REGULARITY OF MAPPINGS OF
G-STRUCTURES OF FROBENIUS TYPE

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Abstract. A notion of Frobenius type for a G-structure is defined. It is shown
that a mapping f between C∞ (resp. Cω) manifolds with a G-structure of
the Frobenius type is C∞ (resp. Cω) if f ∈ Ck, where the integer k depends
on the order of the Frobenius type. It is also shown that a G-structure of finite
order is of the Frobenius type.

0. Introduction

Let G be a Lie subgroup of Gl(n;R). A G-structure on a C∞ manifold
M of dimension n is a C∞ sub-bundle P of the bundle of linear frames
over M with structure group G. Let f be a C1 diffeomorphism of M
with a G-structure P onto another manifold M with a G-structure P. f
is called a G-mapping if for any frame field (e1, ..., en) over M belonging to
P, (f∗e1, ..., f∗en) is a frame field over M belonging to P. We are concern-
ing with regularity of G-mappings. Locally, the above condition is a system of
partial differential equations of order 1 and the question of regularity naturally
arises. Our approach is to reduce the regularity problem of G-mappings to that
of infinitesimal automorphisms of P and then to find conditions on P which
imply the regularity of infinitesimal automorphisms. In cases, regularity of in-
finitesimal automorphisms of a G-structure can be deduced by properties of G
only. A well-known example is that if the associated Lie algebra of G contains
no matrix of rank 1 then an infinitesimal automorphism of a G-structure satis-
ifies a system of elliptic linear partial differential equations with C∞ coefficients,
and therefore is C∞ (see the proof of Theorem 4.1 of [5]). But in general, we
need conditions on a specific G-structure P in addition to the conditions on
G. This paper concerns the cases where regularity of G-mappings follows from
the Frobenius theorem.
Our viewpoint is purely local, so, for example, a "function" must be understood as a germ of a function at a reference point and in §1 we work on an open set \( \mathcal{O} \subseteq \mathbb{R}^n \) instead of a manifold \( M \). Often, we have to think of open subsets of \( \mathcal{O} \), however, we denote them also by \( \mathcal{O} \), for there is no danger of confusion. If the underlying manifolds and \( G \)-structures on them are real analytic we get a real analytic version of this paper by changing every \( C^\infty \) to \( C^\omega \).

1. Mappings of \( G \)-structures of Frobenius type

Let \( \mathcal{O} \) be an open subset of \( \mathbb{R}^n \) and \( f = (f^1, \ldots, f^l) \) be a system of real valued functions of \( \mathcal{O} \). \( f \) is said to satisfy a complete system of order \( m \) if every partial derivative of \( f^j \), \( j = 1, \ldots, l \), of order \( m \) can be expressed as a \( C^\infty \) function of the partial derivatives of \( (f^1, \ldots, f^l) \) of order \( < m \): For each \( j = 1, \ldots, l \), each multi-index \( \alpha \) with \( |\alpha| = m \),

\[
D^\alpha f^j = H^j_\alpha(x, D^\beta f^j : |eta| < m), \quad H^j_\alpha \in C^\infty
\]

where \( D^\beta f^j = (D^\beta f^1, \ldots, D^\beta f^l) \).

We will use summation convention in this section: repeated indices denote the summation over \( 1, \ldots, n \). A complete system for a vector field \( X = \xi^i(\partial / \partial x_i) \) on \( \mathcal{O} \) is a complete system for its components \( (\xi^1, \ldots, \xi^n) \). If \( \mathcal{O} \) is equipped with a \( G \)-structure \( P \) let \( (e_1, \ldots, e_n) \) be a frame belonging to \( P \) and set \( L_X e_j = a_j^i e_i \), where \( L \) is the Lie derivative. Then a \( C^1 \) vector field \( X \) is an infinitesimal automorphism of \( P \) if and only if the matrix \( [a_j^i] \) belongs to \( \mathcal{G} \), the associated Lie algebra of \( G \) (cf. [5]). This condition can be expressed as a system of linear partial differential equations of first order: Let \( X = u^i e_i \) and let

\[
[e_i, e_j] = b_{ij}^k e_k.
\]

Then

\[
L_X e_j = [u^i e_i, e_j] = (-e_j u^i + u^k b_{ij}^k) e_i,
\]

so we have

\[
a_j^i = -e_j u^i + u^k b_{kj}^i.
\]

Since \( \mathcal{G} \) is a linear subspace of \( gl(n; \mathbb{R}) \), it is defined by linear equations, namely,

\[
\mathcal{G} = \{y_j^i \in \mathbb{R}^n : c_{ik}^j y_j^i = 0, \quad \lambda = 1, \ldots, N\},
\]

where \( N \) is the codimension of \( \mathcal{G} \) in \( gl(n; \mathbb{R}) \) and \( c \) are constants. Thus we get

\[
c_{ik}^j (-e_j u^i + u^k b_{kj}^i) = 0, \quad \lambda = 1, \ldots, N.
\]

