HYPERPLANE SECTIONS OF WEAKLY NORMAL VARIETIES
IN POSITIVE CHARACTERISTIC

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ABSTRACT. It is known that the general hyperplane sections of a weakly normal projective algebraic variety are weakly normal if the ground field has characteristic zero. This is shown to be false in positive characteristic; counterexamples can be obtained by intersecting "bad" weakly normal varieties with suitable linear spaces.

INTRODUCTION

Weakly normal (WN) algebraic varieties were introduced by Andreotti and Bombieri in 1969, see [3]. A natural problem is to understand whether the general hyperplane sections of a WN subvariety of \( \mathbb{P}^n \) are WN.

In characteristic zero this problem, studied in [1] and [2], was solved independently and affirmatively in [14] and [7]; a stronger statement also holds, see [8].

In positive characteristic an affirmative answer is known (see [7]) only for the class of those varieties which are \( S_2 \) and "WN1", i.e., WN and, roughly, well behaved in codimension 1 (see [9] and Definition 1 below).

In this paper we show (Theorem 1) that if any sufficiently general hyperplane section of an equidimensional WN variety \( X \subseteq \mathbb{P}^r \) is WN1, then \( X \) is also WN1. This easily implies that if \( X \) is WN but not WN1 a suitable intersection of \( X \) with a linear space (possibly the same \( \mathbb{P}^r \)) in WN, but its general hyperplane sections are not such. And since such varieties do exist (see [9] and Appendix) it follows that the answer to the above question is negative in general.
Our proof is algebraic, and is based on results from [9] and [6] concerning WN1 varieties and analytic branches, and makes use of Hironaka's theory of normal flatness, see [12].

1. THE MAIN RESULT

We work over an algebraically closed field \( k \) and by (algebraic) variety we mean an algebraic \( k \)-scheme, i.e., a scheme of finite type over \( k \). We recall the following definitions.

Definition 1 (see [3]). A reduced variety \( X \) is said to be weakly normal (WN) if every birational morphism \( X' \rightarrow X \), which is also a universal homeomorphism, is indeed an isomorphism. (This is equivalent to say that \( X \) coincides with its "weak normalization," see [3], Theorem 4.)

Definition 2 (see [9]). A variety \( X \) is said to be WN1 if it is WN and moreover the normalization morphism \( \overline{X} \rightarrow X \) is unramified in codimension 1.

The main result in this paper is the following theorem, we shall prove in §2.

Theorem 1. Let \( X \subseteq \mathbb{P}_k' \) be a WN equidimensional projective variety of dimension \( d \geq 2 \). Let us assume that \( X \cap H \) is WN1 for almost all hyperplanes \( H \subseteq \mathbb{P}_k' \).

Then \( X \) is WN1.

(As usual "almost all hyperplanes" or "general hyperplane" means "all the hyperplanes corresponding to a suitable nonempty open subset of \((\mathbb{P}_k')^*\)."

Corollary 1. Let \( X \subseteq \mathbb{P}_k' \) be an equidimensional projective variety which is WN but not WN1. Then there is a linear subspace \( L \subseteq \mathbb{P}_k' \) (possibly \( L = \mathbb{P}_k' \)) such that:

(a) \( Y = L \cap X \) is WN;

(b) a general hyperplane section of \( Y \) is not WN.

Proof. It is known that a WN curve is WN1 (see [9], (3.10)i). The conclusion follows easily by Theorem 1.

Remark. As pointed out in the Introduction, Corollary 1 and the Appendix imply that there are WN integral varieties of dimension \( 0 \geq 2 \) in some projective space whose general hyperplane sections are not WN. However from our method it is not clear if one can find counterexamples of any dimension.

2. PROOF OF THEOREM 1

In order to prove Theorem 1 we need a fact on associated graded rings (Lemma 1) whose geometric translation (Corollary 3) is well known, at least when \( k = \mathbb{C} \).
All rings in this paper are assumed to be commutative, with 1 and noetherian.

**Definition 3.** A prime ideal $\mathfrak{p}$ of a local ring $(R, \mathfrak{m}, k)$ is called admissible in $R$ if the local ring $R/\mathfrak{p}$ is regular and $\text{Gr}_{\mathfrak{p}}(R)$ is a free $R/\mathfrak{p}$-module.

**Lemma 1.** Let $(R, \mathfrak{m}, k)$ be a local ring, let $\mathfrak{p} \subset R$ be an admissible prime ideal and let $x \in R$ be an element, whose image in $R/\mathfrak{p}$ belongs to a regular system of parameters.

Then there is an isomorphism of graded $k$-algebras $\text{Gr}(R) \cong \text{Gr}(R/xR)[T]$ where $T$ is an indeterminate of degree 1.

In particular $\text{Gr}(R)$ and $\text{Gr}(R/xR)$ have the same number of minimal primes.

