ONE PARAMETER SUBMONOIDS IN LOCALLY COMPACT DIFFERENTIABLE MONOIDS

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Abstract. Differentiable semigroups based on generalized manifolds were recently introduced by George Graham. We show that such locally compact monoids in which the multiplication is strongly differentiable at (1, 1) must contain nontrivial one parameter submonoids.

Suppose $H$ is a Banach space, $B$ is a subset of $H$ containing 0, and $V$ is a function from $B \times B$ into $H$ satisfying $V(x, 0) = V(0, x) = x$ for each $x$ in $B$. If $n$ is a positive integer greater than 1, denote $V(x, V(x, \ldots, V(x, x) \ldots))$, the product of $n - x$'s, by $x^n$ whenever the product exists. Define $x^0 = 0$ and $x^1 = x$. $V$ is said to be power associative if and only if $V(x^n, x^m) = x^{n+m}$ whenever each of $n$ and $m$ is a nonnegative integer and $x^{n+m}$ exists.

Suppose $p$ is a positive integer, $D$ is a compact admissible subset of $\mathbb{R}^p$ containing 0, and $V$ is a power associative multiplication from $D \times D$ into $\mathbb{R}^p$ which is strongly differentiable at (0,0). Suppose further that there is a positive number $b$ such that if each of $x$ and $y$ is in $D$ and has norm less than $b$ than $V(x,y)$ is in $D$. It then follows that there exists an $x$ in $D - \{0\}$ and a continuous function $T: [0, 1] \rightarrow D$ satisfying $T(0) = 0$, $T(1) = x$, and $T(s+t) = V(T(s), T(t))$ whenever each of $s$, $t$, and $s+t$ is in $[0, 1]$. This answers Question 4.1 of [G1] regarding the existence of one parameter subsemigroups in locally compact semigroups.

Before proceeding to the main theorem we will indicate some background.

The function $T$ above is called a local one parameter submonoid of $D$. In general, if $S$ is a monoid then a (local) one parameter subsemigroup of $S$ is a (local) homomorphism from $([0, \infty), +)$ into $S$. A (local) one parameter submonoid of $S$ is a (local) one parameter subsemigroup $h$ of $S$ such that $h(0) = 1$. Finally, a (local) one parameter subgroup of $S$ is a (local) homomorphism from the additive group of real numbers into $S$.
A subset $D$ of the Banach space $X$ is said to be admissible provided that each point of $D$ is a limit point of the interior of $D$. Let $f$ be a function with domain the admissible subset $D$ of the Banach space $X$ and codomain contained in the Banach space $Y$. The function $f$ is strongly differentiable at the point $p$ in $D$ provided there is a continuous linear map $T$ from $X$ to $Y$ so that for each positive number $c$ there is a positive number $d$ so that if each of $x$ and $y$ is in $D$ and within $d$ of $p$ then $|f(x) - f(y) - T(x-y)| \leq c|x-y|$. In this case $T$ is unique and is denoted by $f'(p)$.

An Analytical Group, as defined in 1938 by Birkhoff in [B], is an associative multiplication $W$ with domain containing $U \times U$ for some open neighborhood, $U$, of $0$ in a Banach space; $E$, which satisfies $W$ is strongly differentiable at $(0,0)$; and $W(x,0) = W(0,x) = x$ for each $x$ in $U$. Birkhoff used different terminology. In the course of showing $W$ is analytic, Birkhoff shows that for each $x$ in $U$ sufficiently close to $0$ there is a $T_x: [-1,1] \rightarrow U$ satisfying $T_x(0) = 0$, $T_x(1) = x$, and $T_x(s+t) = W(T_x(s), T_x(t))$ whenever each of $s, t$, and $s+t$ is in $[-1,1]$.

