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AUTOMORPHISMS OF GRASSMANNIANS

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ABSTRACT. For a complex vector space $\mathscr{V}$ of dimension n, the group of holomorphic automorphisms of the Grassmannian $\mathrm{Gr}(p, \mathscr{V})$ can be identified with the subgroup of $\mathbf{P}\mathrm{Gl}(\bigwedge^p \mathscr{V})$ preserving the Grassmannian. Using this, Chow showed $\mathrm{Aut}(\mathrm{Gr}(p, \mathscr{V})) = \mathbf{P}\mathrm{Gl}(\mathscr{V})$ for $n \neq 2p$, and $\mathbf{P}\mathrm{Gl}(\mathscr{V})$ is a normal subgroup of index 2 in $\mathrm{Aut}(\mathrm{Gr}(p, \mathscr{V}))$ for $n = 2p$. We prove a version of Chow's result for a separable Hilbert space $\mathscr{H}$. Theorem. $\mathbf{P}\mathrm{Gl}(\mathscr{H})$ is the subgroup of $\mathbf{P}\mathrm{Gl}(\bigwedge^p \mathscr{H})$ which preserves $\mathrm{Gr}(p, \mathscr{H})$. That is, if R is an invertible linear operator on $\bigwedge^p \mathscr{H}$ which preserves decomposable p-vectors, then there exists S, an invertible linear operator on $\mathscr{H}$, such that $R = \bigwedge^p S$.

1. INTRODUCTION

Let $\mathscr{V}$ be a complex vector space of dimension n. Then a holomorphic automorphism of $\mathrm{Gr}(p, \mathscr{V})$, the Grassmannian of p-planes in $\mathscr{V}$, is induced by an endomorphism of $\bigwedge^p \mathscr{V}$ preserving decomposable p-vectors: $\mathrm{Aut}(\mathrm{Gr}(p, \mathscr{V})) = \mathbf{P}\mathrm{Gl}(\bigwedge^p \mathscr{V})_{\mathrm{Gr}(p, \mathscr{V})}$, the subgroup of $\mathbf{P}\mathrm{Gl}(\bigwedge^p \mathscr{V})$ preserving the Grassmannian. For example, S in $\mathrm{Gl}(\mathscr{V})$ induces an automorphism Φ_S of $\mathrm{Gr}(p, \mathscr{V})$, by sending a p-plane W into SW. Classically, Φ_S is called a collineation; the corresponding endomorphism of $\bigwedge^p \mathscr{V}$ is $\bigwedge^p S$. Chow [C] showed all automorphisms are collineations, except when the dual map $*: \mathrm{Gr}(p, \mathscr{V}) \to \mathrm{Gr}(n-p, \mathscr{V})$ is an automorphism—when $n = 2p$. In that case, there are the additional automorphisms of the form $* \circ \Phi_S$, the correlations:

Theorem 1.1 (Chow). $\mathrm{Aut}(\mathrm{Gr}(p, \mathscr{V})) = \mathbf{P}\mathrm{Gl}(\mathscr{V})$ *for* $\dim \mathscr{V} \neq 2p$. $\mathbf{P}\mathrm{Gl}(\mathscr{V})$ *is a normal subgroup of index* 2 *in* $\mathrm{Aut}(\mathrm{Gr}(p, \mathscr{V}))$ *for* $\dim \mathscr{V} = 2p$.

Using the Schubert calculus, Tango [T] gave an alternative proof to Theorem 1.1, generalizing Chow's result to automorphisms of flag manifolds. Kaup [K] used Lie theory to study the Grassmannian of a Banach space. In particular, he showed for a Hilbert space $\mathscr{H}$, every automorphism of $\mathrm{Gr}(p, \mathscr{H})$ is a collineation when $1 \leq p < \infty$.

