A SEMI-FREDHOLM PRINCIPLE
FOR PERIODICALLY FORCED SYSTEMS
WITH HOMOGENEOUS NONLINEARITIES

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Abstract. We show that if the potential in a second-order Newtonian system
of differential equations is positively homogeneous of degree two and positive
semidefinite, and if the unforced system has no nontrivial $T$-periodic solutions
($T > 0$), then for any continuous $T$-periodic forcing, there is at least one
$T$-periodic solution.

In this note we consider a system of second-order differential equations in
which the nonlinear terms have one nice property shared by linear functions:
they are positively homogeneous of degree one. One such system arises from
an idealized model of a suspension bridge considered previously in [4 and 7].

Consider the partial differential equation
\[ U_{tt} + a^2 U_{xxxx} + k(x) U^+ = f(x, t), \]
where $U = U(x, t)$, $0 < x < L$, with boundary conditions
\[ U(0, t) = U(L, t) = U_{xx}(0, t) = U_{xx}(L, t) = 0. \]
If $k(x) \equiv 0$, we have the well-known equation for the transverse vibrations of
a beam of length $L$, which is hinged at the endpoints, subject to an external
force given by $f(x, t)$. If the beam is suspended from above by cables, whose
strengths may vary from point to point, and the displacement $U$ is measured
in the downward direction, we add the term $k(x) U^+$, where $k(x) \geq 0$ and
$U^+$ is the positive part of $U$. This accounts for the fact that cables exert no
restoring force when slack.

When $f$ is periodic in $t$ it is natural to look for time-periodic solutions. We
consider an approximate problem obtained by discretizing in the space variable
$x$. If $N$ is a large positive integer and we consider the mesh points $x_k = kL/N$,
k = 1, $0 \leq k \leq N - 1$, we are lead to the system of ordinary differential equations
\[ u''(t) + Au(t) + Du(t)^+ = p(t), \]
where \( u(t) \) is the \((N-1)\)-dimensional column vector whose \( k \) th component is \( U(x_k,t) \), \( p(t) \) is the vector with \( k \) th component \( f(x_k,t) \), \( u(t)^+ \) has its \( k \) th component equal to \( U(x_k,t)^+ \), \( D \) is a diagonal matrix with nonnegative entries, and \( A \) is the symmetric, positive-definite matrix corresponding to the finite difference approximation to \( U_{xxxx} \), taking into account the boundary conditions. We consider a more general class of systems.

Let \( V \in C^1(\mathbb{R}^n,\mathbb{R}) \) be positively homogeneous of degree two, i.e., \( V(tx) = t^2V(x) \) for \( t \geq 0 \) and \( x \in \mathbb{R}^n \) \((n \geq 1)\). If, for brevity, we set \( V'(x) = \nabla V(x) \), then \( V' \) is positively homogeneous of degree one, i.e.,

\[
V'(tx) = tV'(x)
\]

for \( t \geq 0 \) and \( x \in \mathbb{R}^n \).

**Theorem 1.** Let \( V \in C^1(\mathbb{R}^n,\mathbb{R}) \) be positively homogeneous of degree two and positive semidefinite, i.e.,

\[
V(x) \geq 0.
\]

Let \( T > 0 \). If the system

\[
u''(t) + V'(u(t)) = 0
\]

has no \( T \)-periodic solution other than \( u(t) = 0 \), then for any \( T \)-periodic function \( p \in C^1(\mathbb{R},\mathbb{R}^n) \) the system

\[
u''(t) + V'(u(t)) = p(t)
\]

has at least one \( T \)-periodic solution.

