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THE PSEUDO-ORBIT TRACING PROPERTY AND EXPANSIVENESS ON THE CANTOR SET

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ABSTRACT. The set of all the expansive homeomorphisms with the pseudo-orbit tracing property is dense in the space of all the homeomorphisms of the Cantor set with the topology of uniform convergence. Moreover a topologically transitive (resp. mixing) homeomorphism of the Cantor set is approximated uniformly by topologically transitive (resp. mixing) expansive homeomorphisms with the pseudo-orbit tracing property.

1. INTRODUCTION

Let $\mathscr{H}$ be the space of all the homeomorphisms of the Cantor set C in $[0,1]$ with the topology of uniform convergence. It was shown by M. Sears [3] that the set $\mathscr{E}$ of all the expansive homeomorphisms of C is dense in $\mathscr{H}$. And M. Dateyama [2] showed that the set $\mathscr{P}$ of all the homeomorphisms of C with the pseudo-orbit tracing property (abbrev. POTP) is dense in $\mathscr{H}$. The purpose of this paper is to show that the set $\mathscr{S}$ of all the homeomorphisms which are topologically conjugate to subshifts of finite type is also dense in $\mathscr{H}$. Since $\mathscr{S} = \mathscr{E} \cap \mathscr{P}$ as being shown later, it is a generalization of the results above.

Given an integer $r \geq 1$, we call $[i3^{-r}, (i+1)3^{-r}] \cap C, (i = 0, 1, \dots, 3^{-r} - 1)$ a Cantor subinterval of rank r if $(i \cdot 3^{-r}, (i+1) \cdot 3^{-r}) \cap C \neq \varnothing$. Order the subintervals of rank r by the usual ordering of their left-hand endpoints and denote the kth in this order by $I(k, r)$ $(k = 1, 2, \dots, 2^r)$. Note that $\operatorname{diam} I(k, r) = 3^{-r}$. A Cantor subinterval is homeomorphic to C. More generally, a compact metrizable totally disconnected perfect space is homeomorphic to C.

Let n be a positive integers and let $S_n = \{1, 2, \dots, n\}$ with the discrete topology. We put

$$\Sigma_n = \{x\,; x = (x_i)_{i \in \mathbf{Z}}, x_i \in S_n (i \in \mathbf{Z})\}$$

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with the product topology. Then Σ_n is a compact metrizable totally disconnected perfect space. The shift map $\sigma_n : \Sigma_n \to \Sigma_n$ is defined by $(\sigma_n(x))_i = x_{i+1}$ $(i \in \mathbf{Z})$, where $x = (x_i)_{i\in\mathbf{Z}} \in \Sigma_n$. Then σ_n is a homeomorphism and the pair (Σ_n, σ_n) is called a *full shift* of n symbols. Let $\Lambda \subset \Sigma_n$ be a closed σ_n-invariant set, i.e., $\sigma_n(\Lambda) = \Lambda$. Then the restriction $(\Lambda, \sigma_n|_\Lambda)$ of σ_n on Λ is called a *subshift*. Let A be an $n \times n$ matrix of 0's and 1's. We put

$$\Sigma_A = \{x \in \Sigma_n ; x = (x_i)_{i\in\mathbf{Z}},\ A_{x_i, x_{i+1}} = 1,\ i \in \mathbf{Z}\}$$

and $\sigma_A = \sigma_n|_{\Sigma_A}$. Then (Σ_A, σ_A) is a subshift. A subshift $(\Lambda, \sigma_m|_\Lambda)$ $(m \geq 1)$ is said to be of *finite type* if it is topologically conjugate to (Σ_A, σ_A) for some $n \times n$ matrix A of 0's and 1's $(n \geq 1)$.

A homeomorphism $f \in \mathscr{H}$ is *topologically transitive* if given nonempty open sets U and V of C, $U \cap f^n(V) \neq \varnothing$ for some integer n. An $f \in \mathscr{H}$ is *topologically mixing* if given nonempty open sets U and V of C, there is an $L > 0$ such that $U \cap f^l(V) \neq \varnothing$ for all $l \geq L$. An $f \in \mathscr{H}$ is topologically transitive if and only if it has a dense orbit $o_f(x)$ $(x \in C)$.