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For $\mathscr{H}$ a separable Hilbert space, we consider the holomorphic map

$$\varphi_R\colon \operatorname{Gr}(p,\mathscr{H}) \to \operatorname{Gr}(p,\mathscr{H}),$$

induced by an invertible linear operator R on $\bigwedge^p \mathscr{H}$ which preserves decomposable p-vectors. Such operators arise naturally when studying holomorphic curves in $\operatorname{Gr}(p,\mathscr{H})$ which have the same curvature invariants. We denote by $\mathbf{P}\operatorname{Gl}(\bigwedge^p \mathscr{H})_{\operatorname{Gr}(p,\mathscr{H})}$ the subset of $\mathbf{P}\operatorname{Gl}(\bigwedge^p \mathscr{H})$ consisting of all φ_R which preserve the Grassmannian. If $\mathscr{H}$ is finite dimensional, then $\operatorname{Gr}(p,\mathscr{H})$ is compact and connected, so φ_R is an automorphism. When $\mathscr{H}$ is infinite dimensional, it does not follow directly from our assumptions that R^{-1} preserves the Grassmannian (so that $\mathbf{P}\operatorname{Gl}(\bigwedge^p \mathscr{H})_{\operatorname{Gr}(p,\mathscr{H})}$ is a subgroup of $\operatorname{Aut}\operatorname{Gr}(p,\mathscr{H})$.) Nonetheless, we prove a version of Chow's Theorem which shows

$$\mathbf{P}\operatorname{Gl}(\bigwedge^p \mathscr{H})_{\operatorname{Gr}(p,\mathscr{H})}$$

is indeed a subgroup of $\operatorname{Aut}\operatorname{Gr}(p,\mathscr{H})$:

Theorem 1.2. $\mathbf{P}\operatorname{Gl}(\bigwedge^p \mathscr{H})_{\operatorname{Gr}(p,\mathscr{H})} = \mathbf{P}\operatorname{Gl}(\mathscr{H})$. *That is, if R is an invertible linear operator on $\bigwedge^p(\mathscr{H})$ which preserves decomposable p-vectors, then R is a collineation: $R = \bigwedge^p S$ for an invertible linear operator S on $\mathscr{H}$.*

Note that R determines S up to multiplication by a scalar: Sx spans $\bigcap_{x\in W}\varphi_R(W)$, while Sx and $\widetilde{S}x$ dependent for all x implies $\widetilde{S} = cS$. Consequently, if R is unitary, then $\bigwedge^p(S^*S) = I$ implies $S^*S = cI$, so we have the following corollary:

Corollary 1.3. *If $\widetilde{U}$ is unitary on $\bigwedge^p \mathscr{H}$ and preserves decomposable p-vectors, then $\widetilde{U} = \bigwedge^p U$ for U unitary on $\mathscr{H}$.*

The proof of Theorem 1.2 uses only elementary linear algebra. Along the way we give a short proof of Chow's result using one additional well-known but non-elementary fact: the Picard group of the Grassmannian is $\mathbf{Z}$.

2. Preliminaries

That $\operatorname{Aut}\operatorname{Gr}(p,\mathscr{V})$ equals $\mathbf{P}\operatorname{Gl}(\bigwedge^p \mathscr{V})_{\operatorname{Gr}(p,\mathscr{V})}$ for $\mathscr{V}$ finite dimensional can be proved along the lines of the $\mathbf{P}_n$ case as found in [H]. The crucial fact needed is that the group of equivalence classes of holomorphic line bundles on $\operatorname{Gr}(p,\mathscr{V})$—the Picard group—is isomorphic to $\mathbf{Z}$. This follows from the homology of the Grassmannian in terms of Schubert cycles and from the Hodge decomposition: $H^1(\operatorname{Gr}(p,\mathscr{V}),\mathscr{A})$ equals $H^2(\operatorname{Gr}(p,\mathscr{V}),\mathscr{A}) = 0$, where $\mathscr{A}$ is the sheaf of germs of analytic functions; thus $H^1(\operatorname{Gr}(p,\mathscr{V}),\mathscr{A}^*) \cong H^2(\operatorname{Gr}(p,\mathscr{V}),\mathbf{Z}) \cong \mathbf{Z}$.

