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ONE PARAMETER SUBMONOIDS IN LOCALLY COMPACT DIFFERENTIABLE MONOIDS

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ABSTRACT. Differentiable semigroups based on generalized manifolds were recently introduced by George Graham. We show that such locally compact monoids in which the multiplication is strongly differentiable at $(1,1)$ must contain nontrivial one parameter submonoids.

Suppose H is a Banach space, B is a subset of H containing 0, and V is a function from $B \times B$ into H satisfying $V(x,0) = V(0,x) = x$ for each x in B. If n is a positive integer greater than 1, denote $V(x, V(x, \ldots, V(x,x)\cdots))$, the product of $n-x$'s, by x^n whenever the product exists. Define $x^0 = 0$ and $x^1 = x$. V is said to be power associative if and only if $V(x^n, x^m) = x^{n+m}$ whenever each of n and m is a nonnegative integer and x^{n+m} exists.

Suppose p is a positive integer, D is a compact admissible subset of R^p containing 0, and V is a power asociative multiplication from $D \times D$ into R^p which is strongly differentiable at (0,0). Suppose further that there is a positive number b such that if each of x and y is in D and has norm less than b than $V(x,y)$ is in D. It then follows that there exists an x in $D - \{0\}$ and a continuous function $T\colon [0,1] \to D$ satisfying $T(0) = 0$, $T(1) = x$, and $T(s+t) = V(T(s), T(t))$ whenver each of s, t, and $s+t$ is in $[0,1]$. This answers Question 4.1 of [G1] regarding the existence of one parameter subsemigroups in locally compact semigroups.

Before proceeding to the main theorem we will indicate some background.

The function T above is called a local one parameter submonoid of D. In general, if S is a monoid then a (local) one parameter subsemigroup of S is a (local) homomorphism from $([0,\infty), +)$ into S. A (local) one parameter submonoid of S is a (local) one parameter subsemigroup h of S such that $h(0) = 1$. Finally, a (local) one parameter subgroup of S is a (local) homomorphism from the additive group of real numbers into S.

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A subset D of the Banach space X is said to be admissible provided that each point of D is a limit point of the interior of D. Let f be a function with domain the admissible subset D of the Banach space X and codomain contained in the Banach space Y. The function f is strongly differentiable at the point p in D provided there is a continuous linear map T from X to Y so that for each positive number c there is a positive number d so that if each of x and y is in D and within d of p then $|f(x)-f(y)-T(x-y)| \leq c|x-y|$. In this case T is unique and is denoted by $f'(p)$.

An Analytical Group, as defined in 1938 by Birkhoff in [B], is an associative multiplication W with domain containing $U \times U$ for some open neighborhood, U, of 0 in a Banach space; E, which satisfies W is strongly differentiable at (0,0); and $W(x, 0) = W(0, x) = x$ for each x in U. Birkhoff used different terminology. In the course of showing W is analytic, Birkhoff shows that for each x in U sufficiently close to 0 there is a $T_x\colon [-1, 1] \to U$ satisfying $T_x(0) = 0, T_x(1) = x$, and $T_x(s+t) = W(T_x(s), T_x(t))$ whenever each of s, t, and $s+t$ is in $[-1, 1]$.

Suppose E is a Banach space, U is an open set of E containing 0, and W is a power associative multiplication from $U \times U$ into E. In 1972, Holmes showed in [H1] that if W is continuously Frechet differentiable (this implies strong differentiablity) then there are positive numbers a and c such that if x is in U and $|x| < a$ then there is a unique continuous function T_x from $[0, 1]$ to the ball of radius c centered at 0 satisfying $V(T_x(s), T_x(t)) = T_x(s+t)$ whenever each of s, t and $s+t$ is in $[0, 1], T_x(0) = 0$, and $T_x(1) = x$. In 1977, Holmes in [H2] goes on to show such functions T are continuously Frechet differentiable on $[0, 1]$.

The notion of semigroup with differentiable multiplication based on an ordinary differentiable manifold was studied by Holmes in [H3]. If such a C^k semigroup S, $k \geq 1$, has an idempotent e, $(e^2 = e)$, then there is an open subgroup of eSe which has e as its identity element. Thus, one may appeal to Lie theory for the existence of one parameter subsemigroups. Indeed, each one parameter subsemigroup is contained in a one parameter subgroup. This is not the case with the differentiable semigroups defined by Graham.