Suppose $E$ is a Banach space, $U$ is an open set of $E$ containing $0$, and $W$ is a power associative multiplication from $U \times U$ into $E$. In 1972, Holmes showed in [H1] that if $W$ is continuously Frechet differentiable (this implies strong differentiability) then there are positive numbers $a$ and $c$ such that if $x$ is in $U$ and $|x| < a$ then there is a unique continuous function $T_x$ from $[0,1]$ to the ball of radius $c$ centered at $0$ satisfying $V(T_x(s), T_x(t)) = T_x(s+t)$ whenever each of $s, t$ and $s+t$ is in $[0,1]$, $T_x(0) = 0$, and $T_x(1) = x$. In 1977, Holmes in [H2] goes on to show such functions $T$ are continuously Frechet differentiable on $[0,1]$. The notion of semigroup with differentiable multiplication based on an ordinary differentiable manifold was studied by Holmes in [H3]. If such a $C^k$ semigroup $S$, $k \geq 1$, has an idempotent $e$, $(e^2 = e)$, then there is an open subgroup of $eSe$ which has $e$ as its identity element. Thus, one may appeal to Lie theory for the existence of one parameter subsemigroups. Indeed, each one parameter subsemigroup is contained in a one parameter subgroup. This is not the case with the differentiable semigroups defined by Graham.

In 1983–1984 [G1, G2] Graham developed the theory of generalized differentiable manifolds using the concepts of admissible sets and strong differentiability as follows. The statement that the function $f$ from $D$ into the Banach space $Y$ is $C^1_s$ means that $f$ is strongly differentiable at each point of $D$ and the function $f'$ is continuous as a function from $D$ into $L(X,Y)$, the space of linear transformations from $X$ to $Y$ with the usual norm topology. The statement that $f$ is $C^k_s$ means that $f^{(k-1)}$ is $C^1_s$. A Hausdorff topological space $S$ is a $C^k_s$ manifold based on the Banach space $X$ provided that for each point $p$ of $S$ there is a homeomorphism $g_p$ from a neighborhood, $U$, of $p$ onto an admissible subset $D$ of $X$ containing $0$ so that $g_p(p) = 0$ and the composition $g_p \circ g_q^{-1}$ is $C^k_s$ on its domain for each choice of $p$ and $q$ in $S$. The theory
of these manifolds including the definition of product manifold is elucidated in [G2]. Finally, according to Graham, a topological semigroup is said to be $C^k_s$ provided that it is based on a $C^k_s$ manifold and the multiplication is $C^k_s$ as a function from $S \times S$ into $S$.

This notion of differentiable semigroup includes as examples such things as the real line under real addition or multiplication, the closed interval $[0,1]$ under real multiplication, the unit disk under complex multiplication, and certain closed subsemigroups of Lie groups studied by Hofmann and Lawson in [HL] in 1983.

Much of the calculus on $C^k_s$ manifolds mimics the standard theory. Most of the difference is due to the possible nonconvexity of admissible sets. This nonconvexity also implies that a $C^k$ monoid need have no nontrivial one parameter subsemigroups. For example, from [G1], let $B$ be the subset of the plane to which $(x,y)$ belongs only in case $x$ is positive and $y$ is between 0 and $x^2$ or $(x,y) = (0,0)$. $B$ forms a $C^k$ monoid under vector addition and contains no nontrivial one parameter subsemigroups. This shows that differentiable monoids need not have nontrivial one parameter subsemigroups. However, the closure of $B$ in the plane contains the monoid $\{(x,0): x \geq 0\}$.

A question Graham asks in [G3] is: Under what hypothesis does a $C^\infty$ monoid contain a nontrivial one parameter subsemigroup? He answers this question, in [G3], for finite dimensional $C^\infty$ monoids with smooth boundary.

In 1987 in [H4] Holmes shows that if $S$ is a locally complete $C^k$ monoid, $k \geq 2$, which contains a nontrivial $C^2$ curve starting at 1 then $S$ must contain nontrivial $C^k$ one parameter subsemigroups. In [A], using a much different approach, I improved on this result by requiring only that $S$ be a monoid with multiplication strongly differentiable at $(0,0)$ and that $S$ contains a curve starting at 1 which is strongly differentiable at 0. Finally, Holmes shows in [H5], in 1987, that if $S$ is a locally compact connected $C^k$ monoid then $S$ contains a nontrivial $C^k$ one parameter subsemigroup. Theorem 2 in this paper improves on this result by requiring only that $S$ be a monoid with multiplication strongly differentiable at $(0,0)$. The reader should take note that although the conclusion of Theorem 2 does not imply that $S$ is connected, as in the hypothesis of the theorem by Holmes in [H5], it does imply that the component of 1 in $S$ is nondegenerate.