To fix a choice of the dual map $*$: let $e_1,\dots,e_n$ be a basis for $\mathscr{V}$ and $\delta_1,\dots,\delta_n$ the dual basis. Define $\imath\colon \mathscr{V}^* \to \mathscr{V}$ by $\imath(\delta_j) = e_j$, so that $j\circ\imath = \imath^t$, where $j\colon \mathscr{V} \to \mathscr{V}^{**}$ is the identification map. Then $\imath$ induces a biholomorphic mapping, the dual map $*\colon \operatorname{Gr}(p,\mathscr{V}) \to \operatorname{Gr}(n-p,\mathscr{V})$, as follows: if W is a

p-plane, then $*W = \imath(W^{\perp})$, where $W^{\perp}$ is the annihilator of W. Note $*$ is induced by an isomorphism of $\bigwedge^p \mathscr{V} \to \bigwedge^{n-p} \mathscr{V}$ (a basis p-vector e_I maps to the $(n-p)$-vector $\varepsilon_{IJ} e_J$, where $e_I \wedge e_J = \varepsilon_{IJ} e_1 \wedge \cdots \wedge e_n$). Furthermore, if $S\colon \mathscr{V}^* \to \mathscr{V}$ is a linear transformation, then $S(W^{\perp}) = ((S^t)^{-1} jW)^{\perp}$, so the following holds:

Lemma 2.1. *Let T be in $\mathrm{Gl}(\mathscr{V})$, and let Φ_T be the induced automorphism on $\mathrm{Gr}(p,\mathscr{V})$. Then $* \circ \Phi_T \circ *$ is the automorphism of $\mathrm{Gr}(n-p,\mathscr{V})$ which is induced by $\imath \circ (T^t) \circ \imath^{-1} \in \mathrm{Gl}(\mathscr{V})$. In particular, $*^2 = \mathrm{Id}$.*

Remark 2.2. If $\varphi \in \mathrm{Aut}(\mathrm{Gr}(p,\mathscr{V}))$, then $* \circ \varphi \circ * \in \mathrm{Aut}(\mathrm{Gr}(n-p,\mathscr{V}))$. It thus suffices to prove Theorem 1.1 for $\dim \mathscr{V} \geq 2p$.

3. Adjacency

Let $R\colon \bigwedge^p \mathscr{H} \to \bigwedge^p \mathscr{H}$ be a (bounded) invertible linear operator which preserves the decomposable elements of $\bigwedge^p \mathscr{H}$ (those of the form $v_1 \wedge \cdots \wedge v_p$), and let $\varphi_R\colon \mathrm{Gr}(p,\mathscr{H}) \to \mathrm{Gr}(p,\mathscr{H})$ be the induced map. The most important tool for analyzing R is what Chow calls adjacency: distinct p-planes V and W are *adjacent* if $\dim V \cap W = p-1$.

Lemma 3.1. *Let V and W be p-planes in $\mathscr{H}$ with α and β the corresponding decomposable p-vectors in $\bigwedge^p \mathscr{H}$. Then V and W are adjacent if and only if $\alpha + \beta$ is decomposable.*

Proof. Assume that $\alpha + \beta$ is decomposable—the implication in the other direction is trivial. If $\dim V \cap W = q < p-1$, mod out by $V \cap W$ and replace α and β by appropriate $(p-q)$-vectors; the problem reduces to showing $\dim V \cap W$ cannot equal 0 if $p > 1$ and $\alpha + \beta$ is decomposable.

Assume that $\dim V \cap W = 0$; then $V + W$ is $2p$-dimensional. Now $\alpha + \beta$ is in $\bigwedge^p(V+W)$, which implies that $\alpha + \beta = (v_1 + w_1) \wedge \cdots \wedge (v_p + w_p)$, where $v_i \in V$ and $w_i \in W$. Since $\alpha \wedge (w_1 \wedge \cdots \wedge w_p) = \alpha \wedge (\alpha + \beta) = \alpha \wedge \beta$, then $\beta = w_1 \wedge \cdots \wedge w_p$, and similarly $\alpha = v_1 \wedge \cdots \wedge v_p$; in particular, $v_1, \ldots, v_p, w_1, \ldots, w_p$ are independent. But $\alpha + \beta = v_1 \wedge \cdots \wedge v_p + v_1 \wedge w_1 \wedge \cdots \wedge v_p + \cdots + w_1 \wedge \cdots \wedge w_p$, which implies the mixed terms $v_1 \wedge w_1 \wedge \cdots \wedge v_p$, etc. sum to zero, a contradiction to independence of the v_i and w_i. Thus $\dim V \cap W$ cannot equal 0. □

Remark 3.2. If $\varphi_R\colon \mathrm{Gr}(p,\mathscr{H}) \to \mathrm{Gr}(p,\mathscr{H})$ is induced by $R\colon \bigwedge^p \mathscr{H} \to \bigwedge^p \mathscr{H}$ linear and preserving decomposables, then φ_R preserves adjacency.

