

AUTOMORPHISMS OF GRASSMANNIANS

MICHAEL J. COWEN

(Communicated by Irwin Kra)

ABSTRACT. For a complex vector space \mathcal{V} of dimension n , the group of holomorphic automorphisms of the Grassmannian $\text{Gr}(p, \mathcal{V})$ can be identified with the subgroup of $\text{PGl}(\bigwedge^p \mathcal{V})$ preserving the Grassmannian. Using this, Chow showed $\text{Aut}(\text{Gr}(p, \mathcal{V})) = \text{PGl}(\mathcal{V})$ for $n \neq 2p$, and $\text{PGl}(\mathcal{V})$ is a normal subgroup of index 2 in $\text{Aut}(\text{Gr}(p, \mathcal{V}))$ for $n = 2p$. We prove a version of Chow's result for a separable Hilbert space \mathcal{H} . **Theorem.** $\text{PGl}(\mathcal{H})$ is the subgroup of $\text{PGl}(\bigwedge^p \mathcal{H})$ which preserves $\text{Gr}(p, \mathcal{H})$. That is, if R is an invertible linear operator on $\bigwedge^p \mathcal{H}$ which preserves decomposable p -vectors, then there exists S , an invertible linear operator on \mathcal{H} , such that $R = \bigwedge^p S$.

1. INTRODUCTION

Let \mathcal{V} be a complex vector space of dimension n . Then a holomorphic automorphism of $\text{Gr}(p, \mathcal{V})$, the Grassmannian of p -planes in \mathcal{V} , is induced by an endomorphism of $\bigwedge^p \mathcal{V}$ preserving decomposable p -vectors: $\text{Aut}(\text{Gr}(p, \mathcal{V})) = \text{PGl}(\bigwedge^p \mathcal{V})_{\text{Gr}(p, \mathcal{V})}$, the subgroup of $\text{PGl}(\bigwedge^p \mathcal{V})$ preserving the Grassmannian. For example, S in $\text{Gl}(\mathcal{V})$ induces an automorphism Φ_S of $\text{Gr}(p, \mathcal{V})$, by sending a p -plane W into SW . Classically, Φ_S is called a collineation; the corresponding endomorphism of $\bigwedge^p \mathcal{V}$ is $\bigwedge^p S$. Chow [C] showed all automorphisms are collineations, except when the dual map $*$: $\text{Gr}(p, \mathcal{V}) \rightarrow \text{Gr}(n-p, \mathcal{V})$ is an automorphism—when $n = 2p$. In that case, there are the additional automorphisms of the form $* \circ \Phi_S$, the correlations:

Theorem 1.1 (Chow). $\text{Aut}(\text{Gr}(p, \mathcal{V})) = \text{PGl}(\mathcal{V})$ for $\dim \mathcal{V} \neq 2p$. $\text{PGl}(\mathcal{V})$ is a normal subgroup of index 2 in $\text{Aut}(\text{Gr}(p, \mathcal{V}))$ for $\dim \mathcal{V} = 2p$.

Using the Schubert calculus, Tango [T] gave an alternative proof to Theorem 1.1, generalizing Chow's result to automorphisms of flag manifolds. Kaup [K] used Lie theory to study the Grassmannian of a Banach space. In particular, he showed for a Hilbert space \mathcal{H} , every automorphism of $\text{Gr}(p, \mathcal{H})$ is a collineation when $1 \leq p < \infty$.

Received by the editors December 1, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 32M05; Secondary 14M15.
This research was supported in part by NSF Grant DMS-8602197.