**Proof.** We first assume the stronger condition that \( V \) be positive definite, i.e.,

\[
x \in \mathbb{R}^n \Rightarrow V(x) > 0.
\]

We first show that if \( p \) and \( V \) are as above, then for \( \epsilon > 0 \) the system

\[
u''(t) + \epsilon u'(t) + V'(u(t)) = p(t)
\]

has at least \( T \)-periodic solution. To do this we use the Leray-Schauder continuation method (see, for example, [1 or 6]). Let \( | \cdot | \) denote the usual Euclidean norm on \( \mathbb{R}^n \), and let \( C_T \) and \( C^1_T \) denote the Banach spaces of \( T \)-periodic functions which are continuous and continuously differentiable respectively with norms

\[
|v|_\infty = \sup_{[0,T]} |v(t)|, \quad v \in C_T,
\]

\[
|v|_1 = |v|_\infty + |v'|_\infty, \quad v \in C^1_T.
\]

Since the second-order linear homogeneous differential system

\[
u''(t) + \epsilon u'(t) + u(t) = 0
\]
\((u \in \mathbb{R}^n)\) has no \(T\)-periodic solution other than \(u(t) \equiv 0\), it follows that, for every \(f \in \mathcal{C}_T\), there exists a unique \(T\)-periodic solution of
\[
u''(t) + \varepsilon u'(t) + u(t) = f(t).
\]
Moreover, if we denote the unique \(T\)-periodic solution of the last system by \(K(f)\), then \(K\) may be viewed as a compact linear operator from \(\mathcal{C}_T\) into itself. Let \(N: \mathcal{C}_T \to \mathcal{C}_T\) be the completely continuous operator defined by
\[
N(u) = K(u + p - V'(u)).
\]
We claim that there exists a number \(R > 0\) such that if \(\lambda \in [0, 1]\) and \(u \in \mathcal{C}_T\), then
\[
u = \lambda N(u)
\]
implies that \(\|u\|_{\infty} \leq R\).

Since (1.7) holds if and only if
\[
u''(t) + \varepsilon u'(t) + (1 - \lambda)u(t) + \lambda V'(u(t)) = \lambda p(t),
\]
and since \(\|u\|_{\infty} \leq \|u\|_1\) if \(u \in \mathcal{C}_T^1\), assuming that the above claim is false, we infer the existence of a sequence \(\{u_m\}_{m=1}^\infty\) and a corresponding sequence of numbers \(\{\lambda_m\}_{m=1}^\infty\) such that \(u_m(t)\) is a solution of (1.8) when \(\lambda = \lambda_m\) for \(m = 1, 2, \ldots\), and
\[
u = \lambda N(u)
\]
implies that \(\|u\|_{\infty} \leq R\).