In 1983–1984 [G1, G2] Graham developed the theory of generalized differentiable manifolds using the concepts of admissible sets and strong differentiability as follows. The statement that the function f from D into the Banach space Y is C_s^1 means that f is strongly differentiable at each point of D and the function f' is continuous as a function from D into $L(X, Y)$, the space of linar transformations from X to Y with the usual norm topology. The statement that f is C_s^k means that $f^{(k-1)}$ is C_s^1. A Hausdorff topological space S is a C_s^k manifold based on the Banach space X provided that for each point p of S there is a homeomorphism g_p from a neighborhood, U, of p onto an admissible subset D of X containing 0 so that $g_p(p) = 0$ and the composition $g_p \circ g_q^{-1}$ is C_s^k on its domain for each choice of p and q in S. The theory of these manifolds including the definition of product manifold is elucidated in [G2]. Finally, according to Graham, a topological semigroup is said to be C_s^k provided that it is based on a C_s^k manifold and the multiplication is C_s^k as a function from $S \times S$ into S.

This notion of differentiable semigroup includes as examples such things as the real line under real addition or multiplication, the closed interval [0,1] under real multiplication, the unit disk under complex multiplication, and certain closed subsemigroups of Lie groups studied by Hofmann and Lawson in [HL] in 1983.

Much of the calculus on C_s^k manifolds mimics the standard theory. Most of the difference is due to the possible nonconvexity of admissible sets. This nonconvexity also implies that a C^k monoid need have no nontrivial one parameter subsemigroups. For example, from [G1], let B be the subset of the plane to which (x, y) belongs only in case x is positive and y is between 0 and x^2 or $(x, y) = (0, 0)$. B forms a C^k monoid under vector addition and contains no nontrivial one parameter subsemigroups. This shows that differentiable monoids need not have nontrivial one parameter subsemigroups. However, the closure of B in the plane contains the monoid $\{(x, 0): x \geq 0\}$.

A question Graham asks in [G3] is: Under what hypothesis does a C^∞ monoid contain a nontrivial one parameter subsemigroup? He answers this question, in [G3], for finite dimensional C^∞ monoids with smooth boundary.

In 1987 in [H4] Holmes shows that if S is a locally complete C^k monoid, $k \geq 2$, which contains a nontrivial C^2 curve starting at 1 then S must contain nontrivial C^k one parameter subsemigroups. In [A], using a much different approach, I improved on this result by requiring only that S be a monoid with multiplication strongly differentiable at (0,0) and that S contains a curve starting at 1 which is strongly differentiable at 0. Finally, Holmes shows in [H5], in 1987, that if S is a locally compact connected C^k monoid then S contains a nontrivial C^k one parameter subsemigroup. Theorem 2 in this paper improves on this result by requiring only that S be a monoid with multiplication strongly differentiable at (0,0). The reader should take note that although the conclusion of Theorem 2 does not imply that S is connected, as in the hypothesis of the theorem by Holmes in [H5], it does impliy that the component of 1 in S is nondegenerate.

Let D be an admissible subset of the Banach space X, containing 0. Let V be a power associative multiplication from $D \times D$ into X which is strongly differentiable at (0,0) satisfying $V(x, 0) = V(0, x) = x$ for each x in D. Suppose there is a positive number b such that if each of x and y is in D and has norm less than b then $V(x, y)$ is in D. Such a function is called a strongly differentiable power associative local groupoid.

Theorem 1. *If V is a strongly differentiable power associative local groupoid and D is compact, then there is an x in $D - \{0\}$ and a continuous function*

$T\colon [0,1] \to D$ satisfying $T(0) = 0$, $T(1) = x$, and $T(s+t) = V(T(s), T(t))$ whenever each of s, t, and $s+t$ is in $[0,1]$.

Our strategy for proving Theorem 1 will be to show the existence of an x in D such that if t is a positive integer then $x^{1/2^t}$, the 2^tth root of x, exists. We will then use x to build the function T on the dyadic rationals in [0,1]. The proof of Theorem 1 will follow from a sequence of lemmas. Lemma 1.1 was suggested from arguments in [B].