© 1989 American Mathematical Society
0002-9939/89 \$1.00 + \$.25 per page

For \mathcal{H} a separable Hilbert space, we consider the holomorphic map

$$\varphi_R: \text{Gr}(p, \mathcal{H}) \rightarrow \text{Gr}(p, \mathcal{H}),$$

induced by an invertible linear operator R on $\bigwedge^p \mathcal{H}$ which preserves decomposable p -vectors. Such operators arise naturally when studying holomorphic curves in $\text{Gr}(p, \mathcal{H})$ which have the same curvature invariants. We denote by $\mathbf{P}\text{Gl}(\bigwedge^p \mathcal{H})_{\text{Gr}(p, \mathcal{H})}$ the subset of $\mathbf{P}\text{Gl}(\bigwedge^p \mathcal{H})$ consisting of all φ_R which preserve the Grassmannian. If \mathcal{H} is finite dimensional, then $\text{Gr}(p, \mathcal{H})$ is compact and connected, so φ_R is an automorphism. When \mathcal{H} is infinite dimensional, it does not follow directly from our assumptions that R^{-1} preserves the Grassmannian (so that $\mathbf{P}\text{Gl}(\bigwedge^p \mathcal{H})_{\text{Gr}(p, \mathcal{H})}$ is a subgroup of $\text{Aut Gr}(p, \mathcal{H})$.) Nonetheless, we prove a version of Chow's Theorem which shows

$$\mathbf{P}\text{Gl}(\bigwedge^p \mathcal{H})_{\text{Gr}(p, \mathcal{H})}$$

is indeed a subgroup of $\text{Aut Gr}(p, \mathcal{H})$:

Theorem 1.2. $\mathbf{P}\text{Gl}(\bigwedge^p \mathcal{H})_{\text{Gr}(p, \mathcal{H})} = \mathbf{P}\text{Gl}(\mathcal{H})$. That is, if R is an invertible linear operator on $\bigwedge^p \mathcal{H}$ which preserves decomposable p -vectors, then R is a collineation: $R = \bigwedge^p S$ for an invertible linear operator S on \mathcal{H} .

Note that R determines S up to multiplication by a scalar: Sx spans $\bigcap_{x \in W} \varphi_R(W)$, while Sx and $\tilde{S}x$ dependent for all x implies $\tilde{S} = cS$. Consequently, if R is unitary, then $\bigwedge^p (S^*S) = I$ implies $S^*S = cI$, so we have the following corollary:

Corollary 1.3. If \tilde{U} is unitary on $\bigwedge^p \mathcal{H}$ and preserves decomposable p -vectors, then $\tilde{U} = \bigwedge^p U$ for U unitary on \mathcal{H} .

The proof of Theorem 1.2 uses only elementary linear algebra. Along the way we give a short proof of Chow's result using one additional well-known but non-elementary fact: the Picard group of the Grassmannian is \mathbf{Z} .

2. PRELIMINARIES

That $\text{Aut Gr}(p, \mathcal{V})$ equals $\mathbf{P}\text{Gl}(\bigwedge^p \mathcal{V})_{\text{Gr}(p, \mathcal{V})}$ for \mathcal{V} finite dimensional can be proved along the lines of the \mathbf{P}_n case as found in [H]. The crucial fact needed is that the group of equivalence classes of holomorphic line bundles on $\text{Gr}(p, \mathcal{V})$ —the Picard group—is isomorphic to \mathbf{Z} . This follows from the homology of the Grassmannian in terms of Schubert cycles and from the Hodge decomposition: $H^1(\text{Gr}(p, \mathcal{V}), \mathcal{A})$ equals $H^2(\text{Gr}(p, \mathcal{V}), \mathcal{A}) = 0$, where \mathcal{A} is the sheaf of germs of analytic functions; thus $H^1(\text{Gr}(p, \mathcal{V}), \mathcal{A}^*) \cong H^2(\text{Gr}(p, \mathcal{V}), \mathbf{Z}) \cong \mathbf{Z}$.

To fix a choice of the dual map $*$: let e_1, \dots, e_n be a basis for \mathcal{V} and $\delta_1, \dots, \delta_n$ the dual basis. Define $\iota: \mathcal{V}^* \rightarrow \mathcal{V}$ by $\iota(\delta_j) = e_j$, so that $j \circ \iota = \text{id}$, where $j: \mathcal{V} \rightarrow \mathcal{V}^{**}$ is the identification map. Then ι induces a biholomorphic mapping, the dual map $*$: $\text{Gr}(p, \mathcal{V}) \rightarrow \text{Gr}(n-p, \mathcal{V})$, as follows: if W is a

p -plane, then $*W = \iota(W^\perp)$, where W^\perp is the annihilator of W . Note $*$ is induced by an isomorphism of $\bigwedge^p \mathcal{V} \rightarrow \bigwedge^{n-p} \mathcal{V}$ (a basis p -vector e_I maps to the $(n-p)$ -vector $\varepsilon_{IJ}e_J$, where $e_I \wedge e_J = \varepsilon_{IJ}e_1 \wedge \cdots \wedge e_n$). Furthermore, if $S: \mathcal{V}^* \rightarrow \mathcal{V}$ is a linear transformation, then $S(W^\perp) = ((S^t)^{-1}jW)^\perp$, so the following holds:

Lemma 2.1. *Let T be in $\text{Gl}(\mathcal{V})$, and let Φ_T be the induced automorphism on $\text{Gr}(p, \mathcal{V})$. Then $* \circ \Phi_T \circ *$ is the automorphism of $\text{Gr}(n-p, \mathcal{V})$ which is induced by $\iota \circ (T^t) \circ \iota^{-1} \in \text{Gl}(\mathcal{V})$. In particular, $*^2 = \text{Id}$.*

Remark 2.2. If $\varphi \in \text{Aut}(\text{Gr}(p, \mathcal{V}))$, then $* \circ \varphi \circ * \in \text{Aut}(\text{Gr}(n-p, \mathcal{V}))$. It thus suffices to prove Theorem 1.1 for $\dim \mathcal{V} \geq 2p$.

3. ADJACENCY

Let $R: \bigwedge^p \mathcal{H} \rightarrow \bigwedge^p \mathcal{H}$ be a (bounded) invertible linear operator which preserves the decomposable elements of $\bigwedge^p \mathcal{H}$ (those of the form $v_1 \wedge \cdots \wedge v_p$), and let $\varphi_R: \text{Gr}(p, \mathcal{H}) \rightarrow \text{Gr}(p, \mathcal{H})$ be the induced map. The most important tool for analyzing R is what Chow calls adjacency: distinct p -planes V and W are adjacent if $\dim V \cap W = p - 1$.

Lemma 3.1. *Let V and W be p -planes in \mathcal{H} with α and β the corresponding decomposable p -vectors in $\bigwedge^p \mathcal{H}$. Then V and W are adjacent if and only if $\alpha + \beta$ is decomposable.*

Proof. Assume that $\alpha + \beta$ is decomposable—the implication in the other direction is trivial. If $\dim V \cap W = q < p - 1$, mod out by $V \cap W$ and replace α and β by appropriate $(p-q)$ -vectors; the problem reduces to showing $\dim V \cap W$ cannot equal 0 if $p > 1$ and $\alpha + \beta$ is decomposable.

Assume that $\dim V \cap W = 0$; then $V + W$ is $2p$ -dimensional. Now $\alpha + \beta$ is in $\bigwedge^p (V + W)$, which implies that $\alpha + \beta = (v_1 + w_1) \wedge \cdots \wedge (v_p + w_p)$, where $v_i \in V$ and $w_i \in W$. Since $\alpha \wedge (w_1 \wedge \cdots \wedge w_p) = \alpha \wedge (\alpha + \beta) = \alpha \wedge \beta$, then $\beta = w_1 \wedge \cdots \wedge w_p$, and similarly $\alpha = v_1 \wedge \cdots \wedge v_p$; in particular, $v_1, \dots, v_p, w_1, \dots, w_p$ are independent. But $\alpha + \beta = v_1 \wedge \cdots \wedge v_p + v_1 \wedge w_1 \wedge \cdots \wedge v_p + \cdots + w_1 \wedge \cdots \wedge w_p$, which implies the mixed terms $v_1 \wedge w_1 \wedge \cdots \wedge v_p$, etc. sum to zero, a contradiction to independence of the v_i and w_i . Thus $\dim V \cap W$ cannot equal 0. \square

Remark 3.2. If $\varphi_R: \text{Gr}(p, \mathcal{H}) \rightarrow \text{Gr}(p, \mathcal{H})$ is induced by $R: \bigwedge^p \mathcal{H} \rightarrow \bigwedge^p \mathcal{H}$ linear and preserving decomposables, then φ_R preserves adjacency.