To express the above in terms of local coordinates, let \( X = \xi^i(\partial / \partial x_i) \) and let \( (\partial / \partial x_j) = b_j^i e_i \), then \( u^i = b_j^i \xi^j \) and the above equation becomes

\[
c_{ik}^j [-e_j (b_k^i \xi^j) + b_{ij}^k b_{kj}^i \xi^i] = 0, \quad \lambda = 1, \ldots, N.
\]
(1.1) is a system of linear PDE of first order for $\xi = (\xi^1, \ldots, \xi^n)$ with $C^\infty$ coefficients. By a prolongation of (1.1) we shall mean a system of linear PDE obtained from (1.1) through a process of repeated differentiations, additions and multiplications by $C^\infty$ functions. The main result of this paper is the following.

**Theorem 1.1.** Let $G$ be a Lie subgroup of $GL(n; \mathbb{R})$. Suppose that $\mathcal{O}$ and $\mathcal{P}$ are open neighborhoods of the origin of $\mathbb{R}^n$ with $G$-structure $P$ and $\tilde{P}$, respectively. Let $f: \mathcal{O} \to \mathcal{P}$ be a $G$-mapping of class $C^k$ for some sufficiently large $k$. Suppose that the equation (1.1) for the infinitesimal automorphisms of $P$ has a prolongation to a complete system of order $m$ and that $\tilde{P}$ has the same property. Then $f$ satisfies a complete system of order $m + 1$.

Note that the existence of an infinitesimal automorphism on $\mathcal{O}$ or $\mathcal{P}$ is not assumed. In general, if a system of functions $f = (f^1, \ldots, f^n)$ satisfies a complete system of order $m$, then by introducing new variables for the partial derivatives of order $< m$, we can construct a Pfaffian system on a submanifold of $\mathbb{R}^N$ where $N = n + \text{number of the newly introduced variables}$, so that $f$ may be identified with an integral manifold as in the proof of Theorem 1.1. So the questions of existence, regularity and uniqueness of $f$ reduce to the Frobenius theorem. We will call $G$-structures as in Theorem 1.1 Frobenius type.

**Definition 1.2.** A $G$-structure $P$ on a $C^\infty$ manifold $M$ is a Frobenius type of order $m$ if the equation (1.1) for the infinitesimal automorphisms of $P$ in terms of any local coordinates has a prolongation to a complete system of order $m$.

We have

**Corollary 1.3.** Let $M$ and $\tilde{M}$ be a $C^\infty$ manifolds with $G$-structure $P$ and $\tilde{P}$, respectively, and $f: M \to \tilde{M}$ be a $G$-mapping. If $P$ and $\tilde{P}$ are Frobenius type of order $m$ and $f \in C^{m+1}$ then $f \in C^\infty$ and $f$ is locally determined by $\{D^\beta f(0): |\beta| \leq m\}$ at any point $0 \in M$.

Now let $P$ be a $G$-structure of Frobenius type of order $m$ on an open set $\mathcal{O} \subseteq \mathbb{R}^n$. Let

$$D^\alpha \xi^i = H^i_\alpha(x, D^\beta \xi: |\beta| < m), \quad \forall \alpha \text{ with } |\alpha| = m, \forall i = 1, \ldots, n,$$

be a prolongation of (1.1) to a complete system. Note that each $H^i_\alpha$ is linear in $D^\beta \xi$. To realize the process of prolongation in the jet spaces, we introduce new variables

$$p_\beta = (p^1_\beta, \ldots, p^n_\beta) \text{ for } D^\beta \xi = (D^\beta \xi^1, \ldots, D^\beta \xi^n).$$

For each positive integer $k$, the $k$th order jet space is $J^k \equiv \mathcal{O} \times \mathbb{R}^{(k)} = \{(x, \xi, p)\}$, where $p = (p_\beta: |\beta| \leq k)$ and $(k)$ is the number of the variables $(\xi, p)$. In $J^k$ consider the submanifold $\Delta^k$ which is defined by (1.1) and all
the equations obtained by differentiating (1.1) in all possible ways up to order \( k - 1 \). Let \( X = \xi^i (\partial / \partial x_i) \) be a \( C^k \) vector field. Then the \( k \)-graph of \( X \) is the submanifold of \( J^k \)

\[
j_X^k(x) \equiv (x, \xi(x), D^\beta \xi(x) : |\beta| \leq k)
\]

and \( k \)-th order contact systems is

\[
\Omega^k \equiv \{ w \in T^* J^k : (j_X^k)^* w = 0, \text{ for all vector fields } X \text{ on } \mathcal{O} \}.
\]

