Abstract. We derive a $q$-analogue of the classical formula for the number of derangements of an $n$ element set. Our derivation is entirely analogous to the classical derivation, but relies on a descent set preserving bijection between the set of permutations with a given derangement part and the set of shuffles of two permutations.

A classical application of binomial inversion (more generally the principle of inclusion–exclusion) is the derivation of the formula for the number of derangements (permutations with no fixed points) of an $n$ element set:

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$  

This is obtained by counting permutations according to their number of fixed points and then inverting the resulting equation.

In this note we shall derive a formula of I. Gessel [G] for $q$-counting derangements by the major index statistic in a way entirely analogous to the classical $q = 1$ case. That is, we shall $q$-count permutations with $k$ fixed points and then use Gauss inversion ($q$-binomial inversion or more generally Möbius inversion on the lattice of subspaces of a vector space) to derive the following formula for $q$-derangement numbers:

$$d_n(q) = [n!] \sum_{k=0}^{n} \frac{(-1)^k}{[k!]q^\binom{k}{2}}.$$  

A key step in our derivation and an interesting result in its own right is a descent-preserving bijection between the set of permutations with a given derangement part and the set of shuffles of two permutations. This bijection enables us to use a formula of A. Garsia and I. Gessel for $q$-counting shuffles.

Gessel [G] obtained the formula for $q$-derangement numbers as a corollary of an Eulerian generating function formula for counting permutations by descents, major index, and cycle structure, which is proved via a correspondence
between partitions and permutations. $q$-Derangement numbers have also been interpreted combinatorially on sets of permutations bijectively associated with derangements by A. Garsia and J. Remmel [GR] using the inversion index statistic and by J. Désarménien [D 2] (see [D 1 ] and [DW]) using the major index and inversion index statistics.

We shall briefly review some permutation statistic notation and terminology. For each integer $n \geq 1$, let $[n]$ denote the polynomial $1 + q + q^2 + \cdots + q^{n-1}$ and let $[n]!$ denote the polynomial $[n][n-1] \cdots [1]$. Also $[0]!$ is taken to be 1. The $q$-binomial coefficients are given by

$$\binom{n}{k}^q = \frac{[n]!}{[k]![n-k]!}$$

for $0 \leq k \leq n$.

For any positive integer $n$, let $\langle n \rangle$ denote the set $\{1, 2, \ldots, n\}$. We shall think of permutations in the symmetric group $S_n$ as words with $n$ distinct letters in $\langle n \rangle$. More generally, for a set $A$ of $n$ positive integers, $S_A$ denotes the set of permutations of $A$ or words with $n$ distinct letters in $A$. The descent set of a permutation $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n$ is $\text{des}(\sigma) = \{i \in (n-1)|\sigma_i > \sigma_{i+1}\}$. The major index of $\sigma$ is $\text{maj}(\sigma) = \sum_{i \in \text{des}(\sigma)} i$. Let us recall MacMahon's [M] formula for $\text{maj}$-counting permutations in $S_n$:

$$\sum_{\sigma \in S_n} q^{\text{maj}(\sigma)} = [n]!.$$ 

A letter $i \in A$ is said to be a fixed point of $\sigma \in S_A$ if $\sigma(i) = i$. A permutation with no fixed points is called a derangement. Let $D_n$ denote the set of all derangements in $S_n$. The $q$-derangement numbers are defined by

$$d_n(q) = \sum_{\sigma \in D_n} q^{\text{maj}(\sigma)}.$$ 

It will be convenient to view the empty word $\Lambda$ as a derangement and to define $D_0$ to be the set $\{\Lambda\}$. We also let $\text{maj}(\Lambda) = 0$ and $d_0(q) = 1$.

For any permutation $\alpha \in S_A$, where $A = \{a_1 < a_2 < \cdots < a_k\}$, define the reduction of $\alpha$ to be the permutation in $S_k$ obtained from $\alpha$ by replacing each letter $a_i$ by $i$, $i = 1, 2, \ldots, k$. The derangement part of a permutation $\sigma \in S_n$, denoted $dp(\sigma)$, is the reduction of the subword of nonfixed points of $\sigma$. For example, $dp(5, 3, 1, 4, 7, 6, 2) = 4, 3, 1, 5, 2$. We shall use the convention that the derangement part of the identity permutation is the empty word $\Lambda$. Note that the derangement part of a permutation is a derangement, and that conversely, any derangement in $D_k$ and $k$ element subset of $\langle n \rangle$ determines a permutation in $S_n$ with $n-k$ fixed points. Hence, the number of permutations in $S_n$ with a given derangement part in $D_k$ is $\binom{n}{k}$. Our goal is to $q$-count permutations with a given derangement part.