Proof. If $\dim V \cap W = p - 1$, let $\mu \in \bigwedge^{p-1} \mathcal{H}$ correspond to $V \cap W$; then V corresponds to $\mu \wedge v$ for $v \in \mathcal{H}$ and W corresponds to $\mu \wedge w$. Thus $R(\mu \wedge (v + w))$ is decomposable and the result follows from Lemma 3.1. \square

4. PRESERVING/DUALIZING SCHUBERT CYCLES

Let V be a q -plane in \mathcal{H} . For $p > q$, denote by $\sigma_p(V)$ the Schubert cycle defined by

$$\sigma_p(V) = \{W \in \text{Gr}(p, \mathcal{H}) \mid V \subset W\},$$

and for $p < q$ denote by $\Sigma^p(V)$ the Schubert cycle defined by $\Sigma^p(V) = \{W \in \text{Gr}(p, \mathcal{H}) \mid W \subset V\}$.

Proposition 4.1. *Let $\psi: \text{Gr}(p, \mathcal{H}) \rightarrow \text{Gr}(p, \mathcal{H})$ be a map (not necessarily continuous) preserving adjacency. Let V be a $(p-1)$ -plane, W_1 and W_2 distinct p -planes containing V . Denote by $\psi_{p-1}(V)$ the $(p-1)$ -plane $\psi(W_1) \cap \psi(W_2)$ and by $\psi^{p+1}(V)$ the $(p+1)$ -plane $\psi(W_1) + \psi(W_2)$. Then either (i) ψ preserves the Schubert cycle: $\psi(\sigma_p(V)) \subset \sigma_p(\psi_{p-1}(V))$, so that $\psi_{p-1}(V)$ is independent of the choice of W_1 and W_2 ; or (ii) ψ dualizes the Schubert cycle: $\psi(\sigma_p(V)) \subset \Sigma^p(\psi^{p+1}(V))$, so that $\psi^{p+1}(V)$ is independent of the choice.*

Proof. If (i) does not hold, then there exists a p -plane W_3 containing V such that $\psi(W_3)$ does not contain $\psi_{p-1}(V)$. Let $\psi(W_i) = \widetilde{W}_i$; then \widetilde{W}_3 intersects \widetilde{W}_1 and \widetilde{W}_2 in different $(p-1)$ -planes—otherwise $\widetilde{W}_1 \cap \widetilde{W}_2 \cap \widetilde{W}_3 = \psi_{p-1}(V)$ —and hence $\widetilde{W}_3 \subset \psi^{p+1}(V)$. Let W be any p -plane containing V ; then $\psi(W)$ intersects each of $\widetilde{W}_1, \widetilde{W}_2, \widetilde{W}_3$ in $(p-1)$ -planes, which are not all three the same by the assumption of $\widetilde{W}_1 \cap \widetilde{W}_2 \cap \widetilde{W}_3 \neq \psi_{p-1}(V)$. Thus $\psi(W) \subset \psi^{p+1}(V)$, so (ii) holds. \square

Corollary 4.2. *Let $\varphi_R: \text{Gr}(p, \mathcal{H}) \rightarrow \text{Gr}(p, \mathcal{H})$ be induced by $R: \Lambda^p \mathcal{H} \rightarrow \Lambda^p \mathcal{H}$ invertible and decomposable preserving. For $\dim \mathcal{H} > 2p$, there is a map $\varphi_{p-1}: \text{Gr}(p-1, \mathcal{H}) \rightarrow \text{Gr}(p-1, \mathcal{H})$ such that $\varphi_R(\sigma_p(V)) \subset \sigma_p(\varphi_{p-1}(V))$, with $\varphi_R(\sigma_p(V)) = \sigma_p(\varphi_{p-1}(V))$ if $\dim \mathcal{H} < \infty$. When $\dim \mathcal{H} = 2p$, there exists such a φ_{p-1} either for φ_R or for $* \circ \varphi$.*

Proof. We apply Proposition 4.1 to $\varphi = \varphi_R$, which preserves adjacency. Conditions (i) and (ii) of Proposition 4.1 are equivalent to (i)' $R(\Lambda^{p-1} V \wedge \mathcal{H}) \subset \Lambda^{p-1} \varphi_{p-1}(V) \wedge \mathcal{H}$ and (ii)' $R(\Lambda^{p-1} V \wedge \mathcal{H}) \subset \Lambda^p \varphi^{p+1}(V)$. Now (ii)' implies $\dim \mathcal{H} - (p-1) \leq p+1$, so we are done if $\dim \mathcal{H} > 2p$. If $\dim \mathcal{H} = 2p$, then equality holds in (i)' and (ii)', so (i) and (ii) become $\varphi(\sigma_p(V)) = \sigma_p(\varphi_{p-1}(V))$ and $\varphi(\sigma_p(V)) = \Sigma^p(\varphi^{p+1}(V))$. Since φ is continuous, and the Schubert cycles $\sigma_p(\varphi_{p-1}(V))$ and $\Sigma^p(\varphi^{p+1}(V))$ represent distinct homology classes, only one of these alternatives holds for all $V \in \text{Gr}(p, \mathcal{H})$. If necessary, we can compose R with the endomorphism of $\Lambda^p \mathcal{H}$ which induces the dual map $*$ to achieve $\varphi(\sigma_p(V)) = \sigma_p(\varphi_{p-1}(V))$ for all V . \square