Let $D$ be an admissible subset of the Banach space $X$, containing 0. Let $V$ be a power associative multiplication from $D \times D$ into $X$ which is strongly differentiable at $(0,0)$ satisfying $V(x,0) = V(0,x) = x$ for each $x$ in $D$. Suppose there is a positive number $b$ such that if each of $x$ and $y$ is in $D$ and has norm less than $b$ then $V(x,y)$ is in $D$. Such a function is called a strongly differentiable power associative local groupoid.

**Theorem 1.** If $V$ is a strongly differentiable power associative local groupoid and $D$ is compact, then there is an $x$ in $D - \{0\}$ and a continuous function
Our strategy for proving Theorem 1 will be to show the existence of an \( x \) in \( D \) such that if \( t \) is a positive integer then \( x^{1/2} \), the 2\(^{th}\) root of \( x \), exists. We will then use \( x \) to build the function \( T \) on the dyadic rationals in [0,1]. The proof of Theorem 1 will follow from a sequence of lemmas. Lemma 1.1 was suggested from arguments in [B].

**Lemma 1.1.** If \( c \) is a positive number there is a positive number \( d \) such that if \( n \) is a positive integer and each of \( x_1, x_2, \ldots, x_n \) is in \( D \) and \( \sum_{i=1}^{n} |x_i| < d \), then \( \prod_{i=1}^{n} x_i \) is in \( D \) and \( |\prod_{i=1}^{n} x_i - \sum_{i=1}^{n-1} x_i| \leq c \sum_{i=1}^{n} |x_i| \). Here \( \prod_{i=1}^{n} x_i \) denotes \( V(x_n, V(x_{n-1}, \ldots, V(x_2, x_1) \ldots)) \).

**Proof.** Choose a positive number \( b \) so that if each of \( x \) and \( y \) is in \( D \) and within \( b \) of 0, then \( V(x, y) \) is in \( D \). Suppose \( c \) is a positive number less than 1. Using \( V'(0,0)(x,y) = x + y \), and \( |V(x,y) - x - y| = |V(x,y) - V(x,0) - y| \), choose a positive number \( d_1 < b \) so that if each of \( x \) and \( y \) is in \( D \) and has norm less than \( d_1 \), then \( |V(x,y) - x - y| \leq c|y| \). Let \( d \) be a positive number less than \( d_1/2 \). The proof is by induction on \( n \). If each of \( x_1 \) and \( x_2 \) is in \( D \) and \( |x_1| + |x_2| < d \), then \( V(x_2, x_1) \) is in \( D \), by choice of \( b \), and \( |V(x_2, x_1) - x_2 - x_1| \leq c|x_1| \leq c(|x_1| + |x_2|) \), by choice of \( d_1 \). Next, suppose each of \( x_1, x_2, \ldots, x_n \) is in \( D \) and \( \sum_{i=1}^{n} |x_i| < d \). If \( \prod_{i=1}^{n-1} x_i \) is in \( D \) and \( |\prod_{i=1}^{n-1} x_i - \sum_{i=1}^{n-2} x_i| \leq c \sum_{i=1}^{n-1} |x_i| \), then by the triangle inequality, \( |\prod_{i=1}^{n-1} x_i| \leq 2 \sum_{i=1}^{n-1} |x_i| < d_1 < b \). Therefore, since \( |x_n| \) is also less than \( b \), \( \prod_{i=1}^{n} x_i = V(x_n, \prod_{i=1}^{n-1} x_i) \) is in \( D \). Furthermore, since each of \( |x_n| \) and \( |\prod_{i=1}^{n-1} x_i| \) is less than \( d_1 \), it follows from the triangle inequality that

\[
\left| \prod_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i \right| \leq V \left( x_n, \prod_{i=1}^{n-1} x_i \right) - \left( \sum_{i=1}^{n-1} x_i \right) + \left| \prod_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} x_i \right| \\
\leq c \sum_{i=1}^{n} |x_i|.
\]

Lemma 1.1 now follows from induction. Note that associativity is not used in the proof of Lemma 1.1.