Let $\alpha \in D_k$. There is an obvious bijection between the set $\{\sigma \in S_n|dp(\sigma) = \alpha\}$ and the set $\text{Sh}(\alpha, \beta)$ of all shuffles of $\alpha$ and $\beta = k + 1, k + 2, \ldots, n$, 

...
i.e. permutations in \( S_n \) which contain \( \alpha \) and \( \beta \) as complementary subwords. Indeed, for each permutation \( \sigma \) in the former set, replace the subword of non-fixed points of \( \sigma \) by \( \alpha \) and the complementary subword of fixed points by \( \beta \). A very useful result of Garsia and Gessel [GG, Theorem 3.1] allows us to \( q \)-count the latter set. Unfortunately, since the above-mentioned bijection does not preserve the major index, it does not help us in \( q \)-counting the former set. However, we shall show that there is another bijection between these two sets of permutations which, in fact, preserves descent sets.

Define a letter \( \sigma_i \) of \( \sigma = \sigma_1, \sigma_2, \ldots, \sigma_n \in S_n \) to be an excédant of \( \sigma \) if \( \sigma_i > i \) and a subcedant of \( \sigma \) if \( \sigma_i < i \). Let \( s(\sigma) \) and \( e(\sigma) \) be the number of subcedants and excédants, respectively, of \( \sigma \). We now fix \( n \) and let \( k \leq n \). For \( \sigma \in S_k \), let \( \hat{\sigma} \) be the permutation of \( k \) letters obtained from \( \sigma \) by replacing its \( i \)th smallest subcedant by \( i \), \( i = 1, 2, \ldots, s(\sigma) \), its \( i \)th smallest fixed point by \( s(\sigma) + i \), \( i = 1, 2, \ldots, k - s(\sigma) - e(\sigma) \), and its \( i \)th largest excédant by \( n - i + 1 \), \( i = 1, 2, \ldots, e(\sigma) \). Note that \( \hat{\sigma} \) depends on \( n \) as well as \( \sigma \). For example, if \( \sigma = 32654 \) (with subcedants underlined and excédants overlined) and \( n = 8 \) then \( \hat{\sigma} = 638721 \). If \( k = n \) then \( \hat{\sigma} \in S_n \). If \( \sigma \) is a derangement then \( \hat{\sigma} \in A \), where \( A = \{1, 2, \ldots, s(\sigma)\} \cup \{n - e(\sigma) + 1, n - e(\sigma) + 2, \ldots, n\} \).

**Lemma 1.** Let \( \sigma \in S_k \), \( k \leq n \). Then \( \text{des}(\sigma) = \text{des}(\hat{\sigma}) \).

**Proof.** Suppose \( \sigma = \sigma_1, \sigma_2, \ldots, \sigma_k \) and \( \hat{\sigma} = \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_k \). For each \( i \in \langle k - 1 \rangle \), we shall show \( i \in \text{des}(\sigma) \) if and only if \( i \in \text{des}(\hat{\sigma}) \), by considering the nine possible designations of subcedant (s), excédant (e), and fixed point (f) to \( \sigma_i \) and \( \sigma_{i+1} \). First note that if \( \sigma_i \) is a subcedant of \( \sigma \) then \( \hat{\sigma}_i \leq \sigma_i \) and if \( \sigma_i \) is an excédant of \( \sigma \) then \( \hat{\sigma}_i \geq \sigma_i \).

**Cases 1–3.** Suppose \( (\sigma_i, \sigma_{i+1}) \) is an \( (s, s) \), \( (e, e) \), or \( (f, f) \) pair. It is then clear that \( \sigma_i < \sigma_{i+1} \) if and only if \( \hat{\sigma}_i < \hat{\sigma}_{i+1} \).

**Case 4.** Suppose \( (\sigma_i, \sigma_{i+1}) \) is a \( (s, e) \) pair. Then we have

\[
\hat{\sigma}_i < \sigma_i < i < i + 1 < \sigma_{i+1} < \hat{\sigma}_{i+1},
\]

which shows that \( i \notin \text{des}(\sigma) \) and \( i \notin \text{des}(\hat{\sigma}) \).

**Case 5.** Suppose \( (\sigma_i, \sigma_{i+1}) \) is a \( (s, f) \) pair. Now we have

\[
\sigma_i < i < i + 1 = \sigma_{i+1} \quad \text{and} \quad \hat{\sigma}_i \leq s(\sigma) < \hat{\sigma}_{i+1},
\]

which shows that \( i \notin \text{des}(\sigma) \) and \( i \notin \text{des}(\hat{\sigma}) \).

**Case 6.** Suppose \( (\sigma_i, \sigma_{i+1}) \) is a \( (f, s) \) pair. Then since \( \sigma_{i+1} < i + 1 \) and \( \sigma_i = i \), we have

\[
\sigma_{i+1} < \sigma_i \quad \text{and} \quad \hat{\sigma}_{i+1} \leq s(\sigma) < \hat{\sigma}_i.
\]

This shows that \( i \in \text{des}(\sigma) \) and \( i \in \text{des}(\hat{\sigma}) \).