5. PROOF OF THEOREM 1.1

Assume $\dim \mathcal{V} = n \geq 2p$, so that $\varphi_R(\sigma_p(W)) = \sigma_p(\varphi_{p-1}(W))$ for all $(p-1)$ -planes W (if necessary, compose φ_R with $*$ when $n = 2p$). Thus φ_{p-1} is one-to-one. To show it is an automorphism, it suffices to show it is holomorphic, since $\text{Gr}(p-1, \mathcal{V})$ is compact and connected. Coordinatize $\text{Gr}(p-1, \mathcal{V})$ by letting $\psi(Z)$ be the span of the $v_i + \sum z_{ij} w_j$, where $v_1, \dots, v_{p-1}, w_1, \dots, w_{n-p+1}$ is a basis for \mathcal{V} . Let $W_i(Z)$ denote the p -plane spanned by $\psi(Z)$ and w_i . Then $W_1(Z) \cap W_2(Z) = \psi(Z)$ implies

$\varphi_{p-1}(\psi(Z)) = \varphi_R(W_1(Z)) \cap \varphi_R(W_2(Z))$ is holomorphic, since the intersection has constant dimension.

Theorem 1.1 is true for $p = 1$, so by induction assume φ_{p-1} is induced by an endomorphism S of \mathcal{Z} . If W is a p -plane in \mathcal{Z} , then $W = \bigcap_{V \subset W} \sigma_p(V)$, where $\dim V = p - 1$. Thus

$$\begin{aligned} \varphi(W) &= \bigcap_{V \subset W} \varphi(\sigma_p(V)) = \bigcap_{V \subset W} \sigma_{p-1}(\varphi_{p-1}(V)) \\ &= \bigcap_{V \subset W} \sigma_{p-1}(SV) = \bigcap_{V \subset SW} \sigma_{p-1}(V) = SW. \quad \square \end{aligned}$$

6. $\dim \mathcal{Z} = \infty$

When \mathcal{Z} is infinite dimensional, we need R onto to establish that φ_{p-1} is one-to-one, as the following example shows: Let V be a $(p-1)$ -plane in \mathcal{Z} and λ a $(p-1)$ -vector corresponding to V . Let $T: \bigwedge^p \mathcal{Z} \rightarrow V^\perp$ be an injective linear operator, and define an injective linear operator $R: \bigwedge^p \mathcal{Z} \rightarrow \bigwedge^p \mathcal{Z}$ by $R\alpha = \lambda \wedge T\alpha$. Trivially R preserves decomposables, yet the induced map φ_{p-1} is constant.

Lemma 6.1. *Let $\psi: \text{Gr}(r, \mathcal{Z}) \rightarrow \text{Gr}(r, \mathcal{Z})$ preserve adjacency, $r \geq 2$. Fix Z an $(r-2)$ -plane and L an l -plane, $l \geq r-1$. If $\dim L \cap \psi(W) \geq r-1$ for all $W \in \sigma_r(Z)$, then $L \cap \psi(W) \neq 0$ for every r -plane W .*

Proof. We show for any r -plane $W: \dim W \cap Z = k$ implies $\dim L \cap \psi(W) \geq k+1$. This is trivial if $r = 2$. For $r > 2$, assume true for $k, 0 < k \leq r-2$. Let \widetilde{W} be an r -plane such that $\dim \widetilde{W} \cap Z = k-1$. There is an r -plane W such that $\dim W \cap Z = k$ and $\dim W \cap \widetilde{W} = r-1$. Thus $\dim \psi(W) \cap \psi(\widetilde{W}) = r-1$ and $\dim L \cap \psi(\widetilde{W}) \geq k = (k-1) + 1$. \square