Lemma 1.1. *If c is a positive number there is a positive number d such that if n is a positive integer and each of $x_1, x_2, \ldots, x_n$ is in D and $\sum_{i=1}^n |x_i| < d$, then $\prod_{i=1}^n x_i$ is in D and $|\prod_{i=1}^n x_i - \sum_{i=1}^n x_i| \le c\sum_{i=1}^n |x_i|$. Here $\prod_{i=1}^n x_i$ denotes $V(x_n, V(x_{n-1}, \ldots, V(x_2, x_1)\cdots))$.*

Proof. Choose a positive number b so that if each of x and y is in D and within b of 0, then $V(x,y)$ is in D. Suppose c is a positive number less than 1. Using $V'(0,)(x,y) = x + y$, and $|V(x,y) - x - y| = |V(x,y) - V(x,0) - y|$, choose a positive number $d_1 < b$ so that if each of x and y is in D and has norm less than d_1 then $|V(x,y) - x - y| \le c|y|$. Let d be a positive number less than $d_1/2$. The proof is by induction on n. If each of x_1 and x_2 is in D and $|x_1| + |x_2| < d$, then $V(x_2, x_1)$ is in D, by choice of b, and $|V(x_2,x_1) - x_2 - x_1| \le c|x_1| \le c(|x_1| + |x_2|)$, by choice of d_1. Next, suppose each of $x_1, x_2, \ldots, x_n$ is in D and $\sum_{i=1}^n |x_i| < d$. If $\prod_{i=1}^{n-1} x_i$ is in D and $|\prod_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} x_i| \le c\sum_{i=1}^{n-1} |x_i|$, then by the triangle inequality, $|\prod_{i=1}^{n-1} x_i| \le 2\sum_{i=1}^{n-1} |x_i| < d_1 < b$. Therefore, since $|x_n|$ is also less than b, $\prod_{i=1}^n x_i = V(x_n, \prod_{i=1}^{n-1} x_i)$ is in D. Furthermore, since each of $|x_n|$ and $|\prod_{i=1}^{n-1} x_i|$ is less than d_1, it follows from the triangle inequality that

$$\left|\prod_{i=1}^n x_i - \sum_{i=1}^n x_i\right| \le \left|V\left(x_n, \prod_{i=1}^{n-1} x_i\right) - \left(x_n + \prod_{i=1}^{n-1} x_i\right)\right| + \left|\prod_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} x_i\right|$$
$$\le c\sum_{i=1}^n |x_i|.$$

Lemma 1.1 now follows from induction. Note that associativity is not used in the proof of Lemma 1.1.

Next, consider the function $f\colon D \to X$ defined by $f(x) = x^2$. Since V is strongly differentiable at (0,0), it follows that f is strongly differentiable at 0 and $f'(0)(x) = 2x$ for each x in D. Therefore, let B_1 be a positive number such that if each of x and y is in D and has norm less than B_1, then each of $f(x)$ and $f(y)$ is in D and $|f(x) - f(y) - f'(0)(x-y)| \le \frac{1}{2}|x-y|$. This implies that $|f(x) - f(y) - 2(x-y)| \le \frac{1}{2}|x-y|$. Using the triangle inequality, this yields $|f(x) - f(y)| \ge \frac{3}{2}|x-y|$, which implies f is 1–1 on $A\{x \in D\colon |x| < B_1\}$. Notice, the triangle inequality also implies f is Lipschitz and, hence, continuous on A.

Since V is strongly differentiable at (0,0) we may choose B_2 to be a positive number less than B_1 such that if each of x and y is in D and has norm less than B_2, then $V(x,y)$ is in D and

$$|V(x,y) - x - y| = |V(x,y) - V(x,0) - y| \le |y|.$$

Finally, using Lemma 1.1, let B be a positive number less than $B_2/4$ such that if n is a positive integer and each of $x_1, x_2, \dots, x_n$ is in D and $\sum_{i=1}^n |x_i| < B$, then $\prod_{i=1}^n x_i$ is in D and $|\prod_{i=1}^n x_i - \sum_{i=1}^n x_i| \le \frac{1}{4}\sum_{i=1}^n |x_i|$.

Lemma 1.2. *If n is a positive integer, then there is a positive integer $k, k \ge n$, and a y in A such that if t is a nonnegative integer, $t \le k$, then y^{2^t} is in A, $|y^{2^t}| < \frac{B}{4}$, and $\frac{B}{32} \le y^{2^k}$.*

Proof. Suppose n is a positive integer. Choose y in D such that $2^n|y| < \frac{B}{8}$. Let k be a positive integer, $k \ge n$, such that $\frac{B}{16} \le 2^k|y| < \frac{B}{8}$. Thus, if $t \in \{1,2,\dots,k\}$, then y^{2^t} is in D and $|y^{2^t}| < 2 \cdot 2^t|y| < \frac{B}{4}$, by choice of B and the triangle inequality. Moreover, the choice of B yields $|y^{2^k} - 2^k y| \le \frac{1}{4} \cdot 2^k|y| < 2^{k-1}|y|$. Therefore, using the triangle inequality and the choice of k, we have $\frac{B}{32} \le 2^{k-1}|y| \le y^{2^k}$. Thus, Lemma 1.2 is proved.

Notice, it follows from Lemma 1.2 and the fact that f is 1–1 on A that if n is a positive integer then there is a positive integer k greater than or equal to n and a z in A such that $\frac{B}{32} \le |z| < \frac{B}{4}$ and such that if $t \in \{1,2,\dots,k\}$ then z has a unique 2^tth root, $z^{1/2^t}$, in A. The notation, $z^{1/2^t}$, will henceforth be reserved for the unique 2^tth root of z in A. Therefore, let $\{z_i\}_{i=1}^\infty$ be a sequence in A such that if i is a positive integer then $\frac{B}{32} \le |z_i| < \frac{B}{4}$ and such that if $t \in \{1,2,\dots,i\}$, then z_i has a unique 2^tth root, $z_i^{1/2^t}$, and $|(z_i^{1/2^t})^{2^r}| = |z_i^{1/2^{t-r}}| < \frac{B}{4} < B_1$ for each $r \in \{0,1,\dots,t\}$. Since D is compact, let x be in D and let $\{y_i\}_{i=1}^\infty$ be a convergent subsequence of $\{z_i\}_{i=1}^\infty$ such that $\{y_i\}_{i=1}^\infty \to x$. Notice, $\frac{B}{32} \le \lim_{i-\infty}|y_i| = |x| = \lim_{i-\infty}|y_i| \le \frac{B}{4} < B$, which implies x is in A. Next, using compactness of D, let $\{y_{1,i}\}_{i=1}^\infty$ be a convergent subsequence of $\{y_i^{1/2}\}_{i=1}^\infty$. Since f is continuous on A and $|y_i^{1/2}| < \frac{B}{4}$ for each positive integer i, it follows that x has a unique square root, $x^{1/2} = \lim_{i\to\infty}\{y_{1,i}\}_{i=1}^\infty$. Now suppose n is a positive integer greater than one such that each of $\{y_{n-1,i}\}_{i=1}^\infty$ and $\{y_{n-2,i}\}_{i=1}^\infty$ is defined, $\{y_{n-1,i}\}_{i=1}^\infty$ is a convergent subsequence of $\{y_{n-2,i}^{1/2}\}_{i=1}^\infty$, and $x^{1/2^{n-1}} = \lim_{i-\infty}\{y_{n-1,i}\}_{i=1}^\infty$. Let $\{y_{n,i}\}_{i=1}^\infty$ be a convergent subsequence of $\{y_{n-1,i}^{1/2}\}_{i=1}^\infty$. Then, since $|x^{1/2^{n-1}}| \le \frac{B}{4}$, it again follows from continuity of f on A that $x^{1/2^{n-1}}$ has a unique square root $x^{1/2^n} = \lim_{i\to\infty}\{y_{n,i}\}_{i=1}^\infty$. Therefore, if n is a positive integer then $x^{1/2^n}$ exists, by induction. Moreover, $|x^{1/2^n} = \lim_{i-\infty}|y_{n,i}^{1/2^n}| \le \frac{B}{4} < \frac{B}{2}$, which implies $2|x^{1/2^n}| < B$. Hence, the choice of B yields $|x^{1/2^{n-1}} - 2x^{1/2^n}| \le \frac{1}{4} \cdot 2|x^{1/2^n}|$.

Using the triangle inequality, we have $|x^{1/2^n}| \le \frac{2}{3}|x^{1/2^{n-1}}|$ which, by induction on n, is less than or equal to $(\frac{2}{3})^n|x|$.

Denote by Q the set of dyadic rational numbers in $[0,1]$. For each pair (m,n) of positive integers such that $m \le 2^n$, let $T(m/2^n) = (x^{1/2^n})^m$ and let $T(0) = 0$. The existence of T on Q will be shown in Lemma 1.3. Since it is the case that if n is a positive integer, then $T(2/2^n) = (x^{1/2^n})^2 = x^{1/2^{n-1}}$, it follows that T is well defined on Q. It is also clear that T is nontrivial, since $T(1) = x$ and $|x| \ge \frac{B}{32}$. The next lemma will be used in the proof of Lemma 1.4.

Lemma 1.3. *If (m,n) is a pair of positive integers such that $m \le 2^n$, then $T(m/2^n)$ exists and $|T(m/2^n)| < 4B$.*

Proof. The proof is by induction. We will show that if (m,n) is a pair of positive integers such that $m \le 2^n$, then $|T(m/2^n)| \le 2|x|\cdot\sum_{i=1}^n(\frac{2}{3})^i$. Suppose (m,n) is a pair of positive integers such that $m \le 2^n$. If $n = 1$, then we have already seen that $T(m/2^n)$ exists and $|T(m/2^n)| \le |x| \le 2|x|\cdot\sum_{i=1}^n(\frac{2}{3})^i$. Therefore, assume $n > 1$ and for each positive integer k less than n, assume that $T(m/2^n)$ exists and $|T(s/2^k)| \le 2|x|\cdot\sum_{i=1}^k(\frac{2}{3})^i$ for each $s \in \{0,1,\dots,2^k\}$. Furthermore, assume $T(s/2^n)$ exists and $|T(s/2^n)| \le 2|x|\cdot\sum_{i=1}^n(\frac{2}{3})^i$ for each $s \in \{0,1,\dots,m-1\}$. Then, since $|T(m-1/2^n)| \le 4|x| < 4B < B_2$ and $|T(1/2^n)| < B_2$, it follows from the choice of B_2 that $T(m/2^n)$ exists.

If m is even, then by induction we have

$$\left|T\left(\frac{m}{2^n}\right)\right| = \left|T\left(\frac{m/2}{2^{n-1}}\right)\right| \le 2|x|\cdot\sum_{i=1}^{n-1}\left(\frac{2}{3}\right)^i \le 2|x|\cdot\sum_{i=1}^{n}\left(\frac{2}{3}\right)^i.$$

If n is odd, then $m = 2r+1$ for some nonnegative integer r. Therefore, using the triangle inequality, the inductive hypothesis, the choice of B_2, and the fact that $2|x|\cdot\sum_{i=1}^r(\frac{2}{3})^i \le 4|x| < 4B < B_2$, we obtain

$$\begin{aligned}
\left|T\left(\frac{m}{2^n}\right)\right| &= \left|T\left(\frac{2r+1}{2^n}\right)\right| \\
&\le \left|V\left(T\left(\frac{r}{2^{n-1}}\right), T\left(\frac{1}{2^n}\right)\right) - T\left(\frac{r}{2^{n-1}}\right)\right| + \left|T\left(\frac{r}{2^{n-1}}\right)\right| \\
&\le 2\left|T\left(\frac{1}{2^n}\right)\right| + \left|T\left(\frac{r}{2^{n-1}}\right)\right| \\
&\le 2|x|\cdot\left(\frac{2}{3}\right)^n + 2|x|\cdot\sum_{i=1}^{n-1}\left(\frac{2}{3}\right)^i \\
&\le 2|x|\cdot\sum_{i=1}^{n}\left(\frac{2}{3}\right)^i.
\end{aligned}$$

Thus, Lemma 1.3 is proved. The next lemma shows that T has a unique extension to $[0,1]$.