Setting \(w_m(t) = u_m(t)/\|u_m\|_1\) for \(m = 1, 2, \ldots\), it follows by homogeneity of \(V'\) that
\[
w''_m(t) + \varepsilon w'_m(t) + (1 - \lambda_m)w_m(t) + \lambda_m V'(w_m(t)) = \lambda_m p(t)/\|u_m\|_1
\]
for \(m = 1, 2, \ldots\). Since \(\|w_m\|_1 = 1\) for \(m \geq 1\), it follows from (1.10) that the sequence \(\{w_m\}_{m=1}^\infty\) is bounded. Therefore, both of the sequences \(\{w_m(t)\}_{m=1}^\infty\) and \(\{w'_m(t)\}_{m=1}^\infty\) are equicontinuous and uniformly bounded on \((-\infty, \infty)\) so, by Ascoli's lemma, there exists a subsequence \(\{w_{m_k}(t)\}_{k=1}^\infty\) and a \(w \in \mathcal{C}_T^1\) with \(\|w\|_1 = 1\) such that \(w_m(t) \to w(t)\), \(w'_m(t) \to w'(t)\) as \(m \to \infty\), uniformly on \((-\infty, \infty)\). Since \(0 \leq \lambda_{m_k} \leq 1\) for all \(k \geq 1\), we may assume without loss of generality that \(\lambda_{m_k} \to \lambda^* \in [0, 1]\) as \(k \to \infty\). Therefore, from (1.10), it follows that the sequence \(\{w_{m_k}(t)\}_{k=1}^\infty\) converges uniformly on \((-\infty, \infty)\) so \(w\) is of class \(C^2\) and
\[
w''(t) + \varepsilon w'(t) + (1 - \lambda^*)w(t) + \lambda^* V'(w(t)) = 0.
\]
Taking the inner product of (1.11) with \(w'(t)\) and observing that
\[
\int_0^T (w'(t), w''(t)) dt = \frac{1}{2} \int_0^T \frac{d}{dt} |w'(t)|^2 dt = 0
\]
and
\[
\int_0^T (w'(t), (1 - \lambda^*)w(t) + \lambda^* V'(w(t))) dt
\]
\[
= \int_0^T \frac{d}{dt} [(1 - \lambda^*) |w(t)|^2/2 + \lambda^* V(w(t))] dt = 0,
\]
we find that
\[ \varepsilon \int_0^T |w'(t)|^2 \, dt = 0. \]
Hence, \( w(t) = \xi = \text{constant} \) and according to (1.11)
\[ (1 - \lambda^*) \xi + \lambda^* V'(\xi) = 0. \]
Taking the inner product of this last equation with \( \xi \) and using (1.2), we have
\[ (1 - \lambda^*) |\xi|^2 + 2\lambda^* V(\xi) = 0. \]
Thus, since \( 0 \leq \lambda^* \leq 1 \), it follows from (1.3)* that \( \xi = 0 \). Since, this contradicts the fact that \( |w|_1 = 1 \), the claim that there exists \( R \) independent of \( u \in C_T \) and \( \lambda \in [0, 1] \) such that (1.7) implies that \( |u|_\infty < R \) has been established.

From the Leray-Schauder-Shaefer theorem, it follows that for each \( \lambda \in [0, 1] \) there exists a \( u \in C_T \) which satisfies (1.7). (See, for example, [9, 1, p. 61, or 6, p. 71]). In particular since (1.7) has a solution \( u \) when \( \lambda = 1 \), it follows that (1.6) has at least one \( T \)-periodic solution.

Let \( \{\varepsilon_m\}_1^\infty \) be a sequence of positive numbers such that \( \varepsilon_m \to 0 \) as \( m \to \infty \). By what has been shown for each \( m = 1, 2, \ldots \), there exists \( u_m \in C_T \) such that \( u_m \) is a solution of (1.6) when \( \varepsilon = \varepsilon_m \). We claim that the sequence \( \{|u_m|_1\}_1^\infty \) is bounded. Assuming the contrary, we may suppose without loss of generality that \( |u_m|_1 \to \infty \) as \( m \to \infty \). Setting \( z_m(t) = u_m(t)/|u_m|_1 \), for \( m \geq 1 \), we have, by positive homogeneity of \( V' \),
\[ z_m''(t) + \varepsilon_m z_m'(t) + V'(z_m(t)) = p((t)/|u_m|_1) \]
for \( m = 1, 2, \ldots \). From this it follows that the sequences \( \{z_m(t)\}_1^\infty \) and \( \{z'_m(t)\}_1^\infty \) are equicontinuous and uniformly bounded on \( (-\infty, \infty) \) so there exists \( z \in C_T \) with \( |z|_1 = 1 \) and a subsequence \( \{z_{m_k}(t)\}_1^\infty \) of \( \{z_m(t)\}_1^\infty \) such that \( z_{m_k}(t) \to z(t) \) and \( z_{m_k}'(t) \to z'(t) \) as \( k \to \infty \) uniformly with respect to \( t \in (-\infty, \infty) \). From (1.12), we infer that the sequence \( \{z''_{m_k}(t)\}_1^\infty \) converges uniformly on \( (-\infty, \infty) \). Hence \( z \) is of class \( C^2 \) and
\[ z''(t) + V'(z(t)) = 0. \]
Since \( |z|_1 = 1 \), this contradicts the assumption that (1.4) has no nontrivial \( T \)-periodic solution, our claim that the sequence \( \{|u_m|_1\}_1^\infty \) is bounded has been established.