The main step in proving Theorem 1.2 is the following proposition:

Proposition 6.2. *For $1 \leq q < p$, assume that for all r with $q \leq r \leq p$ we have defined maps $\varphi_r: \text{Gr}(r, \mathcal{Z}) \rightarrow \text{Gr}(r, \mathcal{Z})$, where $\varphi_p = \varphi_R$, which satisfy the following for $r > q$: (1) φ_r preserves adjacency, and (2) $\varphi_r \sigma_r(V) \subset \sigma_r(\varphi_{r-1}(V))$, for each $(r-1)$ -plane V . Then φ_q preserves adjacency. Furthermore, if $q > 1$, then we can define $\varphi_{q-1}: \text{Gr}(q-1, \mathcal{Z}) \rightarrow \text{Gr}(q-1, \mathcal{Z})$ so that (2) holds for $r = q$.*

Before proving Proposition 6.2, we use surjectivity of R to show:

Lemma 6.3. *Under the hypotheses of Proposition 6.2, if Z is a $(q-1)$ -plane and L an l -plane, $l \geq q \geq 1$, then $\dim L \cap \varphi_{q+1}(W) < q$ for some $W \in \sigma_{q+1}(Z)$.*

Proof. If not, then Lemma 6.1 shows that $L \cap \varphi_{q+1}(W) \neq 0$ for all $(q+1)$ -planes W . If U is a $(q+2)$ -plane, then $\varphi_{q+2}(U)$ contains $\varphi_{q+1}(W)$ for any $(q+1)$ -plane $W \subset U$, by hypothesis (2) of the Proposition, and hence $L \cap \varphi_{q+2}(U) \neq 0$. Inductively we obtain that $L \cap \varphi_R(V) \neq 0$ for all $V \in \text{Gr}(p, \mathcal{Z})$.

That is, if we let β be an r -vector which represents L , then $R(\alpha) \wedge \beta = 0$ for all α decomposable, and hence for all α in $\wedge^p \mathcal{H}$. This cannot happen since R is onto (and $\dim \mathcal{H} \geq r + p$). \square

Proof of Proposition 6.2. If φ_q does not preserve adjacency, let V_1 and V_2 be adjacent q -planes such that $\dim \varphi_q(V_1) \cap \varphi_q(V_2) \neq q-1$. Since $W = V_1 + V_2$ is a $(q+1)$ -plane containing both V_1 and V_2 , then (2) implies $\varphi_q(V_1) \subset \varphi_{q+1}(W)$, so $\varphi_q(V_1) = \varphi_q(V_2) = V_0$, say. Let \widetilde{W} be any $(q+1)$ -plane which contains $V_1 \cap V_2$, $\widetilde{W} \neq W$, and V be a q -plane in \widetilde{W} which contains $V_1 \cap V_2$ and is not contained in W . Let $W_i = V_i + V$ for $i = 1, 2$. Then W_1 and W_2 are distinct $(q+1)$ -planes containing V , and $V_i \subset W_i$ implies that $\varphi_q(V_i) \subset \varphi_{q+1}(W_i)$ for $i = 1, 2$. Thus $\varphi_q(V) = \varphi_{q+1}(W_1) \cap \varphi_{q+1}(W_2) = V_0$ and hence $V_0 \subset \varphi_{q+1}(\widetilde{W})$, which contradicts Lemma 6.3 (with $L = V_0$). Thus φ_q preserves adjacency.

To show that (2) holds for $q > 1$: since φ_q preserves adjacency, then either $\varphi_q(\sigma_{q-1}(Z)) \subset \sigma_q(\varphi_{q-1}(Z))$ or $\varphi_q(\sigma_{q-1}(Z)) \subset \Sigma^q(\varphi_{q+1}(W))$. If the latter, then $\varphi_q(V) \subset \varphi_{q+1}(W)$ for all q -planes V which contain Z . But if \widetilde{W} is any $(q+1)$ -plane containing Z , let V be a q -plane with $Z \subset V \subset \widetilde{W}$. Then $\varphi_q(V) \subset \varphi_{q+1}(\widetilde{W})$, so $\dim \varphi_{q+1}(W) \cap \varphi_{q+1}(\widetilde{W}) \geq q$ for all \widetilde{W} containing Z , contradicting Lemma 6.3 (with $L = \varphi_{q+1}(W)$). \square

Proposition 6.4. *Let v_1, \dots, v_q be independent. Then*

$$\varphi_q([v_1, \dots, v_q]) = \varphi_1([v_1]) + \dots + \varphi_1([v_q]),$$

where $[v_1, \dots, v_r]$ is the r -plane spanned by v_1, \dots, v_r .