**Proof.** Let $r := \dim R/\mathfrak{p}$ and let $x_1 := x, x_2, \ldots, x_r$ be elements of $R$, whose images mod $\mathfrak{p}$ are a regular system of parameters of $R/\mathfrak{p}$.

Let us consider the commutative diagram

$$
\begin{array}{ccc}
\oplus_{n \geq 0} \mathfrak{p}^n/m^r \mathfrak{p}^n [x_1, \ldots, x_r] & \xrightarrow{\psi} & \text{Gr}(R) \\
\downarrow^\alpha & & \downarrow^\beta \\
\oplus_{n \geq 0} \mathfrak{p}^n/m^r \mathfrak{p}^n [x_2, \ldots, x_r] & \xrightarrow{\psi_1} & \text{Gr}(R/xR)
\end{array}
$$

where $\psi$ is the Hironaka isomorphism defined by $\psi(x_i) := x_i^* \in m/m^2$ (see [12] Chap. II, Prop. 1, p. 183) and $\alpha$ is defined by $\alpha(x_1) = 0, \alpha(x_i) = x_i$ for each $i = 2, \ldots, r$. The map $\beta$ is induced by the canonical homomorphism $R \to R/xR$ and $\psi_1$ is induced by $\psi$, since $\beta(\psi(x)) = \beta(x^*) = 0$.

Since $\psi$ is an isomorphism, $x^* = \psi(x)$ is a nonzero divisor in $\text{Gr}(R)$, hence $\text{Gr}(R/xR) \cong \text{Gr}(R)/(x^*)$ (see [13] (1.1)); therefore $\psi_1$ is an isomorphism and the conclusion follows.

**Corollary 2.** Let the notation and the assumptions be as in Lemma 1. Then $R$ and $R/xR$ have the same multiplicity.

**Proof.** By Lemma 1 the Hilbert polynomial of $\text{Gr}(R/xR)$ is the difference function of the Hilbert polynomial of $\text{Gr}(R)$. The conclusion follows by definition of multiplicity.

**Corollary 3.** Let $X \subseteq \mathbb{P}_k^r$ be a variety; let $Y \subseteq X$ be a 1-codimensional irreducible subvariety.

Then there exists a nonempty open subset $U \subset Y$ such that for every closed point $x \in U$ and every general hyperplane $H$ through $x$ one has $T_{X,x} \cong T_{X \cap H,x} \times \mathbb{A}^1$ where $T_{X,x}$ stands for tangent cone of $X$ at $x$.

**Proof.** It follows by Lemma 1 and [12] Chap. II, Cor. p. 189.

**Proof of Theorem 1.** Let $Y \subseteq X$ be an irreducible subvariety of codimension 1 and multiplicity $s$. By [9] (2.1) and (2.3) we have to prove that there exists a nonempty open subset $U \subset Y$ such that $X$ has exactly $s$ distinct branches (which are linear by the multiplicity formula) at every closed point $x \in U$. 

Let $U \subset Y$ be a nonempty open subset such that:

(i) $X$ is admissible along $Y$ at every $x \in U$;

(ii) for every closed point $x \in U$ there exists a canonical surjective map
\{
\text{branches of } X \text{ at } x \} \to \text{Min}(\text{Gr}(\mathcal{O}_{X,x}));

(iii) the number of branches of $X$ at every closed point $x \in U$ is constant.

Such a nonempty open subset $U$ of $Y$ exists by [12] (see Chap. II, Cor. p. 189),
[6] (see (3.1)), [10] (see (1.6)); then by (i) and [12] the multiplicity at each closed
$x \in U$ is $s$, and hence by (ii) and [10] (2.5) we have $\#\text{Min}(\text{Gr}(\mathcal{O}_{X,x})) \leq \#
\text{\{branches at } x \leq s\} \text{ for all closed } x \in U$. Hence by (iii) it is sufficient to
show that there is a closed point $z \in U$ such that $\#\text{Min}(\text{Gr}(\mathcal{O}_{X,z})) = s$.

Let $H$ be a general hyperplane of $\mathbb{P}_k^r$. We have:

(a) $H \cap U \neq \emptyset$;

(b) $H \cap X' =: X' \text{ is WN1};$

(c) every closed point of $U \cap X' \subset X'$ has multiplicity $s$ as a point of $X'$
(see Cor. 2);

(d) there exists a nonembedded irreducible component $Z$ of $Y' := Y \cap H$
such that $Z \cap U \neq \emptyset$ (see [7] (3.4)).

By (b), (c), (d) and [9] (see (2.3)) there exists a closed nonsingular point
$z \in Z \cap U$ such that
$$Gr(\mathcal{O}_{X',z}) \simeq k[x_1, \ldots, x_{s+d-1}]/(..., x_i \cdot x_j \ldots)_{1 \leq i < j \leq t}.$$ The conclusion follows by Lemma 1 applied to the local ring $R := \mathcal{O}_{X,z}$, the
prime ideal $\mathfrak{p}$ of $R$ associated to $Y$ and the local equation $t \in R$ of the
hyperplane $H$.

