla \sigma|^2}{\sigma} - 2\sigma^2 - 2\sigma(H^2 + c) \geq \frac{1}{2} \frac{|\nabla \sigma|^2}{\sigma} - 2\sigma^2.
\]

Combining (3.9) with (3.10), we have $\bar{K} \leq 1$, and $\bar{K} \equiv 1$ if and only if $H^2 + c = 0$. Q.E.D.

(3.11) Proof of Theorem 2. Assume that $D$ is not strongly stable. By the Smale’s version of the Morse index theorem [6], there exists a domain $D' \subset D$ and a function $f : D' \rightarrow (0, \infty)$ so that $\Delta f - 2f(K - 2c - 2H^2) = 0$ in $D'$, and $f|\partial D' = 0$. Let $g$ be the induced metric, from Theorem 1, the Gaussian curvature $\bar{K} \leq 1$ of $\bar{g} = (2H^2 + 2c - K)g$. By Proposition 3.13 of [1], $2 \geq \lambda_1(D^*)$, where $D^*$ is a geodesic disk in a sphere $S^2(1)$ with curvature $1$ and area of $D^*$ is equal to the area of $D'$ in the metric $\bar{g}$. Here $\lambda_1(D^*)$ is the first eigenvalue of the Laplacian of the sphere $S^2(1)$ on $D^*$. Since $\int_{B'(2H^2 + 2c - K)} dA < 2\pi$, the area of $D'$ in the metric $\bar{g}$ is smaller than $2\pi$. It follows that $D^*$ is contained in a hemisphere of $S^2(1)$ the first eigenvalue of which is 2. Thus
\[
2 \geq \lambda_1(D^*) > 2,
\]
which is a contradiction. Q.E.D.

(3.12) Proof of Theorem 3. We first observe that, since $c < 0$ and $H^2 + c \leq 0$,
\[
(3.13) \quad I_D(f) = \int_D [||\nabla f||^2 - 2(2c + 2H^2 - K)f^2] dA
\]
\[
\geq \int_D [||\nabla f||^2 + 2Kf^2] dA = \bar{I}_D(f)
\]
for all functions $f : D \rightarrow R$ such that $f|\partial D = 0$. 

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To show that $D$ is strongly stable, it suffices to show $\tilde{T}_D(f) > 0$ for all such $f$. The proof is similar to the proofs of Theorem 1.3 of [1] and above Theorem 2. We omit it here. Q.E.D.

(3.14) Proof of Theorem 4. We assume $M$ is not totally umbilic, then

$$\sigma = H^2 - K + c = |B|^2/2 - H^2 = \frac{1}{4}(\lambda_1 - \lambda_2)^2 > 0.$$  

We can define a conformal metric $\bar{g} = \sigma g$ on $M$.

(3.3) and (3.15) yield:

$$-\sigma K = \sigma - H^2 - c + \frac{1}{2}\Delta\sigma - \frac{1}{2\sigma^2}|
abla\sigma|^2.$$

From (2.7), (3.15) and Proposition 2.1, we get by a direct computation:

$$\frac{1}{2}\Delta\sigma = \frac{1}{2}\frac{|
abla\sigma|^2}{\sigma} - 2\sigma^2 + 2\sigma(H^2 + c).$$

Combining (3.16) with (3.17), we have

$$-\sigma K = -\sigma + H^2 + c,$$  i.e.  $K = 1 - \frac{H^2 + c}{H^2 - K + c}$

and $K \equiv 1$ if and only if $H^2 + c = 0$. Q.E.D.

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**References**