From the differential equation
\[ u_m''(t) + \varepsilon_m u_m'(t) + V'(u_m(t)) = p(t), \]
it follows that the sequence \( \{u_m(t)\}_1^\infty \) is also uniformly bounded on \( (-\infty, \infty) \). Therefore, using the same type of argument used above, we infer the existence of a subsequence of \( \{u_m(t)\}_1^\infty \) such that this subsequence, as well as the corresponding sequences of first and second derivatives converges uniformly on
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Since the limit of this subsequence is a $T$-periodic solution of (1.5), the proof of the theorem under assumption (1.3)* is complete.

To prove the theorem with (1.3)* replaced by (1.3), we observe that for all sufficiently small $\delta > 0$, the system

$$u''(t) + \delta u(t) + V'(u(t)) = 0$$

has no nontrivial $T$-periodic solution. Indeed, in the contrary case, for $\delta > 0$ arbitrarily small, we could find a solution with $C_T$-norm equal to 1. A compactness argument, similar to those used above, would give a nontrivial solution of (1.4), contradicting one of our hypotheses.

Therefore, since $\delta|x|^2/2 + V(x) > 0$ for $x \neq 0$, by what we have shown, for small $\delta > 0$ there exists a $T$-periodic solution of

$$u'' + \delta u + V'(u) = p(t).$$

The $C_T$-norms of these solutions are bounded as $\delta \to 0$ since (1.4) has no nontrivial $T$-periodic solution. Therefore, by the same type of compactness argument as used above, we obtain a $T$-periodic solution of (1.5).

Remark. It does not seem possible to prove the theorem more directly by connecting (1.5) rather than (1.6) to a linear equation by a homotopy.

Remark. The theorem remains true if the condition (1.3) is replaced by

$$(1.3)' \quad x \in \mathbb{R}^N, \quad x \neq 0 \Rightarrow V(x) < 0.$$  

In the proof one would consider the parameter-dependent differential equation

$$u''(t) + \varepsilon u'(t) - (1 - \lambda)u(t) + \lambda V'(u(t)) = \lambda p(t)$$

instead of (1.8) and use the fact that the linear differential equation

$$u''(t) + \varepsilon u'(t) - u(t) = f(t),$$

where $\varepsilon > 0$, has a unique $T$-periodic solution for any $f \in C_T$.

Examples 1. The homogeneous nonlinearity in the differential equation

$$(1.13) \quad u''(t) + |u(t)| = p(t) \equiv p(t + T),$$

where $n = 1$, does not satisfy the condition (1.3), since in this case $V(x) = \frac{1}{2}(\text{sgn} x)x^2$. If $p(t) \neq 0$ and $u(t)$ is a $T$-periodic solution of (1.13), then

$$\int_0^T |u(t)| \, dt = \int_0^T p(t) \, dt,$$

so, in order that (1.13) have a $T$-periodic solution, it is necessary that the mean value of $p$ be positive. If $q(t)$ is a continuous $T$-periodic function with mean value zero, and $p(t) = c + q(t)$ where $c$ is a constant, then by using the well-known upper and lower solution method for periodic solutions of second-order differential equations (see, for example, [10]) one can easily adapt the methods of [2] (which concerns a boundary value problem for a P.D.E.) to prove the
existence of $\bar{c} = \bar{c}(q)$ such that (1.13) has a $T$-periodic solution if and only if $c \geq \bar{c}$.