Proof. Since φ_1 preserves adjacency, then $\varphi_1([v_1])$ and $\varphi_1([v_2])$ are distinct lines in $\varphi_2([v_1, v_2])$, so the Proposition is true when $q = 2$. Assume true for q and $q-1$ with $2 \leq q < p$. If not true for $q+1$, then we may assume that $\varphi_1([v_{q+1}]) \subset \varphi_q([v_1, \dots, v_q])$. But $\varphi_1([v_{q+1}]) \subset \varphi_q([v_1, \dots, v_{q-1}, v_{q+1}])$. Thus $\varphi_1([v_{q+1}]) \subset \varphi_{q-1}([v_1, \dots, v_{q-1}])$ by definition of φ_{q-1} , so that

$$\varphi_q([v_1, \dots, v_{q-1}, v_{q+1}]) \neq \varphi_1([v_1]) + \dots + \varphi_1([v_{q-1}]) + \varphi_1([v_{q+1}]),$$

contrary to assumption. \square

Remark 6.5. If V and W are vector spaces, T and $\tilde{T}: V \rightarrow W$ linear transformations such that for each $x \in V$, $\tilde{T}x = c_x Tx$, where $c_x \neq 0$ is a scalar, then $\tilde{T} = cT$. (Assume $\ker T = 0$, then use $c_x Tx + c_y Ty = c_{x+y}(Tx + Ty)$ to get $c_x = c_y$.)

Proof of Theorem 1.2. Fix e_1, \dots, e_{p-1} orthonormal in \mathcal{H} , and choose $\tilde{e}_i \in \varphi_1([e_i])$. Let $\alpha_i = e_1 \wedge \dots \wedge e_i$, $\tilde{\alpha}_i = \tilde{e}_1 \wedge \dots \wedge \tilde{e}_i$, and V_i be spanned by e_1, \dots, e_i . By Proposition 6.4, for $x \neq 0$ in V_{p-1}^\perp there is a unique $Sx \in \varphi_1([x])$ such that $R(\alpha_{p-1} \wedge x) = \tilde{\alpha}_{p-1} \wedge Sx$. Define $S0 = 0$.

We show S is linear: if x and y in V_{p-1}^\perp are independent, then so are $R(\alpha_{p-1} \wedge x) = \tilde{\alpha}_{p-1} \wedge Sx$ and $R(\alpha_{p-1} \wedge y) = \tilde{\alpha}_{p-1} \wedge Sy$. Now $S(x+y) = aSx + bSy$ since $\varphi_1([x+y]) \subset \varphi_2([x,y])$. Linearity of R implies that $\tilde{\alpha}_{p-1} \wedge (S(x+y) - Sx - Sy) = 0$, so $a = b = 1$ and $S(x+y) = Sx + Sy$. Similarly $S(cx) = cSx$.

Assume that we have defined S linear on V_i^\perp so that $Sx \in \varphi_1([x])$ for $x \neq 0$ and $R(\alpha_i \wedge x_{i+1} \wedge \cdots \wedge x_p) = \tilde{\alpha}_i \wedge Sx_{i+1} \wedge \cdots \wedge Sx_p$; then Proposition 6.4 implies $R(\alpha_{i-1} \wedge x_i \wedge \cdots \wedge x_p) = \lambda(x_i, \dots, x_p) \tilde{\alpha}_{i-1} \wedge Sx_i \wedge \cdots \wedge Sx_p$, where $\lambda(x_i, \dots, x_p)$ is a nonzero scalar depending on x_i, \dots, x_p independent in V_i^\perp . From Remark 6.5, applied to fixed x_i, \dots, x_{p-1} with x_p varying, we see that λ only depends on x_i, \dots, x_{p-1} —hence λ is constant by symmetry.