Proof. If $\dim V \cap W = p-1$, let $\mu \in \bigwedge^{p-1} \mathscr{H}$ correspond to $V \cap W$; then V corresponds to $\mu \wedge v$ for $v \in \mathscr{H}$ and W corresponds to $\mu \wedge w$. Thus $R(\mu \wedge (v + w))$ is decomposable and the result follows from Lemma 3.1. □

4. Preserving/dualizing Schubert cycles

Let V be a q-plane in $\mathscr{H}$. For $p > q$, denote by $\sigma_p(V)$ the Schubert cycle defined by

$$\sigma_p(V) = \{W \in \mathrm{Gr}(p,\mathscr{H}) | V \subset W\},$$

and for $p < q$ denote by $\Sigma^p(V)$ the Schubert cycle defined by $\Sigma^p(V) = \{W \in \mathrm{Gr}(p, \mathscr{H}) | W \subset V\}$.

Proposition 4.1. *Let $\psi\colon \mathrm{Gr}(p,\mathscr{H}) \to \mathrm{Gr}(p,\mathscr{H})$ be a map (not necessarily continuous) preserving adjacency. Let V be a $(p-1)$-plane, W_1 and W_2 distinct p-planes containing V. Denote by $\psi_{p-1}(V)$ the $(p-1)$-plane $\psi(W_1) \cap \psi(W_2)$ and by $\psi^{p+1}(V)$ the $(p+1)$-plane $\psi(W_1) + \psi(W_2)$. Then either* (i) *ψ preserves the Schubert cycle: $\psi(\sigma_p(V)) \subset \sigma_p(\psi_{p-1}(V))$, so that $\psi_{p-1}(V)$ is independent of the choice of W_1 and W_2; or* (ii) *ψ dualizes the Schubert cycle: $\psi(\sigma_p(V)) \subset \Sigma^p(\psi^{p+1}(V))$, so that $\psi^{p+1}(V)$ is independent of the choice.*

Proof. If (i) does not hold, then there exists a p-plane W_3 containing V such that $\psi(W_3)$ does not contain $\psi_{p-1}(V)$. Let $\psi(W_i) = \widetilde{W}_i$; then $\widetilde{W}_3$ intersects $\widetilde{W}_1$ and $\widetilde{W}_2$ in different $(p-1)$-planes—otherwise $\widetilde{W}_1 \cap \widetilde{W}_2 \cap \widetilde{W}_3 = \psi_{p-1}(V)$—and hence $\widetilde{W}_3 \subset \psi^{p+1}(V)$. Let W be any p-plane containing V; then $\psi(W)$ intersects each of $\widetilde{W}_1$, $\widetilde{W}_2$, $\widetilde{W}_3$ in $(p-1)$-planes, which are not all three the same by the assumption of $\widetilde{W}_1 \cap \widetilde{W}_2 \cap \widetilde{W}_3 \neq \psi_{p-1}(V)$. Thus $\psi(W) \subset \psi^{p+1}(V)$, so (ii) holds. □

Corollary 4.2. *Let $\varphi_R\colon \mathrm{Gr}(p,\mathscr{H}) \to \mathrm{Gr}(p,\mathscr{H})$ be induced by $R\colon \bigwedge^p \mathscr{H} \to \bigwedge^p \mathscr{H}$ invertible and decomposable preserving. For $\dim \mathscr{H} > 2p$, there is a map $\varphi_{p-1}\colon \mathrm{Gr}(p-1,\mathscr{H}) \to \mathrm{Gr}(p-1,\mathscr{H})$ such that $\varphi_R(\sigma_p(V)) \subset \sigma_p(\varphi_{p-1}(V))$, with $\varphi_R(\sigma_p(V)) = \sigma_p(\varphi_{p-1}(V))$ if $\dim \mathscr{H} < \infty$. When $\dim \mathscr{H} = 2p$, there exists such a φ_{p-1} either for φ_R or for $* \circ \varphi$.*