Theorem. *$\mathscr{S}$ is dense in $\mathscr{H}$. Moreover, if $f \in \mathscr{H}$ is topologically transitive (resp. mixing), then f is approximated uniformly by elements of $\mathscr{S}$ which are also topologically transitive (resp. mixing).*

2. Preliminaries

For a homeomorphism of a compact space, the expansiveness and POTP is independent of the metric used. A homeomorphism of a Cantor set is expansive if and only if it is topologically conjugate to a subshift. And a subshift is of finite type if and only if it has POTP (Theorem 1. of P. Walters [4]). Therefore, we get the following

Proposition 1. $\mathscr{E} \cap \mathscr{P} = \mathscr{S}$.

Definition. An $n \times n$ matrix A of 0's and 1's is irreducible if, for every a, b $(1 \leq a, b \leq n)$, there is an $l > 0$ such that $A^l_{a,b} > 0$, where $A^l_{a,b}$ is the (a, b) component of the matrix $A^l = A \times \cdots \times A$ (l-times).

Remark. If A is an irreducible $n \times n$ matrix, then, for every $L > 0$ and $1 \leq a, b \leq n$, there is an $l \geq L$ such that $A^l_{a,b} > 0$.

The following Lemma 2 is well known, so we omit a proof.

Lemma 2. *Let A be an irreducible $n \times n$ matrix (of 0's and 1's). Then for any nonempty open sets U and V of Σ_A, there is an arbitrarily large $m > 0$ such that $\sigma_A^m(U) \cap V \neq \varnothing$.*

Lemma 3. *Let A be a nondegenerate $n \times n$ matrix of 0's and 1's. Then (Σ_A, σ_A) is topologically mixing if and only if there is an $m > 0$ such that $A^m > 0$ (i.e., $A^m_{a,b} > 0$ for all $1 \leq a, b \leq n$).*

Proof. Lemma 1.3 of R. Bowen [1].

3. A PROOF OF THE MAIN RESULT

The following is a proof of our main result. Let $f \in \mathscr{H}$ and $\varepsilon > 0$. Fix an integer $r > 0$ with $3^{-r} < \varepsilon/2$ such that $|x - y| \le 3^{-r}$ $(x, y \in C)$ implies $|f(x) - f(y)| < \varepsilon/2$ $(x, y \in C)$. Let $n = 2^r$. Define an $n \times n$ matrix $A = A_f$ as follows. For $1 \le a, b \le n$,

$$A_{a,b} = \begin{cases} 1 & \text{if } f(I(a,r)) \cap I(b,r) \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Then for each a $(1 \le a \le n)$, there exists b and c $(1 \le b, c \le n)$ such that $A_{b,a} = A_{a,c} = 1$. Thus if we put

$$(a)_A = \{x \in \Sigma_A ; x_0 = a\} \qquad (1 \le a \le n),$$

then each $(a)_A$ $(1 \le a \le n)$ is not empty.

First suppose that each $(a)_A$ $(1 \le a \le n)$ is perfect. Then there is a homeomorphism φ from Σ_A onto C such that $\varphi((a_A)) = I(a, r)$ $(1 \le a \le n)$. Put $g = \varphi \cdot \sigma_A \cdot \varphi^{-1}$. We shall show that $|f(x) - g(x)| < \varepsilon$ $(x \in C)$. Let $x \in C$. Suppose that $x \in I(a, r)$ and that $g(x) \in I(b, r)$. Then since

$$\sigma_A(\varphi^{-1}(x)) = \varphi^{-1}(g(x)) \in \sigma_A((a)_A) \cap (b)_A \neq \varnothing,$$

we get $A_{a,b} = 1$. Hence there is a $y \in I(a, r)$ such that $f(y) \in I(b, r)$. Thus we get

$$\begin{aligned} |f(x) - g(x)| &\le |f(x) - f(y)| + |f(y) - g(x)| \\ &< \varepsilon/2 + 3^{-r} \\ &< \varepsilon. \end{aligned}$$