Next, consider the function \( f: D \to X \) defined by \( f(x) = x^2 \). Since \( V \) is strongly differentiable at \( (0,0) \), it follows that \( f \) is strongly differentiable at 0 and \( f'(0)(x) = 2x \) for each \( x \) in \( D \). Therefore, let \( B_1 \) be a positive number such that if each of \( x \) and \( y \) is in \( D \) and has norm less than \( B_1 \), then each of \( f(x) \) and \( f(y) \) is in \( D \) and \( |f(x) - f(y) - f'(0)(x-y)| \leq \frac{1}{3}|x-y| \). This implies that \( |f(x) - f(y) - 2(x-y)| \leq \frac{1}{3}|x-y| \). Using the triangle inequality, this yields \( |f(x) - f(y)| \geq \frac{1}{3}|x-y| \), which implies \( f \) is \( 1-1 \) on \( A(\{ x \in D : |x| < B_1 \}) \). Notice, the triangle inequality also implies \( f \) is Lipschitz and, hence, continuous on \( A \).
Since $V$ is strongly differentiable at $(0,0)$ we may choose $B_2$ to be a positive number less than $B_1$ such that if each of $x$ and $y$ is in $D$ and has norm less than $B_2$, then $V(x,y)$ is in $D$ and 

$$|V(x,y) - x - y| = |V(x,y) - V(x,0) - y| \leq |y|.$$  

Finally, using Lemma 1.1, let $B$ be a positive number less than $B_2/4$ such that if $n$ is a positive integer and each of $x_1, x_2, \ldots, x_n$ is in $D$ and $\sum_{i=1}^n |x_i| < B$, then $\prod_{i=1}^n x_i$ is in $D$ and $|\prod_{i=1}^n x_i - \sum_{i=1}^n x_i| \leq \frac{1}{4} \sum_{i=1}^n |x_i|$. 

**Lemma 1.2.** If $n$ is a positive integer, then there is a positive integer $k, k \geq n$, and a $y$ in $A$ such that if $t$ is a nonnegative integer, $t \leq k$, then $y^{2^t}$ is in $A$, $|y^{2^t}| < \frac{B}{8}$, and $\frac{B}{32} \leq y^{2^t}$.

**Proof.** Suppose $n$ is a positive integer. Choose $y$ in $D$ such that $2^n|y| < \frac{B}{8}$. Let $k$ be a positive integer, $k \geq n$, such that $\frac{B}{16} < 2^k|y| < \frac{B}{8}$. Thus, if $t \in \{1, 2, \ldots, k\}$, then $y^{2^t}$ is in $D$ and $|y^{2^t}| < 2 \cdot 2^t|y| < \frac{B}{4}$, by choice of $B$ and the triangle inequality. Moreover, the choice of $B$ yields $|y^{2^k} - 2^k y| < \frac{1}{4} \cdot 2^k|y| < 2^{k-1}|y|$. Therefore, using the triangle inequality and the choice of $k$, we have $\frac{B}{32} \leq 2^{k-1}|y| \leq y^{2^k}$. Thus, Lemma 1.2 is proved.