Proof. We apply Proposition 4.1 to $\varphi = \varphi_R$, which preserves adjacency. Conditions (i) and (ii) of Proposition 4.1 are equivalent to (i)$'$ $R(\bigwedge^{p-1} V \wedge \mathscr{H}) \subset \bigwedge^{p-1} \varphi_{p-1}(V) \wedge \mathscr{H}$ and (ii)$'$ $R(\bigwedge^{p-1} V \wedge \mathscr{H}) \subset \bigwedge^p \varphi^{p+1}(V)$. Now (ii)$'$ implies $\dim \mathscr{H} - (p-1) \leq p+1$, so we are done if $\dim \mathscr{H} > 2p$. If $\dim \mathscr{H} = 2p$, then equality holds in (i)$'$ and (ii)$'$, so (i) and (ii) become $\varphi(\sigma_p(V)) = \sigma_p(\varphi_{p-1}(V))$ and $\varphi(\sigma_p(V)) = \Sigma^p(\varphi^{p+1}(V))$. Since φ is continuous, and the Schubert cycles $\sigma_p(\varphi_{p-1}(V))$ and $\Sigma^p(\varphi^{p+1}(V))$ represent distinct homology classes, only one of these alternatives holds for all $V \in \mathrm{Gr}(p,\mathscr{H})$. If necessary, we can compose R with the endomorphism of $\bigwedge^p \mathscr{H}$ which induces the dual map $*$ to achieve $\varphi(\sigma_p(V)) = \sigma_p(\varphi_{p-1}(V))$ for all V. □

5. Proof of theorem 1.1

Assume $\dim \mathscr{V} = n \geq 2p$, so that $\varphi_R(\sigma_p(W)) = \sigma_p(\varphi_{p-1}(W))$ for all $(p-1)$-planes W (if necessary, compose φ_R with $*$ when $n = 2p$). Thus φ_{p-1} is one-to-one. To show it is an automorphism, it suffices to show it is holomorphic, since $\mathrm{Gr}(p-1,\mathscr{V})$ is compact and connected. Coordinatize $\mathrm{Gr}(p-1,\mathscr{V})$ by letting $\psi(Z)$ be the span of the $v_i + \Sigma z_{ij} w_j$, where $v_1, \ldots, v_{p-1}, w_1, \ldots, w_{n-p+1}$ is a basis for $\mathscr{V}$. Let $W_i(Z)$ denote the p-plane spanned by $\psi(Z)$ and w_i. Then $W_1(Z) \cap W_2(Z) = \psi(Z)$ implies

$\varphi_{p-1}(\psi(Z)) = \varphi_R(W_1(Z)) \cap \varphi_R(W_2(Z))$ is holomorphic, since the intersection has constant dimension.

Theorem 1.1 is true for $p = 1$, so by induction assume φ_{p-1} is induced by an endomorphism S of $\mathscr{V}$. If W is a p-plane in $\mathscr{V}$, then $W = \bigcap_{V \subset W} \sigma_p(V)$, where $\dim V = p - 1$. Thus

$$\begin{aligned}\varphi(W) &= \bigcap_{V \subset W} \varphi(\sigma_p(V)) = \bigcap_{V \subset W} \sigma_{p-1}(\varphi_{p-1}(V)) \\ &= \bigcap_{V \subset W} \sigma_{p-1}(SV) = \bigcap_{V \subset SW} \sigma_{p-1}(V) = SW. \quad \square\end{aligned}$$

6. $\dim \mathscr{H} = \infty$

When $\mathscr{H}$ is infinite dimensional, we need R onto to establish that φ_{p-1} is one-to-one, as the following example shows: Let V be a $(p-1)$-plane in $\mathscr{H}$ and λ a $(p-1)$-vector corresponding to V. Let $T\colon \bigwedge^p \mathscr{H} \to V^\perp$ be an injective linear operator, and define an injective linear operator $R\colon \bigwedge^p \mathscr{H} \to \bigwedge^p \mathscr{H}$ by $R\alpha = \lambda \wedge T\alpha$. Trivially R preserves decomposables, yet the induced map φ_{p-1} is constant.