In the general case, we shall use a product $(\Sigma_A \times \Sigma_2, \sigma_A \times \sigma_2)$, which is naturally topologically conjugate to a subshift of finite type. Since each $(a)_A \times \Sigma_2$ $(1 \le a \le n)$ is perfect, we get a homeomorphism φ from $\Sigma_A \times \Sigma_2$ to C such that

$$\varphi((a)_A \times \Sigma_2) = I(a, r) \qquad (1 \le a \le n).$$

Putting $g = \varphi \circ (\sigma_A \times \sigma_2) \circ \varphi^{-1}$ we proceed as before to get an inequality; $|f(x) - g(x)| < \varepsilon$ $(x \in C)$. Thus we have proved the first half of the Theorem.

We shall show that g is topologically transitive (resp. mixing) when f is topologically transitive (resp. mixing). Suppose that f is topologically transitive. Then there is a dense orbit $o_f(x)$ $(x \in C)$. Since each point of C is not isolated, $o_f(x)$ is a set of first category. Thus $C - o_f(x)$ is also dense in C. Since each point of $C - o_f(x)$ is a limit point of $o_f(x)$, we get $C = \alpha_f(x) \cup \omega_f(x)$, where $\alpha_f(x)$ (resp. $\omega_f(x)$) is the α (resp. ω-) limit set of x by f. Since both $\alpha_f(x)$ and $\omega_f(x)$ are closed f-invariant sets, $U = C - \alpha_f(x)$ and $V = C - \omega_f(x)$ are disjoint open f-invariant sets. Thus, by topological transitivity, either U or V must be empty. Hence either $\alpha_f(x) = C$ or $\omega_f(x) = C$ holds. In either case, for any $1 \le a, b \le n$, there is an $l_{a,b} > 0$ such that $f^{l_{a,b}}(I(a,r)) \cap (I(b,r)) \neq \emptyset$. Then it is easy to check that $A_{a,b}^{l_{a,b}} > 0$ $(1 \leq a, b \leq n)$, where $A = A_f$. Thus A $(= A_f)$ is irreducible. Since (Σ_2, σ_2) is topologically mixing, both (Σ_A, σ_A) and $(\Sigma_A \times \Sigma_2, \sigma_A \times \sigma_2)$ are topologically transitive by Lemma 2. Hence g is topologically transitive in either case. Next suppose that f is topologically mixing. It is enough to show that (Σ_A, σ_A) is topologically mixing and that Σ_A is perfect. For any $1 \leq a, b \leq n$, there is an $L_{a,b} > 0$ such that $f^l(I(a,r)) \cap I(b,r) \neq \emptyset$ for all $l \geq L_{a,b}$. Thus $A_{a,b}^l > 0$ for all $l \geq L_{a,b}$ $(1 \leq a, b \leq n)$. Hence (Σ_A, σ_A) is topologically mixing, by Lemma 3. Suppose that Σ_A has an isolated point p. Then, since $\{p\}$ is an open set, there is an $L > 0$ such that $p = f^l(p)$ for all $l \geq L$. Thus p is a fixed point. This contradicts the fact that (Σ_A, σ_A) is topologically mixing, for $\Sigma_A - \{p\}$ is also open and f-invariant. □

Remark. In the above proof, one could use another perfect subshift say (Σ, σ) in place of (Σ_2, σ_2). If a property of (Σ, σ) is not lost by taking a product with any subshift of finite type (Σ_A, σ_A), then elements of $\mathscr{H}$ will be approximated uniformly by subshifts with this property.

After I finished writing this paper, I accepted T. Kimura [4], where the density of $\mathscr{E} \cap \mathscr{P}$ in $\mathscr{H}$ is proved independently.

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