Notice, it follows from Lemma 1.2 and the fact that $f$ is $1$-1 on $A$ that if $n$ is a positive integer then there is a positive integer $k$ greater than or equal to $n$ and a $z$ in $A$ such that $\frac{B}{32} \leq |z| < \frac{B}{4}$ and such that if $t \in \{1, 2, \ldots, k\}$ then $z$ has a unique $2^t$th root, $z^{1/2^t}$, in $A$. The notation, $z^{1/2^t}$, will henceforth be reserved for the unique $2^t$th root of $z$ in $A$. Therefore, let $\{z_i\}_{i=1}^\infty$ be a sequence in $A$ such that if $i$ is a positive integer then $\frac{B}{32} \leq |z_i| < \frac{B}{4}$ and such that if $t \in \{1, 2, \ldots, i\}$, then $z_i$ has a unique $2^t$th root, $z_i^{1/2^t}$, and $|(z_i^{1/2^t})^{2^t}| = |z_i^{1/2^{2t}}| < \frac{B}{4} < B$, for each $r \in \{0, 1, \ldots, t\}$. Since $D$ is compact, let $x$ be in $D$ and let $\{y_{i,j}\}_{i=1}^\infty$ be a convergent subsequence of $\{z_i\}_{i=1}^\infty$ such that $\lim_{i=1}^\infty \left| y_{i,j} \right| = x$. Notice, $\frac{B}{32} \leq \lim_{i=1}^\infty \left| y_{i,j} \right| = \lim_{i=1}^\infty \left| y_{i,j} \right| \leq \frac{B}{4} < B$, which implies $x$ is in $A$. Next, using compactness of $D$, let $\{y_{i,j}\}_{i=1}^\infty$ be a convergent subsequence of $\{y_{i,j}\}_{i=1}^\infty$. Since $f$ is continuous on $A$ and $|f_{i,j}| < \frac{B}{4}$ for each positive integer $i$, it follows that $x$ has a unique square root, $x^{1/2} = \lim_{i=\infty} \{y_{1,i}\}_{i=1}^\infty$. Now suppose $n$ is a positive integer greater than one such that each of $\{y_{n-1,i}\}_{i=1}^\infty$ and $\{y_{n-2,i}\}_{i=1}^\infty$ is defined, $\{y_{n-1,i}\}_{i=1}^\infty$ is a convergent subsequence of $\{y_{n-2,i}\}_{i=1}^\infty$, and $x^{1/2^{n-1}} = \lim_{i=\infty} \{y_{n-1,i}\}_{i=1}^\infty$. Let $\{y_{n,i}\}_{i=1}^\infty$ be a convergent subsequence of $\{y_{n-1,i}\}_{i=1}^\infty$. Then, since $|x^{1/2^{n-1}}| \leq \frac{B}{4}$, it again follows from continuity of $f$ on $A$ that $x^{1/2^{n-1}}$ has a unique square root $x^{1/2^n} = \lim_{i=\infty} \{y_{n,i}\}_{i=1}^\infty$. Therefore, if $n$ is a positive integer then $x^{1/2^n}$ exists, by induction. Moreover, $|x^{1/2^n} = \lim_{i=\infty} \left| y_{n,i}^{1/2^n} \right| \leq \frac{B}{4} \leq \frac{B}{4}$, which implies $2|x^{1/2^n}| < B$. Hence, the choice of $B$ yields $|x^{1/2^{n-1}} - 2x^{1/2^n}| \leq \frac{1}{4} \cdot 2|x^{1/2^n}|$. 
Using the triangle inequality, we have \( |x^{1/2^n}| \leq \frac{1}{2}|x^{1/2^{n-1}}| \) which, by induction on \( n \), is less than or equal to \( \left( \frac{1}{3} \right)^n |x| \).

Denote by \( Q \) the set of dyadic rational numbers in \([0,1]\). For each pair \((m,n)\) of positive integers such that \( m \leq 2^n \), let \( T(m/2^n) = (x^{1/2^n})^m \) and let \( T(0) = 0 \). The existence of \( T \) on \( Q \) will be shown in Lemma 1.3. Since it is the case that if \( n \) is a positive integer, then \( T(2/2^n) = (x^{1/2^n})^2 = x^{1/2^{n-1}} \), it follows that \( T \) is well defined on \( Q \). It is also clear that \( T \) is nontrivial, since \( T(1) = x \) and \( |x| \geq \frac{B}{32} \). The next lemma will be used in the proof of Lemma 1.4.

**Lemma 1.3.** If \((m,n)\) is a pair of positive integers such that \( m \leq 2^n \), then \( T(m/2^n) \) exists and \( |T(m/2^n)| \leq 4B \).