Lemma 6.1. *Let $\psi\colon \mathrm{Gr}(r, \mathscr{H}) \to \mathrm{Gr}(r, \mathscr{H})$ preserve adjacency, $r \geq 2$. Fix Z an $(r-2)$-plane and L an l-plane, $l \geq r - 1$. If $\dim L \cap \psi(W) \geq r - 1$ for all $W \in \sigma_r(Z)$, then $L \cap \psi(W) \neq 0$ for every r-plane W.*

Proof. We show for any r-plane W: $\dim W \cap Z = k$ implies $\dim L \cap \psi(W) \geq k+1$. This is trivial if $r = 2$. For $r > 2$, assume true for k, $0 < k \leq r-2$. Let $\widetilde{W}$ be an r-plane such that $\dim \widetilde{W} \cap Z = k - 1$. There is an r-plane W such that $\dim W \cap Z = k$ and $\dim W \cap \widetilde{W} = r - 1$. Thus $\dim \psi(W) \cap \psi(\widetilde{W}) = r - 1$ and $\dim L \cap \psi(\widetilde{W}) \geq k = (k-1) + 1$. $\square$

The main step in proving Theorem 1.2 is the following proposition:

Proposition 6.2. *For $1 \leq q < p$, assume that for all r with $q \leq r \leq p$ we have defined maps $\varphi_r\colon \mathrm{Gr}(r, \mathscr{H}) \to \mathrm{Gr}(r, \mathscr{H})$, where $\varphi_p = \varphi_R$, which satisfy the following for $r > q$: (1) φ_r preserves adjacency, and (2) $\varphi_r \sigma_r(V) \subset \sigma_r(\varphi_{r-1}(V))$, for each $(r-1)$-plane V. Then φ_q preserves adjacency. Furthermore, if $q > 1$, then we can define $\varphi_{q-1}\colon \mathrm{Gr}(q-1, \mathscr{H}) \to \mathrm{Gr}(q-1, \mathscr{H})$ so that (2) holds for $r = q$.*

Before proving Proposition 6.2, we use surjectivity of R to show:

Lemma 6.3. *Under the hypotheses of Proposition 6.2, if Z is a $(q-1)$-plane and L an l-plane, $l \geq q \geq 1$, then $\dim L \cap \varphi_{q+1}(W) < q$ for some $W \in \sigma_{q+1}(Z)$.*

Proof. If not, then Lemma 6.1 shows that $L \cap \varphi_{q+1}(W) \neq 0$ for all $(q+1)$-planes W. If U is a $(q+2)$-plane, then $\varphi_{q+2}(U)$ contains $\varphi_{q+1}(W)$ for any $(q+1)$-plane $W \subset U$, by hypothesis (2) of the Proposition, and hence $L \cap \varphi_{q+2}(U) \neq 0$. Inductively we obtain that $L \cap \varphi_R(V) \neq 0$ for all $V \in \mathrm{Gr}(p, \mathscr{H})$.

That is, if we let β be an r-vector which represents L, then $R(\alpha)\wedge\beta = 0$ for all α decomposable, and hence for all α in $\bigwedge^p \mathscr{H}$. This cannot happen since R is onto (and $\dim \mathscr{H} \geq r+p$). □

Proof of Proposition 6.2. If φ_q does not preserve adjacency, let V_1 and V_2 be adjacent q-planes such that $\dim \varphi_q(V_1)\cap\varphi_q(V_2) \neq q-1$. Since $W = V_1+V_2$ is a $(q+1)$-plane containing both V_1 and V_2, then (2) implies $\varphi_q(V_1) \subset \varphi_{q+1}(W)$, so $\varphi_q(V_1) = \varphi_q(V_2) = V_0$, say. Let $\widetilde{W}$ be any $(q+1)$-plane which contains $V_1 \cap V_2$, $\widetilde{W} \neq W$, and V be a q-plane in $\widetilde{W}$ which contains $V_1\cap V_2$ and is not contained in W. Let $W_i = V_i + V$ for $i = 1, 2$. Then W_1 and W_2 are distinct $(q+1)$-planes containing V, and $V_i \subset W_i$ implies that $\varphi_q(V_i) \subset \varphi_{q+1}(W_i)$ for $i = 1, 2$. Thus $\varphi_q(V) = \varphi_{q+1}(W_1) \cap \varphi_{q+1}(W_2) = V_0$ and hence $V_0 \subset \varphi_{q+1}(\widetilde{W})$, which contradicts Lemma 6.3 (with $L = V_0$). Thus φ_q preserves adjacency.