Lemma 1.4. *T is uniformly continuous on Q.*

Proof. Recall that B_2 is a positive number such that if each of x and y is in D and has norm less than B_2 then $|V(x,y)-x-y| \leq |y|$ and recall that B is a positive number less than $B_2/4$. Suppose (m,n) is a pair of nonnegative integers such that $m < 2^n$ and a is in Q such that $a-(m/2^n) \leq (1/2^n)$. Let $\{b_i\}_{i=1}^k$ be a finite sequence in $\{0,1\}$ such that $a=(m/2^n)+\sum_{i=1}^k(b_i/2^{n+i})$. Let $a_i = (m/2^n)+\sum_{j=1}^i(b_j/2^{n+j})$ for each $i \in \{0,1,\dots,k\}$. Then, since $|T(a_i)| < 4B < B_2$ for each $i \in \{0,1,\dots,k\}$ by lemma 1.3, it follows from the triangle inequality and the choice of B_2 that

$$\begin{aligned} \left|T\left(\frac{m}{2^n}\right)\right| &\leq \sum_{i=1}^k |T(a_{i-1})-T(a_i)| \\ &\leq 2|x|\sum_{i=1}^k \left|T\left(\frac{1}{2^{n-i}}\right)\right| \\ &\leq 2|x|\sum_{i=1}^k \left(\frac{2}{3}\right)^{n+1} \\ &\leq 6|x|\left(\frac{2}{3}\right)^n. \end{aligned}$$

This implies that T is uniformly continuous on Q and hence completes the proof of Lemma 1.4.

It is clear by the construction and the continuity of T that $T(0)=0$, $T(1)=1$, and $T(s+t)=V(T(s),T(t))$ whenever each of s, t, and $s+t$ is in $[0,1]$. Thus, Theorem 1 is proved.

Locally compact monoids

A C_s^k monoid in which $k \geq 1$ and which has a neighborhood, U, of 1 so that $g_1(U)$ is a compact subset of X is called a locally compact monoid. It is clear, since $g_1(U)$ is compact in X and each C_s^k monoid is a strongly differentiable groupoid, that Theorem 1 can be applied to the setting of differentiable semigroups as defined by Graham. We proceed with Theorem 2.

Theorem 2. *If S is a C_s^k locally compact monoid, $k \geq 1$, then S has a C_s^k one parameter submonoid T.*

Proof. Choose a compact (in X) neighborhood, D, of 0 in $g_1(S)$ so that $V(x,y) = g_1(g_1^{-1}(x)\cdot g_1^{-1}(y))$ exists for each x and y in D. Since V is clearly a strongly differentiable groupoid, Theorem 1 provides the existence of an x in D and a function $R\colon [0,1] \to D$ such that $R(0)=0$, $R(1)=x$, and $R(s+t)=V(R(s),R(t))$ whenever each of s, t, and $s+t$ is in $[0,1]$. Define $T\colon [0,1] \to S$ by $T(t)=g_1^{-1}(R(t))$ for each t in $[0,1]$. Clearly, T has a unique continuous extension to R^+ since S is a monoid and, hence is algebraically closed. Since T is obviously a one parameter submonoid of S, it only remains to be seen that T is C_s^k on R^+. However, it was shown in [H5, Theorem 2] by Holmes that if S is a C_s^k monoid and T is a continuous homomorphism from R^+ into S with $T(0) = 1$ then T is C_s^k. Thus, Theorem 2 is proved.

Although Theorem 1 shows that very little differentiability is required in order to guarantee the existence of one parameter submonoids in locally compact monoids, the author knows of no example of a locally compact monoid with multiplication strongly differentiable at $(1,1)$ which is not analytic. Indeed, in the case of an Analytical group (see introduction), Birkhoff in [B] showed that no such example exists. The author is presently considering the following questions:

(1) If S is a monoid, based on an admissible subset of the Banach space, E, with multiplication V which is strongly differentiable at $(1,1)$, then is there an extension, W, of V to an open subset of $E \times E$ containing $(1,1)$ which is a monoid tht is strongly differentiable at $(1,1)$?

(2) If T is a one parameter submonoid of the monoid S which has multiplication strongly differentiable at $(1,1)$, is T strongly differentiable at 0?

(3) If S is a locally compact monoid which has multiplication strongly differentiable at $(1,1)$, is the component of 1 an admissible set?

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