**Proof.** The proof is by induction. We will show that if \((m,n)\) is a pair of positive integers such that \( m \leq 2^n \), then \( |T(m/2^n)| \leq 2|x| \cdot \sum_{i=1}^{n} \left( \frac{2}{3} \right)^i \). Suppose \((m,n)\) is a pair of positive integers such that \( m \leq 2^n \). If \( n = 1 \), then we have already seen that \( T(m/2^n) \) exists and \( |T(m/2^n)| \leq |x| \leq 2|x| \cdot \sum_{i=1}^{n} \left( \frac{2}{3} \right)^i \). Therefore, assume \( n > 1 \) and for each positive integer \( k \) less than \( n \), assume that \( T(m/2^n) \) exists and \( |T(s/2^n)| \leq 2|x| \cdot \sum_{i=1}^{k} \left( \frac{2}{3} \right)^i \) for each \( s \in \{0,1,\ldots,2^k\} \).

Furthermore, assume \( T(s/2^n) \) exists and \( |T(s/2^n)| \leq 2|x| \cdot \sum_{i=1}^{n} \left( \frac{2}{3} \right)^i \) for each \( s \in \{0,1,\ldots,m-1\} \). Then, since \( |T(m-1/2^n)| \leq 4|x| < 4B < B_2 \) and \( |T(1/2^n)| < B_2 \), it follows from the choice of \( B_2 \) that \( T(m/2^n) \) exists.

If \( m \) is even, then by induction we have

\[
|T\left(\frac{m}{2^n}\right)| = |T\left(\frac{m/2}{2^{n-1}}\right)| \leq 2|x| \cdot \sum_{i=1}^{n-1} \left( \frac{2}{3} \right)^i \leq 2|x| \cdot \sum_{i=1}^{n} \left( \frac{2}{3} \right)^i .
\]

If \( n \) is odd, then \( m = 2r + 1 \) for some nonnegative integer \( r \). Therefore, using the triangle inequality, the inductive hypothesis, the choice of \( B_2 \), and the fact that \( 2|x| \cdot \sum_{i=1}^{n} \left( \frac{2}{3} \right)^i \leq 4|x| < 4B < B_2 \), we obtain

\[
|T\left(\frac{m}{2^n}\right)| = |T\left(\frac{2r+1}{2^n}\right)|
\]
\[
\leq \left| V\left( \frac{r}{2^{n-1}}, \frac{r}{2^n}, \frac{1}{2^n} \right) \right| - T\left(\frac{r}{2^n-1}\right)| - |T\left(\frac{r}{2^{n-1}}\right)|
\]
\[
\leq 2 \left| T\left(\frac{2}{2^n}\right) \right| + \left| T\left(\frac{r}{2^{n-1}}\right) \right|
\]
\[
\leq 2|x| \cdot \left( \frac{2}{3} \right)^n + 2|x| \cdot \sum_{i=1}^{n-1} \left( \frac{2}{3} \right)^i
\]
\[
\leq 2|x| \cdot \sum_{i=1}^{n} \left( \frac{2}{3} \right)^i .
\]

Thus, Lemma 1.3 is proved. The next lemma shows that \( T \) has a unique extension to \([0,1]\).
Lemma 1.4. \( T \) is uniformly continuous on \( Q \).

Proof. Recall that \( B_2 \) is a positive number such that if each of \( x \) and \( y \) is in \( D \) and has norm less than \( B_2 \) then \( |V(x, y) - x - y| \leq |y| \) and recall that \( B \) is a positive number less than \( B_2/4 \). Suppose \((m, n)\) is a pair of nonnegative integers such that \( m < 2^n \) and \( a \) is in \( Q \) such that \( a - (m/2^n) < 1/2^n \). Let \( \{b_i\}_{i=1}^k \) be a finite sequence in \( \{0, 1\} \) such that \( a = (m/2^n) + \sum_{i=1}^k (b_i/2^{n+i}) \).