To show that (2) holds for $q > 1$: since φ_q preserves adjacency, then either $\varphi_q(\sigma_{q-1}(Z)) \subset \sigma_q(\varphi_{q-1}(Z))$ or $\varphi_q(\sigma_{q-1}(Z)) \subset \Sigma^q(\varphi_{q+1}(W))$. If the latter, then $\varphi_q(V) \subset \varphi_{q+1}(W)$ for all q-planes V which contain Z. But if $\widetilde{W}$ is any $(q+1)$-plane containing Z, let V be a q-plane with $Z \subset V \subset \widetilde{W}$. Then $\varphi_q(V) \subset \varphi_{q+1}(\widetilde{W})$, so $\dim \varphi_{q+1}(W) \cap \varphi_{q+1}(\widetilde{W}) \geq q$ for all $\widetilde{W}$ containing Z, contradicting Lemma 6.3 (with $L = \varphi_{q+1}(W)$). □

Proposition 6.4. *Let* $v_1, \dots, v_q$ *be independent. Then*

$$\varphi_q([v_1, \dots, v_q]) = \varphi_1([v_1]) + \cdots + \varphi_1([v_q]),$$

where $[v_1, \dots, v_r]$ *is the r-plane spanned by* $v_1, \dots, v_r$.

Proof. Since φ_1 preserves adjacency, then $\varphi_1([v_1])$ and $\varphi_1([v_2])$ are distinct lines in $\varphi_2([v_1, v_2])$, so the Proposition is true when $q = 2$. Assume true for q and $q-1$ with $2 \leq q < p$. If not true for $q+1$, then we may assume that $\varphi_1([v_{q+1}]) \subset \varphi_q([v_1, \dots, v_q])$. But $\varphi_1([v_{q+1}]) \subset \varphi_q([v_1, \dots, v_{q-1}, v_{q+1}])$. Thus $\varphi_1([v_{q+1}]) \subset \varphi_{q-1}([v_1, \dots, v_{q-1}])$ by definition of φ_{q-1}, so that

$$\varphi_q([v_1, \dots, v_{q-1}, v_{q+1}]) \neq \varphi_1([v_1]) + \cdots + \varphi_1([v_{q-1}]) + \varphi([v_{q+1}]),$$

contrary to assumption. □

Remark 6.5. If V and W are vector spaces, T and $\widetilde{T}\colon V \to W$ linear transformations such that for each $x \in V$, $\widetilde{T}x = c_x Tx$, where $c_x \neq 0$ is a scalar, then $\widetilde{T} = cT$. (Assume $\ker T = 0$, then use $c_x Tx + c_y Ty = c_{x+y}(Tx+Ty)$ to get $c_x = c_y$.)

Proof of Theorem 1.2. Fix $e_1, \dots, e_{p-1}$ orthonormal in $\mathscr{H}$, and choose $\tilde{e}_i \in \varphi_1([e_i])$. Let $\alpha_i = e_1 \wedge \cdots \wedge e_i$, $\tilde{\alpha}_i = \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_i$, and V_i be spanned by $e_1, \dots, e_i$. By Proposition 6.4, for $x \neq 0$ in $V_{p-1}^{\perp}$ there is a unique $Sx \in \varphi_1([x])$ such that $R(\alpha_{p-1} \wedge x) = \tilde{\alpha}_{p-1} \wedge Sx$. Define $S0 = 0$.

We show S is linear: if x and y in $V_{p-1}^{\perp}$ are independent, then so are $R(\alpha_{p-1} \wedge x) = \tilde{\alpha}_{p-1} \wedge Sx$ and $R(\alpha_{p-1} \wedge y) = \tilde{\alpha}_{p-1} \wedge Sy$. Now $S(x+y) = aSx + bSy$ since $\varphi_1([x+y]) \subset \varphi_2([x, y])$. Linearity of R implies that $\tilde{\alpha}_{p-1} \wedge (S(x+y) - Sx - Sy) = 0$, so $a = b = 1$ and $S(x+y) = Sx + Sy$. Similarly $S(cx) = cSx$.