2. Consider the differential equation

$$(1.14) \quad u'' + bu^+ - au^- = p(t) \equiv p(t + 2\pi),$$

where $n = 1$, $a > 0$ and $b > 0$. If $u$ is a nontrivial solution of

$$(1.15) \quad u'' + bu^+ - au^- = 0,$$

then the distance between two consecutive zeros of $u$ which border an interval on which $u$ is positive is $\pi/\sqrt{b}$, since $u'' + bu = 0$ on such an interval. Similarly the distance between two consecutive zeros of $u$ which border an interval on which $u$ is negative is $\pi/\sqrt{a}$. It follows that every nontrivial solution of (1.15) is periodic with least period $\pi/\sqrt{a} + \pi/\sqrt{b}$. Therefore, since the potential for the nonlinearity in (1.15), $V(x) = \frac{b(x^+)^2 + a(x^-)^2}{2}$, satisfies (1.3), it follows that if

$$\pi/\sqrt{b} + \pi/\sqrt{a} \neq 2\pi/m$$

for $m = 1, 2, \ldots$, then, for any continuous $2\pi$-periodic function $p(t)$, (1.14) has a $2\pi$-periodic solution.

3. Suppose that for some integer $m \geq 1$ we have

$$\pi/\sqrt{b} + \pi/\sqrt{a} = 2\pi/m,$$

and in addition that

$$(1.16) \quad (m-1)^2 < a < b < (m+1)^2.$$ 

Let $H$ be the Hilbert space consisting of $2\pi$-periodic functions defined on $(-\infty, \infty)$ whose restrictions to $[-\pi, \pi]$ belong to $L^2[-\pi, \pi]$ with the $L^2[\pi]$ inner product. Let $W$ be the two-dimensional subspace of $H$ spanned by $\cos mt$ and $\sin mt$ and let $P: H \rightarrow W$ denote orthogonal projection. Since the spectrum of the linear operator $A: D(A) \subset H \rightarrow H$ defined by $Au = -u''$ is $\{k^2 | k = 0, 1, \ldots\}$ it follows from the Liapunov-Schmidt technique and (1.16) that for any $w_1 \in W$ there exists a unique $w_2 \in (I - P)W$ such that

$$(1.17) \quad w_2'' + (I - P)[b(w_1 + w_2)^+ - a(w_1 + w_2)^-] = 0$$

and $w_2(t + 2\pi) \equiv w_2(t)$. (See [5], or the proof of Proposition 2.1 of [3] for more details.) If $u_0$ is a nonzero solution of (1.15), then $u_0$ is $(2\pi/m)$-periodic and $Pu_0 \neq 0$, for otherwise, since $w_2 \equiv 0$ solves (1.17) when $w_1 \equiv 0$, we would have $(I - P)u_0$, and hence $u_0$, identically zero.

We claim that if $c^2 + d^2 \neq 0$ then there exists no $2\pi$-periodic solution of

$$(1.18) \quad u'' + bu^+ - au^- = c \cos mt + d \sin mt.$$ 

Assuming, on the contrary, that there exists a $2\pi$-periodic solution $u^*$ of (1.18), the reasoning used above shows that $Pu^* \neq 0$. Therefore, since both $Pu_0$ and $Pu^*$ are linear combinations of $\cos mt$ and $\sin mt$, both have the form
\( r \sin(mt + \delta) \) for some \( r > 0 \). Therefore there exist numbers \( \alpha > 0 \) and \( \gamma \) such that \( \dot{u}(t) = \alpha u_0(t + \gamma) \), then \( Pu = Pu^\ast \). Therefore, since by homogeneity \( \dot{u} \) is also a solution of (1.15), it follows that if \( w_1 = P\dot{u} \), then both \( w_2 = (I - P)\dot{u} \) and \( w_2 = (I - P)u^\ast \) solve (1.17), so by uniqueness \( (I - P)\dot{u} = (I - P)u^\ast \). Therefore we have the absurdity \( \dot{u} = u^\ast \). This contradiction proves the claim.

This phenomenon is, of course, well known as \textit{resonance} in the linear case \( a = b = m^2 \).

The work of Podolak [8] shows that if \( n = 1 \) and periodic boundary conditions are replaced by Dirichlet boundary conditions, then the statement of the theorem does not remain true.

**References**


\textit{Added in Proof}. The result in Example 2 is also in: