EXISTENCE OF MULTIPLE PERIODIC SOLUTIONS
FOR A SEMILINEAR EVOLUTION EQUATION

NORIMICHI HIRANO

(Communicated by Walter D. Littman)

Abstract. In this paper, we consider the existence of multiple periodic solutions for the problem
\[ \frac{du}{dt} + Lu = g(u) + h, \quad t > 0, \]
\[ u(0) = u(T), \]
where \( L \) is a uniformly strongly elliptic operator with domain \( D(L) = H^m_0(\Omega) \), \( g: \mathbb{R} \to \mathbb{R} \) is a continuous mapping, \( T > 0 \) and \( h: (0, T) \to H^m_0(\Omega) \) is a measurable function.

1. Introduction

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with a smooth boundary \( \partial \Omega \) and
\[
L := \sum_{0 \leq |\alpha|, |\beta| \leq m} (-1)^{|\beta|} D^\beta (a_{\alpha \beta}(x) D^\alpha)
\]
a differential operator of order \( 2m \) with domain \( D(L) = H^m_0(\Omega) \) satisfying the following conditions.

(L1) \( L \) is uniformly strongly elliptic, that is, there exists \( C > 0 \) such that
\[
\sum_{|\alpha| = |\beta| = m} a_{\alpha \beta}(x) \xi^\alpha \xi^\beta \geq C |\xi|^m
\]
for all \( \xi \in \mathbb{R}^n \) and \( x \in \Omega \).

(L2) \( a_{\alpha \beta} = a_{\alpha \beta} \) are real valued functions in \( C^\infty(\Omega) \).

Our purpose in the present paper is to consider the existence of multiple solutions for the problem of the form
\[
\left( P \right) \quad \frac{du}{dt} + Lu = g(u) + h(t), \quad t > 0, \]
\[ u(0) = u(T), \]
where \( T > 0 \), \( h: (0, \infty) \to L^2(\Omega) \) is a \( T \)-periodic function and, \( g: \mathbb{R} \to \mathbb{R} \) is a continuous function with \( g(0) = 0 \).

Received by the editors August 1, 1988.
Key words and phrases. Periodic solution, elliptic operator, evolution equation.

©1989 American Mathematical Society
0002-9939/89 $1.00 + $.25 per page
The existence of periodic solutions for problems of this kind has been studied by many authors (see Nieto [8] which also contains many references). But few seem to be known about the multiplicity of the periodic solutions. In [3], Amann established a multiplicity result for the problem \( P \) for the case in which \( L \) is second order. The method employed in [1] is based on the super-subsolution method. Our argument in the present paper is based on the Leray-Schauder fixed point theorem.

To state our result, we need some notations. Throughout the rest of this paper, we assume that \( L \) satisfies the condition (L1) and (L2). Let \( \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \) denote the eigenvalues of the selfadjoint realization in \( L^2(\Omega) \) of \( L \). If \( X \) is a Banach space, we denote by \( L^p(0, T ; X) \) \( (1 \leq p \leq \infty) \) the space of all \( X \)-valued strongly measurable functions such that \( \int_0^T \|u(t)\|^p \, dt < \infty \). \( W^{1,p}(0, T ; X) \) denotes the space of functions \( u \in L^p(0, T ; X) \) such that \( du/dt \in L^p(0, T ; X) \), where \( du/dt \) is the derivative in the sense of distribution (see [4]). We denote by \( | \cdot | \) the norm of \( L^2(\Omega) \).

We suppose that \( g \) satisfies the following condition

\[
-\infty < a_* \leq a^* < \lambda_1 < b_* \leq b^* < \lambda_2,
\]

where

\[
a_* = \inf_{s \neq t} \frac{g(t) - g(s)}{t - s}, \quad a^* = \limsup_{|t| \to \infty} \frac{g(t)}{t}, \quad b_* = \liminf_{|t| \to \infty} \frac{g(t)}{t}, \quad b^* = \sup_{s \neq t} \frac{g(t) - g(s)}{t - s}.
\]

**Theorem.** Suppose that \( g \) satisfies the condition (G). Then there exists \( M > 0 \) such that for each \( T \)-periodic function \( h \in W^{1,\infty}(0, T ; L^2(\Omega)) \) with

\[
\sup \{|h(t)| : t \in (0, T)\} \leq M
\]

the problem \( P \) possesses at least two solutions in \( W^{1,\infty}(0, T ; L^2(\Omega)) \).

**Remark.** In (G), we have implicitly supposed \( \lambda_1 \) is single.

2. Preliminaries

In the following, we write \( H \), \( V \) and \( V^* \) instead of \( L^2(\Omega) \), \( H_0^m(\Omega) \) and \( H^{-m}(\Omega) \), respectively. The pairing between \( H_0^m(\Omega) \) and \( H^{-m}(\Omega) \) is denoted by \( \langle \cdot, \cdot \rangle \). If \( u, v \in L^2(\Omega) \), then \( \langle u, v \rangle \) is the ordinary inner product of \( L^2(\Omega) \). Here we observe that we may assume without any loss of generality that \( \lambda_1 > 0 \). In fact, the problem \( P \) is equivalent to \( du/dt + L_1 u = g_1(u) + h, \ u(0) = u(T) \), where \( L_1 u = Lu + \lambda u \) and \( g_1(u) = \lambda u + g(u) \). If \( \lambda + \lambda_1 > 0 \), the first eigenvalue of \( L_1 \) is positive, and \( g_1 \) satisfies the condition (G) with \( \lambda_1 \) replaced by \( \lambda + \lambda_1 \), \( \lambda + \lambda_2 \), respectively. Also we may assume that \( 0 < a_* \). We denote by \( \| \cdot \| \) the norm of \( H_0^m(\Omega) \) defined by \( \|u\|^2 = \langle Lu, u \rangle \) for \( u \in H_0^m(\Omega) \). From the observation above, the norm \( \| \cdot \| \) is equivalent to the Sobolov norm on \( H_0^m(\Omega) \). The norm of \( H^{-m}(\Omega) \) is denoted by \( \| \cdot \|_* \). Let \( \Phi \)
and $V_2$ be the subspaces of $H_0^m(\Omega)$ spanned by the eigenspaces corresponding to the eigenvalues $\{\lambda_2, \lambda_3, \ldots\}$ and $\{\lambda_1\}$, respectively. Then $V_1$ and $V_2$ are orthogonal in $L^2(\Omega)$. Let $\phi$ be a normalized eigenfunction corresponding to $\lambda_1$. Then $\phi \in L^\infty(\Omega)$ and $V_2 = \{k\phi: k \in \mathbb{R}\}$. We denote by $P_1$ and $P_2$ the projections from $L^2$ onto $V_1$ and $V_2$, respectively. We denote by $|\Omega|$ the measure of $\Omega$. Suppose that $g$ satisfies the condition (G). Then there exist positive constants $\alpha, \beta, \rho, \delta$ such that

(2.1) \[ \alpha < \lambda_1, \quad g(t)/t \leq \alpha \quad \text{for all } t \quad \text{with } |t| \geq \delta. \]

and

(2.2) \[ \beta > \lambda_1, \quad g(t)/t \geq \beta \quad \text{for all } t \quad \text{with } 0 < |t| \leq \rho. \]

**Lemma 1.** There exists $\omega_1 > 0$ such that

\[ \langle Lu - g(u), u \rangle \geq \omega_1 \|u\|^2 - \delta \cdot \max\{g(t): |t| \leq \delta\} |\Omega| \]

for all $u \in V$.

**Proof.** From (2.1), we have that

\[ \langle Lu - g(u), u \rangle \geq \|u\|^2 - \alpha \|u\|^2 - \int_{|u| < \delta} a(u)u \, dx \geq \|u\|^2 - \alpha \|u\|^2 - \delta \cdot \max\{g(t): |t| \leq \delta\} |\Omega| \]

for each $u \in V$. Then since $\alpha < \lambda_1$, the inequality (2.3) follows:

**Lemma 2.** There exist $c_0 > 0$ and $\omega_2 > 0$ such that

(2.4) \[ \langle Lu - g(u), u - 2P_2u \rangle \geq \omega_2 \|u\|^2 \quad \text{for all } u \in V \quad \text{with } \|P_2u\| \leq c_0. \]

**Proof.** Let $b$ be a positive constant with $b^* < b < \lambda_2$. We choose $d > 0$ so small that

(2.5) \[ (b - b^*)p^2 - (b - a_*)d^2 > 0 \quad \text{for all } p \quad \text{with } p \geq \rho - d. \]

We next choose $c_0 > 0$ such that $\sup_{x \in \Omega} |w(x)| \leq d$ for all $w \in V_2$ with $\|w\| \leq c_0$. In fact, we can choose such $c_0 > 0$ because $\phi \in L^\infty(\Omega)$. Let $u \in V$ such that $\|P_2u\| \leq c$. For simplicity, we put $v = P_1u$ and $w = P_2u$. Also we set $A = \{x \in \Omega: |u(x)| > \rho\}$. Then we have

(2.6) \[ \langle Lu - g(u), u - 2P_2u \rangle \geq \|v\|^2 - \lambda_1 |w|^2 - \int_\Omega g(u)(v - w) \, dx. \]

Let $x \in A$. If $|w(x)| > |v(x)|$, then noting that $u(x)(v - w)(x) < 0$ we find by (G) that

\[ -g(u(x))(v - w)(x) \geq a_* |w(x)|^2 - a_* |v(x)|^2 \geq a_* |w(x)|^2 - b^* |v(x)|^2. \]

If $|w(x)| \leq |v(x)|$, then we have again by (G) that

\[ -g(u(x))(v - w)(x) \geq a_* |w(x)|^2 - b^* |v(x)|^2. \]
Then from (2.5) and the inequalities above, we have that
\[-\lambda_1|w(x)|^2 - g(u(x))(v - w)(x) \geq (b - \lambda_1)|w(x)|^2 - b|v(x)|^2
+ \{(b - b^*)|v(x)|^2 - (b - a_*)(w(x))^2\}
\geq (b - \lambda_1)|w(x)|^2 - b|v(x)|^2.

Let \( x \in \Omega \setminus A \). Then we have by (2.2) and (G)
\[ (2.8) \quad -g(u(x))(v - w)(x) \geq \beta|w(x)|^2 - b^*|v(x)|^2. \]
Then we find that
\[ (2.9) \quad -\lambda_1|w(x)|^2 - g(u(x))(v - w)(x) \geq (\beta - \lambda_1)|w(x)|^2 - b^*|v(x)|^2. \]
Then combining (2.7), (2.9) with (2.6), we obtain that
\[ (2.10) \quad \langle Lu - g(u), u - 2P_2u \rangle \geq \|v\|^2 - \lambda_1|w|^2 - \int_\Omega g(u)(v - w) \, dx, \]
\[ \geq \|v\|^2 - b|v|^2 + (\beta - \lambda_1)|w|^2. \]
Since \( b < \lambda_2 \) and \( (\beta - \lambda_1) > 0 \), we can choose \( \omega_2 > 0 \) such that
\[ \|v\|^2 - b|v|^2 + (\beta - \lambda_1)|w|^2 \geq \omega_2\|u\|^2 \]
and this completes the proof.

We now define a functional \( \Phi: V \to \mathbb{R} \) by
\[ (2.11) \quad \Phi(v) = \frac{1}{2} \langle Lv, v \rangle - \int_\Omega \int_0^{u(x)} g(\tau) \, d\tau \, dx \quad \text{for each } v \in V. \]
It is easy to see that \( \langle \Phi'(v), w \rangle = \langle Lv - g(v), w \rangle \) for \( v, w \in V \). It is also easy to verify that for each \( w \in V_2 \), the mapping \( \Phi(\cdot + w): V \to \mathbb{R} \) is strictly convex. In fact, we have from (G) that
\[ (2.12) \quad \langle \Phi'(v_1 + w) - \Phi'(v_2 + w), v_1 - v_2 \rangle \geq \|v_1 - v_2\|^2 - b^*|v_1 - v_2|^2 \]
\[ \geq \omega_2\|v_1 - v_2\|^2 \]
for each \( w \in V_2 \) and each \( v_1, v_2 \in V_1 \). This implies that \( \Phi(\cdot + w) \) is strictly convex. Then we have that for each \( w \in V_1 \), there exists a unique element \( v_0 \in V_1 \) such that
\[ \Phi(v_0 + w) = \min\{\Phi(v + w) : v \in V_1\}. \]
Here we set \( \theta(w) = v_0 \). Then \( \theta: V_2 \to V_1 \) is a continuous mapping satisfying that
\[ (2.13) \quad \langle L(\theta(w) + w) - g(\theta(w) + w), z \rangle = 0 \quad \text{for all } z \in V_1. \]
Let \( w \in V_2 \). Then it also follows from the strict convexity of \( \Phi(\cdot + w) \) that
\[ (\ast) \quad v = \theta(w) \text{ if and only if } \langle L(v + w) - g(v + w), z \rangle = 0 \quad \text{for all } z \in V_1. \]
We next define a continuous function \( F : \mathbb{R} \to \mathbb{R} \) by
\[
F(s) = \Phi(\theta(s\phi) + s\phi) \quad \text{for } s \in \mathbb{R}.
\]
Then it is not difficult to see that
\[
F'(x) = \langle L(\theta(s\phi) + s\phi) - g(\theta(s\phi) + s\phi), \phi \rangle,
\]
(See Theorem 2.3 of [2].) It also follows from (2.13) and (2.15) that if \( F(s) = \min\{F(\tau) : \tau \in \mathbb{R}\} \), then
\[
L(\theta(s\phi) + s\phi) - g(\theta(s\phi) + s\phi) = 0.
\]

**Lemma 3.** (1) There exists \( r > 0 \) such that \( F(s) \geq 0 \) for all \( s \) with \( |s| \geq r \);
(2) \( \min\{F(s) : s \in (0, r)\} < 0 \) and \( \min\{F(s) : s \in (-r, 0)\} < 0 \).

**Proof.** We have by (2.13) and Lemma 1 that
\[
F'(s)s = \langle L(\theta(s\phi) + s\phi) - g(\theta(s\phi) + s\phi), s\phi \rangle
\]
\[
\geq \omega_1 \|\theta(s\phi) + s\phi\|^2 - \delta \cdot \max\{g(\tau) : |\tau| \leq \delta\}|\Omega|
\]
\[
\geq s^2 \omega_1 \|\phi\|^2 - \delta \cdot \max\{g(\tau) : |\tau| \leq \delta\}|\Omega|.
\]
Then from the inequality above, we can choose \( r > 0 \) such that \( F(s) \geq 0 \) for all \( s \) with \( |s| \geq r \). We next show that (2) holds. From (2.13) and Lemma 2, we have that
\[
F'(s)s = \langle L(\theta(s\phi) + s\phi) - g(\theta(s\phi) + s\phi), s\phi \rangle < 0
\]
for all \( s \) with \( 0 < |s| \leq c_0 \). Then since \( F(0) = 0 \), (2) follows from (1) and the inequality above.

**3. Proof of theorem**

The Proof of Theorem will be accomplished by a series of lemmas. From Lemma 3, we can see that there exist constants \( \alpha_\pm, \beta_\pm, \alpha_\pm^*, \) and \( \beta_\pm^* \) such that
\[
\alpha_-^* < \alpha_- < \beta_- < \alpha_-^* < \alpha_+ < \beta_+ < \beta_-^*.
\]
(3.1)
\[
F(s) = m_+ = \min\{F(\tau) : \tau \in (0, r)\} \quad \text{on } (\alpha_+, \beta_+),
\]
\[
F(s) = m_- = \min\{F(\tau) : \tau \in (-r, 0)\} \quad \text{on } (\alpha_-, \beta_-) \quad \text{and}
\]
\[
|F'(s)| > 0 \quad \text{for all } s \in (\alpha_+^*, \beta_-^*) \setminus (\alpha_-, \beta_+).
\]
For simplicity, we will assume that \( m_- = m_+ = m \). Here we choose a negative number \( c \) such that \( m < c < \min\{F(\alpha_+^*), F(\beta_-^*)\} \). Then from the definition above, \( V_\pm = \{\tau \in \mathbb{R} : F(\tau) \leq c\} \cap (\alpha_+^*, \beta_-^*) \) are closed intervals which contain \((\alpha_+, \beta_+)\) and \((\alpha_-, \beta_-)\), respectively. We set
\[
K_\pm = \{v \in V : \Phi(v) \leq c\} \cap \{v \in V : P_2 v \in V_\pm\} \quad \text{and} \quad \tilde{K}_\pm = \overline{\Phi} K_\pm,
\]
where \( \overline{\Phi} A \) denotes the closed convex hull of a subset \( A \) of \( V \). Then from (2.12), we can see that \( K_\pm \) are bounded closed subsets of \( V \). Since \( V \) is compactly embedded in \( H \), \( K_\pm \) is compact in \( H \). We also have by the definition that \( \tilde{K}_\pm \subset \{v \in V : P_2 v \in V_\pm\} \).
Lemma 4. There exist mappings $Q_\pm : \tilde{K}_\pm \to K_\pm$ such that $Q_\pm$ are continuous with respect to the norm of $H$ and $Q_\pm x = x$ for each $x \in K_\pm$.

Proof. Let $w \in V_+$. Then since the functional $\Phi(\cdot + w)$ is convex, we have that $P_2^{-1}(w) \cap K_+$ is compact convex in $H$. Here we denote by $Q_w$ the metric projection from $H$ onto $P_2^{-1}(w) \cap K_+$. Then $Q_w$ is continuous with respect to the norm of $H$. We define a mapping $Q_+$ from $\tilde{K}_+$ onto $K_+$ by $Q_+ x = Q_{P_2 x} x$ for each $x \in \tilde{K}_+$. Then it is easy to verify that the mapping $Q_+$ is continuous on $\tilde{K}_+$ and that $Q_+ x = x$ for $x \in K_+$. By the same argument, we can find a continuous mapping $Q_-$ satisfying the required property.

We next choose open sets $U_+ \subset V$ such that

$$\{\langle \theta(\cdot + t\phi) + t\phi : t \in (\alpha_\pm, \beta_\pm) \} \subset U_+ \subset K_+ \quad \text{and} \quad \sup\{\Phi(v) : v \in U_+\} < c.$$ 

Then we have the following lemma:

Lemma 5. There exist $d_+ > 0$ such that

$$\|L v - g(v)\| \geq d_+ \quad \text{for all } v \in K_+ \setminus U_+.$$ 

Proof. Suppose that there exists a sequence $\{v_n\} \subset K_+ \setminus U_+$ such that $\|L v_n - g(v_n)\| \to 0$, as $n \to \infty$. Since $K_+$ is a bounded subset of $V$, we may assume that $v_n \to v \in K_+$ weakly in $V$. Then from the compactness of $K_+$ in $L^2$, we have that $v_n \to v$ strongly in $L^2$. Thus we find that $L v - g(v) = 0$. Then from (*), it follows that $v = \theta(||P_2 v||\|\phi\| + ||P_2 v||\|\phi\|)$. Here we suppose that $||P_2 v|| \in (\alpha_+, \beta_+) \setminus (\alpha_+, \beta_+)$. Then we have by (3.1) that

$$0 = \langle L v - g(v), P_2 v \rangle = |F'(\|P_2 v\|)| > 0.$$ 

This is a contradiction. Therefore we have that $||P_2 v|| \in (\alpha_+, \beta_+)$ and $v \in U_+$. Then from (2.12) and (2.16), we find that

$$\|L w - g(w)\| ||w - v|| \geq \langle L w - g(w), w - v \rangle \geq \omega_2 ||w - v||^2$$ 

for all $w \in V$ with $P_2 w = P_2 v$. Noting that $P_2 v_n \to P_2 v$, we see that

$$\lim \|L w_n - g(w_n)\|_\ast = \lim \|L v_n - g(v_n)\|_\ast = 0,$$ 

where $w_n = v_n - P_2 v_n + P_2 v$ for each $n \geq 1$. Then combining (3.4) with (3.3), we obtain that $v_n$ converges to $v$ strongly in $V$. Since $K_+ \setminus U_+$ is strongly closed in $V$, we find that $v \in K_+ \setminus U_+$. This is a contradiction. Thus we have shown that there exists $d_+ > 0$ such that (3.2) holds for all $v \in K_+ \setminus U_+$. By the same argument, we have that there exists $d_- > 0$ satisfying (3.2) for all $v \in K_- \setminus U_-$. 

Hence we consider the initial value problem of the form

$$\frac{du}{dt} + Lu = g(u) + h, \quad 0 < t < T, \quad u(0) = u_0.$$
SOLUTIONS FOR A SEMILINEAR EVOLUTION EQUATION

where \( u_0 \in H \). Let \( h \in W^{1,1}(0,T;H) \) and \( u_0 \in H \). Then since \( g \) is Lipschitz continuous, the problem (3.5) has a unique solution \( u \in W^{1,\infty}(0,T;H) \) such that \( u(t) \in V \), a.e. \((0,\infty)\). (See, for example, [4, 5 or 6].) Noting that \( du/dt \in L^\infty(0,T;H) \) and

\[
\Phi(u(t)) - \Phi(u(s)) = \int_s^t \langle \Phi'(u(\tau)), u'(\tau) \rangle \, d\tau = \int_s^t \left( h(\tau) - \frac{du}{dt}, \frac{du}{dt} \right) \, d\tau
\]

for \( 0 < s \leq t < T \), we can see that the mapping \( t \to \Phi(u(t)) \) is continuous.

For each \( h \in W^{1,\infty}(0,T;H) \) and \( w \in H \), we set \( T_h w = u(T) \), where \( u \) is the solution of (3.5) with \( u_0 = w \). Then the mapping \( T_h : H \to H \) is continuous. (See also [6].)

**Lemma 6.** There exists \( M > 0 \) such that \( T_h w \in K_\pm \), for all \( w \in K_\pm \) and all \( h \in W^{1,\infty}(0,T;H) \) with \( \sup\{|h(t)|: t \in (0,T)\} < M \).

**Proof.** By Lemma 5, it is easy to verify that there exist a positive number \( M > 0 \) and a negative number \( c^- > c \) such that

\[
(3.6) \quad \|Lu - g(v)\|_* \geq M \quad \text{for all} \quad v \in W_\pm.
\]

where \( W_\pm = \{v \in V: \Phi(v) < c_*, P_\pm v \in (\alpha_\pm, \beta_\pm)\} \setminus U_\pm \). Let \( w \in K_\pm \) and \( h \in W^{1,\infty}(0,T;H) \) such that \( \sup\{|H(t)|: t \in (0,T)\} \leq M \). Let \( u : (0,T) \to V \) be the solution of the problem (3.5) with \( u_0 = w \). To prove Lemma 6, it is sufficient to show that \( \Phi(u(t)) \leq c \) for all \( t \in (0,T) \). Suppose that \( \Phi(u(t)) \leq c \) on \((0,s)\) and \( \Phi(u(s)) = c \) for some \( 0 < s < T \). Then we have that \( u(s) \in K_\pm \setminus U_+ \). Since the mapping \( t \to \Phi(u(t)) \) is continuous, we find that there exists \( s^* \in (s,T) \) such that \( u(t) \in W_+ \) on \((s,s^*)\). Then from (3.6), we find that

\[
\Phi(u(t)) - \Phi(u(s)) = \int_s^t \langle \Phi'(u(\tau)), u'(\tau) \rangle \, d\tau = \int_s^t \left( h(\tau) - (Lu(\tau) - g(u(\tau))), - (Lu(\tau) - g(u(\tau))) \right) \, d\tau \\
\leq \int_s^t (-|Lu(\tau) - g(u(\tau))|^2 + |Lu(\tau) - g(u(\tau))||h(\tau)|) \, d\tau \\
\leq \int_s^t (-\|Lu(\tau) - g(u(\tau))\|^2_\ast + \|Lu(\tau) = g(u(\tau))\|_\ast |h(\tau)|) \, d\tau \\
\leq -M^2(t-s) + M \cdot \sup\{|h(t)|: t \in (0,T)\}(t-s) \leq 0
\]

for all \( t \in (s,s^*) \). Therefore \( \Phi(u(t)) \leq c \) for all \( t \in (s,s^*) \). Thus we have shown that \( \Phi(u(t)) \leq c \) on \((0,T)\). This completes the proof.

We can now complete

**Proof of Theorem.** Let \( h \in W^{1,\infty}(0,T;H) \) such that \( \sup\{|h(t)|: t \in (0,T)\} \leq M \). From the definition of \( T_h \), the fixed points of \( T_h \) are the periodic
solutions of (P). Here we consider the mapping $ET_hQ_+: \tilde{K}_+ \to \tilde{K}_+$. From the observation above, $T_hQ_+$ is continuous with respect to the norm of $H$. Then since $\tilde{K}_+$ is compact and convex in $H$, $T_hQ_+$ has a fixed point $w_+ \in \tilde{K}_+$. Noting that $Q_+w_+ \in K_+ \subset \{v \in V : \Phi(v) \leq c\}$, we have by Lemma 6 that $w_+ = T_hQ_+w_+ \in K_+$. Then $w_+ = Q_+w_+$ and $w_+ = T_hw_+$. Therefore the solution $u_+$ of the problem (3.5) with the initial value $u_0 = w_+$ is a $T$-periodic solution. By the same argument, we have that there exists a solution $u_-$ of (P) satisfying $u_-(0) \in K_-$. Then since $K_+ \cap K_- = \{\phi\}$, the problem (P) has at least two solutions.

REFERENCES


Department of Mathematics, Faculty of Engineering, Yokohama National University, Yokohama, Japan