Let \( a_i = (m/2^n) + \sum_{j=1}^i (b_j/2^{n+j}) \) for each \( i \in \{0, 1, \ldots, k\} \). Then, since \( |T(a_i)| < 4B < B_2 \) for each \( i \in \{0, 1, \ldots, k\} \) by lemma 1.3, it follows from the triangle inequality and the choice of \( B_2 \) that

\[
|T\left(\frac{m}{2^n}\right)| = \sum_{i=1}^k |T(a_{i-1}) - T(a_i)| \\
\leq 2|x| \sum_{i=1}^k \left| \frac{1}{2^{n-i}} \right| \\
\leq 2|x| \sum_{i=1}^k \left( \frac{2}{3} \right)^{n+1} \\
\leq 6|x| \left( \frac{2}{3} \right)^n.
\]

This implies that \( T \) is uniformly continuous on \( Q \) and hence completes the proof of Lemma 1.4.

It is clear by the construction and the continuity of \( T \) that \( T(0) = 0, T(1) = 1 \), and \( T(s + t) = V(T(s), T(t)) \) whenever each of \( s, t, \) and \( s + t \) is in \([0, 1]\). Thus, Theorem 1 is proved.

Locally compact monoids

A \( C^k_s \) monoid in which \( k \geq 1 \) and which has a neighborhood, \( U \), of 1 so that \( g_1(U) \) is a compact subset of \( X \) is called a locally compact monoid. It is clear, since \( g_1(U) \) is compact in \( X \) and each \( C^k_s \) monoid is a strongly differentiable groupoid, that Theorem 1 can be applied to the setting of differentiable semigroups as defined by Graham. We proceed with Theorem 2.

Theorem 2. If \( S \) is a \( C^k_s \) locally compact monoid, \( k \geq 1 \), then \( S \) has a \( C^k_s \) one parameter submonoid \( T \).

Proof. Choose a compact (in \( X \)) neighborhood, \( D \), of 0 in \( g_1(S) \) so that \( V(x, y) = g_1(g_1^{-1}(x) \cdot g_1^{-1}(y)) \) exists for each \( x \) and \( y \) in \( D \). Since \( V \) is clearly a strongly differentiable groupoid, Theorem 1 provides the existence of an \( x \) in \( D \) and a function \( R: [0, 1] \to D \) such that \( R(0) = 0, R(1) = x \), and \( R(s + t) = V(R(s), R(t)) \) whenever each of \( s, t, \) and \( s + t \) is in \([0, 1]\). Define \( T: [0, 1] \to S \) by \( T(t) = g_1^{-1}(R(t)) \) for each \( t \) in \([0, 1]\). Clearly, \( T \) has a unique continuous extension to \( R^+ \) since \( S \) is a monoid and, hence is algebraically closed. Since \( T \) is obviously a one parameter submonoid.
of $S$, it only remains to be seen that $T$ is $C^k_s$ on $R^+$. However, it was shown in [H5, Theorem 2] by Holmes that if $S$ is a $C^k_s$ monoid and $T$ is a continuous homomorphism from $R^+$ into $S$ with $T(0) = 1$ then $T$ is $C^k_s$. Thus, Theorem 2 is proved.

Although Theorem 1 shows that very little differentiability is required in order to guarantee the existence of one parameter submonoids in locally compact monoids, the author knows of no example of a locally compact monoid with multiplication strongly differentiable at $(1,1)$ which is not analytic. Indeed, in the case of an Analytical group (see introduction), Birkhoff in [B] showed that no such example exists. The author is presently considering the following questions:

1. If $S$ is a monoid, based on an admissible subset of the Banach space, $E$, with multiplication $V$ which is strongly differentiable at $(1,1)$, then is there an extension, $W$, of $V$ to an open subset of $E \times E$ containing $(1,1)$ which is a monoid that is strongly differentiable at $(1,1)$?

2. If $T$ is a one parameter submonoid of the monoid $S$ which has multiplication strongly differentiable at $(1,1)$, is $T$ strongly differentiable at $0$?

3. If $S$ is a locally compact monoid which has multiplication strongly differentiable at $(1,1)$, is the component of 1 an admissible set?

REFERENCES