Assume that we have defined S linear on $V_i^{\perp}$ so that $Sx \in \varphi_1([x])$ for $x \neq 0$ and $R(\alpha_i \wedge x_{i+1} \wedge \cdots \wedge x_p) = \tilde{\alpha}_i \wedge Sx_{i+1} \wedge \cdots \wedge Sx_p$; then Proposition 6.4 implies $R(\alpha_{i-1} \wedge x_i \wedge \cdots \wedge x_p) = \lambda(x_i, \ldots, x_p)\tilde{\alpha}_{i-1} \wedge Sx_i \wedge \cdots \wedge Sx_p$, where $\lambda(x_i, \ldots, x_p)$ is a nonzero scalar depending on $x_i, \ldots, x_p$ independent in $V_i^{\perp}$. From Remark 6.5, applied to fixed $x_i, \ldots, x_{p-1}$ with x_p varying, we see that λ only depends on $x_i, \ldots, x_{p-1}$—hence λ is constant by symmetry.

Replacing S by $1/aS$, where $a^{p-i+1} = \lambda$, we then have

$$R(\alpha_{i-1} \wedge x_i \wedge \cdots \wedge x_p) = \tilde{\alpha}_{i-1} \wedge Sx_i \wedge \cdots \wedge Sx_p,$$

for $x_i, \ldots, x_p$ independent in $V_i^{\perp}$. Now extend S to a linear map on $V_{i-1}^{\perp}$ by setting $Se_i = a^{p-i}\tilde{e}_i$. Then $R(\alpha_{i-1} \wedge x_i \wedge \cdots \wedge x_p) = \tilde{\alpha}_{i-1} \wedge Sx_i \wedge \cdots \wedge Sx_p$, for $x_i, \ldots, x_p$ independent in $V_{i-1}^{\perp}$. We construct S inductively so that $R(x_1 \wedge \cdots \wedge x_p) = Sx_1 \wedge \cdots \wedge Sx_p$ for all $x_1, \ldots, x_p$ independent, and hence for all $x_1, \ldots, x_p$. Thus $R = \bigwedge^p S$. Since R is invertible, S is one-to-one.

We need only show that S is bounded and onto when $\mathscr{H}$ is infinite dimensional. We use that if W_1 and W_2 are disjoint finite dimensional subspaces of $\mathscr{H}$, and P_i is projection on $W_i^{\perp}$, then $\Pi = P_1 + P_2$ is invertible. To see this, let $W = W_1 + W_2$, so that $\Pi|_{W^\perp} = 2\,\mathrm{Id}_{W^\perp}$. If $x \in W$, then $P_i x = x - w_i$ where $w_i \in W_i$, hence $\Pi W \subset W$. Since P_i is nonnegative, $\ker \Pi = \ker P_1 \cap \ker P_2 = 0$, so that $\Pi|_W$ is invertible.

Now fix $x_1, \ldots, x_{p-1}$ independent in $\mathscr{H}$. Let P denote projection on the orthogonal complement of the x_i, and Q projection on the orthogonal complement of the Sx_i. Let $\alpha = x_1 \wedge \cdots \wedge x_{p-1}$ and $\beta = Sx_1 \wedge \cdots \wedge Sx_{p-1}$. Then $\|\beta\|\,\|QSy\| = \|Sx_1 \wedge \cdots \wedge Sx_{p-1} \wedge Sy\| \leq \|R\|\,\|\alpha \wedge y\| = \|R\|\,\|\alpha\|\,\|Py\| \leq \|R\|\,\|\alpha\|\,\|y\|$. If we then take $\tilde{x}_1, \ldots, \tilde{x}_{p-1}$ such that $x_1, \ldots, x_{p-1}, \tilde{x}_1, \ldots, \tilde{x}_{p-1}$ are independent, and let $\widetilde{P}$ and $\widetilde{Q}$ be defined for the $\tilde{x}_i$, then

$$\|(Q + \widetilde{Q})Sy\| \leq \|R\|(\|\alpha\|/\|\beta\| + \|\tilde{\alpha}\|/\|\tilde{\beta}\|)\|y\|.$$

The boundedness of S follows from invertibility of $Q + \widetilde{Q}$.

To see that S is invertible: let $\mathscr{H}_0$ be the closure of range S. The range of R is contained in $\bigwedge^p \mathscr{H}_0$, so $\mathscr{H}_0 = \mathscr{H}$, and S has dense range. But if Sy_n converges, then $R(\alpha \wedge y_n) = \beta \wedge Sy_n$ converges. This implies $\alpha \wedge y_n$ converges, and hence so does Py_n; similarly, so does $\widetilde{P}y_n$. Thus $(P + \widetilde{P})y_n$ converges, which implies y_n converges and range S is closed. □

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