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AN APPLICATION OF HADAMARD-LÉVY'S THEOREM TO A SCALAR INITIAL VALUE PROBLEM

SORIN RĂDULESCU AND MARIUS RĂDULESCU

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ABSTRACT. A global inversion theorem due to Hadamard and Lévy is applied to a Volterra integral operator acting in a space of continuous functions to obtain a one-sided criterion for global existence and uniqueness of a scalar initial value problem.

The aim of the present paper is to establish an existence and uniqueness theorem for a scalar initial value problem, by using a global inversion theorem due to Hadamard and Lévy.

We obtain a one-sided condition which improves a special case of a previous result established in [8].

The following theorem is known as the Hadamard-Lévy theorem.

Theorem 1. *Let E, F be two Banach spaces and $f\colon E \to F$ be a C^1 map for which*

(1) $f'(x) \in \operatorname{Isom}(E, F)$ *for every* $x \in E$. *If there exists a continuous map* $\omega\colon \mathbf{R}_+ \to \mathbf{R}_+^*$ *such that*
(2) $\int_0^\infty \frac{ds}{\omega(s)} = +\infty$,
(3) $\|[f'(x)]^{-1}\| \leq \omega(\|x\|)$, $x \in E$,

then f is a C^1 global diffeomorphism. Here by a C^1 global diffeomorphism we understand a one-to-one map which is C^1, onto and whose inverse is also a C^1 map. By $\operatorname{Isom}(E, F)$ *we denote the set of all linear continuous isomorphisms of E onto F.*

One can easily see that for some positive constants a, b the maps $\omega_1(t) = 1$, $\omega_2(t) = at + b$, $\omega_3(t) = (at+b)\ln(t+2)$, $\omega_4(t) = (at+b)\ln(t+2)\ln\ln(t+3)$, $t \geq 0$ satisfy condition (2).

A proof of Theorem 1 may be found in [2, 6–9, 11]. Here we have presented a somewhat more general statement cf. [5]. In [1, 3, 4, 8–11], the Hadamard-Lévy

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theorem was used to obtain existence and uniqueness theorems for nonlinear boundary value problems.

Throughout the paper J will denote the closed bounded interval $[a, b]$ and F will denote the Banach space of all real continuous functions defined on J, endowed with the usual sup-norm.

Lemma 2. *Let* $v \in F$, $v_1 = \frac{1}{2}(v + |v|) = \max(v, 0)$ *and* $c = \exp(\sup_{t\in J} \int_a^t v(s)ds)$. *Consider the linear bounded operator* $T: F \to F$ *defined as*

$$(4)\qquad (Tk)(t) = k(t) + \int_a^t \exp\left(\int_s^t v(\tau)\,d\tau\right) v(s)k(s)\,ds \qquad k \in F, t \in J.$$

Then the following assertions hold

(i) $c \le \|T\| \le 2\exp(\int_a^b v_1(s)\,ds)$;

(ii) $\|T\| = c$ *provided that* $v \ge 0$;

(iii) *the map* $h = Tk$ *is the unique solution in* F *of the integral equation*

$$(5)\qquad h(t) - \int_a^t v(s)h(s)\,ds = k(t) \qquad t \in J.$$

Proof. Denote $w(s, t) = \exp(\int_s^t v(\tau)\,d\tau)$, $w_1(s, t) = \exp(\int_s^t v_1(\tau)\,d\tau), s, t \in J$. One can easily see that

$$(6)\qquad \int_a^t w(s, t)v(s)\,ds = w(a, t) - 1 \qquad t \in J$$

$$(7)\qquad \int_a^t w_1(s, t)v_1(s)\,ds = w_1(a, t) - 1 \qquad t \in J$$

$$(8)\qquad w(s, t) \le w_1(s, t) \qquad a \le s \le t \le b.$$

Let $t \in J$ and $k \in F$. Then by (6), (7) and (8) we obtain

$$\begin{aligned} |(Tk)(t)| &\le \|k\| \left(1 + \int_a^t w(s, t)|v(s)|\,ds\right) \\ &= \|k\| \left(1 + \int_a^t w(s, t)(2v_1(s) - v(s))\,ds\right) \\ &= \|k\| \left(1 + 2\int_a^t w(s, t)v_1(s)\,ds - \int_a^t w(s, t)v(s)\,ds\right) \\ &\le \|k\| \left(2 - w(a, t) + 2\int_a^t w_1(s, t)v_1(s)\,ds\right) \\ &= \|k\|(2w_1(a, t) - w(a, t)) \le 2\|k\|w_1(a, b). \end{aligned}$$

Thus the right inequality in assertion (i) is proved. Let $k_0 \in \mathbf{R}$ and $k(t) = k_0$ for every $t \in J$. Then

$$(Tk)(t) = k_0\left(1 + \int_a^t w(s, t)v(s)\,ds\right) = k_0 w(a, t) \qquad t \in J$$

whence $\|Tk\| = c\|k\|$. Thus the proof of the left inequality in (i) is complete. Assertion (ii) follows at once from (i) and from the inequality

$$|(Tk)(t)| \le \|k\| \left(1 + \int_a^t w(s,t)v(s)\,ds\right) \qquad t \in J, k \in F.$$

By Banach's contraction principle (cf. [8, Lemma 3.1]), equation (5) has a unique solution. Integration by parts shows that the map $h = Tk$ verifies (5).

It would be interesting to know whether assertion (ii) holds without the assumption $v \ge 0$.

Theorem 3. *Let $f\colon J \times \mathbf{R} \to \mathbf{R}$ be a continuous map for which its partial derivative with respect to the second argument, denoted by f'_x, exists and is continuous. Suppose that there exist an integrable map $g\colon J \to \mathbf{R}_+$ and a continuous increasing map $\omega\colon \mathbf{R}_+ \to (1,\infty)$ for which the following conditions hold*

$$\int_0^\infty \frac{ds}{\omega(s)} = +\infty \tag{9}$$

$$f'_x(t,x) \le g(t)\ln\omega(|x|) \qquad t \in J, x \in \mathbf{R}. \tag{10}$$

The for every $x_0 \in \mathbf{R}$ the initial value problem

$$\dot{x} = f(t,x), \qquad x(a) = x_0 \tag{11}$$

has a unique global solution $x\colon J \to \mathbf{R}$.

Proof. Let $x_0 \in \mathbf{R}$ and define $V\colon F \to F$ as

$$(Vx)(t) = x(t) - x_0 - \int_a^t f(s,x(s))\,ds, \qquad t \in J, x \in F.$$

Note that V is of C^1 class,

$$V'(x)(h)(t) = h(t) - \int_a^t f'_x(s,x(s))h(s)ds, \qquad t \in J, x, h \in F$$

and $V'(x) \in \operatorname{Isom}(F,F)$ for every $x \in F$.

Consider the equation $V'(x)(h) = k$ where $x, k \in F$ are supposed to be known. Then by Lemma 2 the unique solution of the above equation is

$$h(t) = k(t) + \int_a^t \exp\left(\int_s^t f'_x(\tau,x(\tau))\,d\tau\right)\cdot f'_x(s,x(s))k(s)\,ds, \qquad t \in J,$$

and

$$\|[V'(x)]^{-1}(k)\| = \|h\| \le 2\exp\left(\int_a^b \max(f'_x(s,x(s)),0)\,ds\right).$$

By (10) we obtain

$$\|[V'(x)]^{-1}\| \le 2\exp\left(\int_a^b g(s)\ln\omega(\|x\|)\,ds\right) \qquad x \in F. \tag{12}$$

If $\int_a^b g(s)\,ds \le 1$, then it follows from (11) that

$$\|[V'(x)]^{-1}\| \le 2\omega(\|x\|), \qquad x \in F.$$

Then Hadamard-Lévy theorem implies that V is a C^1 global diffeomorphism, whence the equation $Vx = 0$ has a unique solution $x \in F$, which is the unique solution of (11).

Next, consider the case $\int_a^b g(s)\,ds > 1$. Let $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a division of J such that

$$\int_{t_i}^{t_{i+1}} g(s)\,ds \le 1, \qquad i = 0, 1, \ldots, n-1. \tag{13}$$

Note that

$$f'_x(t, x) \le g(t) \ln \omega(|x|), \qquad t \in [t_i, t_{i+1}], x \in \mathbf{R}, \quad i \in \{0, 1, \ldots, n-1\}. \tag{14}$$

By (13) and (14) there exists a unique solution $\varphi_0\colon [t_0, t_1] \to \mathbf{R}$ for the initial value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0. \tag{IVP$_0$}$$

Denote $x_1 = \varphi_0(t_1)$. By (13) and (14) there exists a unique solution φ_1: $[t_1, t_2] \to \mathbf{R}$ for the intial value problem

$$\dot{x} = f(t, x), \qquad x(t_1) = x_1. \tag{IVP$_1$}$$

Construct inductively the solutions $\varphi_i\colon [t_i, t_{i+1}] \to \mathbf{R}$, $i = 1, 2, \ldots, n-1$, of the initial value problems

$$\dot{x} = f(t, x), \qquad x(t_i) = x_i, \tag{IVP$_i$}$$

where $x_i = \varphi_{i-1}(t_i)$.

One can easily see that the map $\varphi(t) = \varphi_i(t)$, $t \in [t_i, t_{i+1}]$, $i \in \{0, 1, \ldots, n-1\}$ is of C^1 class, verifies the equations $\dot{\varphi}(t) = f(t, \varphi(t))$, $t \in J$, $\varphi(a) = x_0$, and is the unique map possessing these properties.

The above theorem improves a special case (namely $E = \mathbf{R}$) of the following theorem:

Theroem 4[8, Theorem 3.3]. *Let E be a Banach space and $f\colon [0, a] \times E \to E$ a continuous map for which its partial derivative with respect to the second argument, denoted by f'_x, exists and is continuous.*

Suppose that there exist a constant $k > 0$ and a continuous increasing map $\omega\colon \mathbf{R}_+ \to [1, \infty)$ satisfying condition (9) and a constant $k > 0$ such that

$$\|f'_x(t, x)\| \le k \ln \omega(\|x\|), \qquad t \in [0, a],\ x \in E.$$

Then for every $x_0 \in E$ the initial value problem

$$\dot{x} = f(t, x), \qquad x(0) = x_0$$

has a unique global solution $x\colon [0,a] \to E$.

Remarks. One can easily see that there is a class of functions $f\colon J \times \mathbf{R} \to \mathbf{R}$ for which the growth condition (10) on f'_x holds but the Wintner condition [12]:

$$|f(t,x)| \le g(t)\omega(|x|), \qquad t \in J,\ x \in \mathbf{R} \tag{14}$$

where g is integrable and ω verifies (9), does not hold. See for example rapidly decreasing maps such as $f_1(t,x) = -e^x$ or $f_2(t,x) = -x^n|x|$, $t \in J$, $x \in \mathbf{R}$ where n is a positive odd number.

Note that there are also functions $f\colon J \times \mathbf{R} \to \mathbf{R}$ such as $f(t,x) = \sin x^2$, $t \in J, x \in \mathbf{R}$ for which Wintner condition (14) holds but which does not verify the growth condition (10) with ω subject to (9).

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Institute of Mathematics, str.Academiei 14, 70109 Bucharest, Romania

Centre of Mathematical Statistics, Bd. Magheru 22, 70158, Bucharest, Romania