Replacing S by $1/aS$, where $a^{p-i+1} = \lambda$, we then have

$$R(\alpha_{i-1} \wedge x_i \wedge \cdots \wedge x_p) = \tilde{\alpha}_{i-1} \wedge Sx_i \wedge \cdots \wedge Sx_p,$$

for x_i, \dots, x_p independent in V_i^\perp . Now extend S to a linear map on V_{i-1}^\perp by setting $Se_i = a^{p-i} \tilde{e}_i$. Then $R(\alpha_{i-1} \wedge x_i \wedge \cdots \wedge x_p) = \tilde{\alpha}_{i-1} \wedge Sx_i \wedge \cdots \wedge Sx_p$, for x_i, \dots, x_p independent in V_{i-1}^\perp . We construct S inductively so that $R(x_1 \wedge \cdots \wedge x_p) = Sx_1 \wedge \cdots \wedge Sx_p$ for all x_1, \dots, x_p independent, and hence for all x_1, \dots, x_p . Thus $R = \bigwedge^p S$. Since R is invertible, S is one-to-one.

We need only show that S is bounded and onto when \mathcal{H} is infinite dimensional. We use that if W_1 and W_2 are disjoint finite dimensional subspaces of \mathcal{H} , and P_i is projection on W_i^\perp , then $\Pi = P_1 + P_2$ is invertible. To see this, let $W = W_1 + W_2$, so that $\Pi|_{W^\perp} = 2\text{Id}_{W^\perp}$. If $x \in W$, then $P_i x = x - w_i$ where $w_i \in W_i$, hence $\Pi W \subset W$. Since P_i is nonnegative, $\ker \Pi = \ker P_1 \cap \ker P_2 = 0$, so that $\Pi|_W$ is invertible.

Now fix x_1, \dots, x_{p-1} independent in \mathcal{H} . Let P denote projection on the orthogonal complement of the x_i , and Q projection on the orthogonal complement of the Sx_i . Let $\alpha = x_1 \wedge \cdots \wedge x_{p-1}$ and $\beta = Sx_1 \wedge \cdots \wedge Sx_{p-1}$. Then $\|\beta\| \|QSy\| = \|Sx_1 \wedge \cdots \wedge Sx_{p-1} \wedge Sy\| \leq \|R\| \|\alpha \wedge y\| = \|R\| \|\alpha\| \|Py\| \leq \|R\| \|\alpha\| \|y\|$. If we then take $\tilde{x}_1, \dots, \tilde{x}_{p-1}$ such that $x_1, \dots, x_{p-1}, \tilde{x}_1, \dots, \tilde{x}_{p-1}$ are independent, and let \tilde{P} and \tilde{Q} be defined for the \tilde{x}_i , then

$$\|(Q + \tilde{Q})Sy\| \leq \|R\|(\|\alpha\|/\|\beta\| + \|\tilde{\alpha}\|/\|\tilde{\beta}\|)\|y\|.$$

The boundedness of S follows from invertibility of $Q + \tilde{Q}$.

To see that S is invertible: let \mathcal{H}_0 be the closure of range S . The range of R is contained in $\bigwedge^p \mathcal{H}_0$, so $\mathcal{H}_0 = \mathcal{H}$, and S has dense range. But if Sy_n converges, then $R(\alpha \wedge y_n) = \beta \wedge Sy_n$ converges. This implies $\alpha \wedge y_n$ converges, and hence so does Py_n ; similarly, so does $\tilde{P}y_n$. Thus $(P + \tilde{P})y_n$ converges, which implies y_n converges and range S is closed. \square

ACKNOWLEDGMENT

Before beginning, I would like to thank Ron Donagi, James Faran, Mark Green, Richard Hain, and Harald Upmeyer for many useful discussions.

REFERENCES

- [C] W.-L. Chow, *On the geometry of algebraic homogeneous spaces*, Ann. of Math. **50** (1949), 32–67.
- [H] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977.
- [K] W. Kaup, *Über die Automorphismen Graßmannscher Mannigfaltigkeiten unendlicher Dimension*, Math. Z. **144** (1975), 75–96.
- [T] H. Tango, *On the automorphisms of Flag manifolds*, Bull. Kyoto Univ. of Ed., B, **49** (1976), 1–9.

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, BUFFALO, NEW YORK 14214