APPENDIX

For sake of completeness we show that there are projective integral varieties
which are WN but not WN1. The main idea of the construction is in [9], but
we give a more complete and simpler account here.

Lemma 2. Let $A'$ be a finitely generated integral normal $k$-algebra, $\mathfrak{p} \neq 0$
be a prime ideal of $A'$, $R' := A'/\mathfrak{p}$, $f : A' \to R'$ the natural homomorphism.

Let $R \subseteq R'$ be a $k$-subalgebra of $R'$ such that $R'$ is integral over $R$; let $K,$
$K'$ be the fractions fields of $R$, $R'$, respectively, and assume $R = K \cap R'$.

Let $A := \{a \in A'/f(a) \in R\}$. Then:

(i) $A$ is a finitely generated $k$-algebra with normalization $A'$;

(ii) $A$ is SN;

(iii) $A$ is WN if and only if $K = K_{\text{rad}}$, where $K_{\text{rad}} := \{a \in K'/a^{p^e} \in K \text{ for some } e \geq 0\}$
($p$ is the characteristic exponent of $k$);

(iv) if furthermore $ht_{\mathfrak{p}} = 1$, $K = K_{\text{rad}}$ and $K'/K$ is inseparable, then $A$ is
WN but not WN1.
Proof. A direct computation shows that, since \( R \subseteq R' \), the ideal \( \mathfrak{m} \) is the conductor of \( A \) in \( A' \); and since \( \mathfrak{m} \neq 0 \) the rings \( A \) and \( A' \) have the same quotient field.

(i) Let \( 0 \neq x \in \mathfrak{m} \). Then \( A' \subseteq Ax^{-1} \) and hence \( A' \) is integral over \( A \) by [4] (5.1)(iv), so \( A' = \overline{A} \) by the above remark, since \( A' \) is normal. Moreover \( A \) is a finitely generated \( k \)-algebra by [4] (7.8).

(ii) Let \( t \in A' \) be such that \( t^2, t^3 \in A \). Then \( f(t) = f(t^3)/f(t^2) \in K \cap R' = R \), whence \( t \in A \). Then \( A \) is SN by [11].

(iii) Assume \( K = K_{\text{rad}} \). Let \( t \in A' \) be such that \( t^p \in A \). Then \( f(t)^p \in R = R' \cap K \), whence \( f(t)^p \in K = K_{\text{rad}} \), hence \( f(t) \in K \cap R' = R \), i.e., \( t \in A \). Thus \( A \) is WN by [15] Cor. p. 651.

Conversely assume that \( A \) is WN and let \( x \in K' \) be such that \( x^{p^r} \in K \). Since \( R' \) is integral over \( R \), we have \( x = a/s \), \( a \in R' \), \( s \in R \). Then \( a^{p^r} \in R' \cap K = R \). Put \( a = f(t) \), \( t \in A' \). Then \( t^{p^r} \in A \), whence \( t \in A \) by [15] Cor. p. 651. It follows that \( a = f(t) \in R \) and hence \( x \in K \). Then \( K = K_{\text{rad}} \).

(iv) It follows by (iii) and [9] (1.1) and (1.2).

Corollary 4. There are projective integral varieties which are WN but not WN1.

Proof. Let \( k \) be an algebraically closed field of characteristic \( p > 2 \), let \( K := k(X, Y) \subseteq K' := K(Z)/(Z^{2p} + XZ^p + Y) \). By [5], Chap. V, §9, Ex. 7, p. 136, \( K'/K \) is a finite inseparable field extension and \( K_{\text{rad}} = K \).

Let \( R := k[X, Y] \) and \( R' := k[X, Y, Z]/(Z^{2p} + XZ^p + Y) \), \( A' := R'[T] \), \( \mathfrak{m} := (T) \), \( f : A' \to R' = A'/\mathfrak{m} \) the natural homomorphism. We observe that \( A' \) is regular and the quotient fields of \( R \) and \( R' \) are \( K \) and \( K' \), respectively.

By Lemma 2 the ring \( A := \{ a \in A'/f(a) \in R \} \) is a finitely generated WN but not WN1 \( k \)-algebra.

Let \( \overline{V} \) be a projective closure of \( V := \text{spec} A \) and let \( W \) be the weak normalizaton of \( \overline{V} \). Then \( W \) is WN and contains an open subset isomorphic to \( V \) (see [3]), hence it is not WN1. Moreover \( W \) is projective by [7] and the conclusion follows.

Remark. From the proof of Corollary 4 we see that there are WN varieties which are not WN1 of any dimension \( \geq 3 \) (add indeterminates if necessary). It is not clear to us whether one can have two-dimensional examples.

We observe that by the proof of Corollary 4 and Lemma 2 (iv) this is equivalent to the existence of field extensions \( k \subseteq K \subseteq K' \) as in Lemma 2 (iv), with \( \text{tr. deg} K/k = 1 \).

References


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