**Cases 7–9.** The remaining three cases are that \( (\sigma_i, \sigma_{i+1}) \) is a \( (f, e) \), \( (e, s) \), or \( (e, f) \) pair. These cases are handled similarly to the previous three cases and are left to the reader. \( \Box \)
Theorem 2. Let $\alpha \in D_k$, $k \leq n$, and $\gamma = s(\alpha) + 1, s(\alpha) + 2, \ldots, n - e(\alpha)$. Then the map $\varphi: \{\sigma \in \mathcal{S}_n | dp(\sigma) = \alpha\} \rightarrow Sh(\hat{\alpha}, \gamma)$ defined by $\varphi(\sigma) = \hat{\sigma}$ is a bijection which preserves descent sets, i.e. $des(\sigma) = des(\varphi(\sigma))$. Consequently, for all $J \subseteq (n - 1)$, 
\[
|\{\sigma \in \mathcal{S}_n | dp(\sigma) = \alpha, des(\sigma) = J\}| = |\{\sigma \in Sh(\hat{\alpha}, \gamma) | des(\sigma) = J\}|.
\]
Proof. In view of Lemma 1, we need only show that $\varphi$ is an invertible map whose image is $Sh(\hat{\alpha}, \gamma)$. First, we claim that if $dp(\sigma) = \alpha$ then $\hat{\sigma}$ is obtained from $\sigma$ by replacing the subword of nonfixed points of $\sigma$ by $\hat{\alpha}$ and the subword of fixed points of $\sigma$ by $\gamma$. Indeed, the subword of fixed points of $\sigma$ is replaced by the word $s(\sigma) + 1, s(\sigma) + 2, \ldots, n - e(\sigma)$, which is precisely $\gamma$ since $s(\sigma) = s(\alpha)$ and $e(\sigma) = e(\alpha)$. Also since $\alpha$ is the reduction of the subword of nonfixed points of $\sigma$, the position of the $i$th smallest subcedant of $\alpha$ is the same as the position of the $i$th smallest subcedant of $\alpha$ in the subword of nonfixed points. The same is true for the $i$th smallest excédant. Hence each subcedant and excédant of $\sigma$ is replaced by the same letter that replaces the corresponding subcedant or excédant of $\alpha$. This means that the subword of subcedants and excédants of $\sigma$ is replaced by $\hat{\alpha}$. We may now conclude that $\hat{\sigma} \in Sh(\hat{\alpha}, \gamma)$.

The above description of $\hat{\sigma}$ as a shuffle of $\hat{\alpha}$ and $\gamma$ also implies that $\varphi$ is invertible. Indeed, if we replace the $\hat{\alpha}$ subword of any $\tau \in Sh(\hat{\alpha}, \gamma)$ by the permutation, of the subword positions, whose reduction is $\alpha$, and the letters of the $\gamma$ subword by their positions, we obtain a unique permutation $\sigma \in \mathcal{S}_n$ such that $dp(\sigma) = \alpha$ and $\varphi(\sigma) = \tau$. $\square$

Remark. Although a descent set preserving bijection between $\{\sigma \in \mathcal{S}_n | dp(\sigma) = \alpha\}$ and $Sh(\alpha, \beta)$, where $\beta = k + 1, k + 2, \ldots, n$, will not be needed in the sequel, we should point out here that one can be constructed by composing the bijection $\varphi$ with a bijection between $Sh(\alpha, \beta)$ and $Sh(\hat{\alpha}, \gamma)$ described in [BW, Proof of Proposition 4.1].

Corollary 3. Let $\alpha \in D_k$ and $k \leq n$. Then
\[
\sum_{\substack{dp(\sigma) = \alpha \\
\sigma \in \mathcal{S}_n}} q^{maj(\sigma)} = q^{maj(\alpha)} \binom{n}{k}.
\]
Proof. Since $maj(\sigma)$ depends only on $des(\sigma)$, it follows from Theorem 2 that
\[
\sum_{dp(\sigma) = \alpha} q^{maj(\sigma)} = \sum_{\sigma \in Sh(\hat{\alpha}, \gamma)} q^{maj(\sigma)} = q^{maj(\hat{\alpha})} \binom{n}{k},
\]
with the last step following from Garsia-Gessel [GG, Theorem 3.1]. (For a bijective alternative proof and generalization of the Garsia-Gessel result, see [BW].) By Lemma 1, $maj(\hat{\alpha}) = maj(\alpha)$, which completes the proof. $\square$

Theorem 4. For all $n \geq 0$,
\[
d_n(q) = [n!] \sum_{k=0}^{n} \frac{(-1)^k}{[k]!} q^{\binom{k}{2}}.
\]
Proof. By maj-$q$-counting the permutations in $\mathcal{S}_n$ according to derangement part and applying Corollary 3, we obtain

$$[n]! = \sum_{\sigma \in \mathcal{S}_n} q^\text{maj}(\sigma)$$

$$= \sum_{k=0}^{n} \sum_{\alpha \in D_k} \sum_{d(p(\sigma)) = \alpha} q^\text{maj}(\sigma)$$

$$= \sum_{k=0}^{n} \sum_{\alpha \in D_k} q^\text{maj}(\alpha) \binom{n}{k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} d_k(q).$$

Gauss inversion [A, p. 96] on the resulting equation yields,

$$d_n(q) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} q^{\binom{n-k}{2}} [k]!$$

$$= \sum_{k=0}^{n} \frac{[n]!}{[n-k]!} (-1)^{n-k} q^{\binom{n-k}{2}},$$

which is equivalent to the desired formula. □

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References


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