Then \( \Omega^k \) is spanned by

\[
w^k \equiv d\xi^k - p^i j dx^i \text{ and } w^i_\beta \equiv dp^i_\beta - p^i_{(\beta,j)} j dx^j,
\]

(\( \beta, j \)) denotes the multi-index \((\beta_1, \ldots, \beta_j + 1, \ldots, \beta_n)\) if \( \beta = (\beta_1, \ldots, \beta_n) \), (cf. [8 and 9]). We prove

**Lemma 1.4.** Let \( P \) be a \( G \)-structures on \( \mathcal{O} \) and let \( J^k, \Delta^k \) and \( \Omega^k \) be as above. Then the following are equivalent:

(i) \( P \) is of Frobenius type of order \( m \);

(ii) the \((m - 1)\)th contact system \( \Omega^{m-1} \) defines an \( n \)-dimensional distribution \( \mathcal{D} \) on \( \Delta^{m-1} \), with \( dx^1 \wedge \cdots \wedge dx^n \neq 0 \) on each integral element of \( \mathcal{D} \).

**Proof of Lemma 1.4.** (i) \( \Rightarrow \) (ii): Let (1.2) be a prolongation of (1.1) to a complete system. (1.2) is equivalent to the total differential equation

\[
d(D^\beta \xi^i) = H^i_{(\beta,j)}(x, \xi, D^\gamma \xi : |\gamma| \leq m - 1) dx^j, \quad \forall \beta \text{ with } |\beta| = m - 1,
\]

\( \forall i = 1, \ldots, n \). This implies that on \( \Delta^{m-1} \)

\[
\Omega^i_\beta \equiv dp^i_\beta - H^i_{(\beta,j)}(x, \xi, p) dx^j = 0,
\]

\( \forall \beta \text{ with } |\beta| = m - 1, \forall i = 1, \ldots, n \). Let \( \mathcal{D} \) be the distribution on \( \Delta^{m-1} \) given by \( \Omega^{(m-1)} = 0 \). Then on each integral element of \( \mathcal{D} \) we have

\[
d\xi^i = p^i_j dx^j,
\]

\[
dp^i_\beta = p^i_{(\beta,j)} dx^j, \quad \forall \beta \text{ with } |\beta| < m - 1,
\]

\[
dp^i_\beta = H^i_{(\beta,j)}(x, \xi, p) dx^j, \quad \forall \beta \text{ with } |\beta| = m - 1, \text{ and }
\]

\[
dx^1 \wedge \cdots \wedge dx^n \neq 0.
\]

Therefore, \( \mathcal{D} \) is an \( n \)-dimensional distribution.

(ii) \( \Rightarrow \) (i): Let \( \mathcal{D} \) be the distribution as in (ii). Let \( \varphi^1, \ldots, \varphi^\nu \) be differential 1-forms on \( \Delta^{m-1} \) which generate the differential ideal associated with \( \mathcal{D} \), where \( \nu = (\text{dimension of } \Delta^{m-1}) - n \). Set

\[
\varphi^j = a^j dx + b^j d\xi + c^j dp, \quad j = 1, \ldots, \nu,
\]
where \( a^i, b^j \) and \( c^j \) are row vectors and \( dx, \) \( d\xi \) and \( dp \) are column vectors so that \( a^i dx = a_1^i dx^1 + \cdots + a_n^i dx^n, \) etc. Since each integral element of \( D \) is \( n \)-dimensional subspace of \( T(\Delta^{m-1}) \) on which \( dx^1 \wedge \cdots \wedge dx^n \neq 0, \) we can solve \( \phi^j = 0, \) \( j = 1, \ldots, \nu, \) for \( d\xi \) and \( dp \) we get

\[
\begin{cases}
  d\xi^i = h^i_j dx^j \\
  dp^i_\beta = h^i_{(\beta, j)} dx^j, \quad \forall \beta \text{ with } |\beta| \leq m - 1, \quad \forall i = 1, \ldots, n,
\end{cases}
\]

where \( h \) are \( C^\infty \) functions on \( \Delta^{m-1}. \)

This implies that if \( \xi = (\xi^1, \ldots, \xi^\nu) \) satisfies (1.1) then

\[
d(D^\beta \xi^i) - h^i_{(\beta, j)}(x, D^\gamma \xi: |\gamma| \leq m - 1) dx^j = 0,
\]

for all \( \beta \) with \( |\beta| \leq m - 1. \) In particular, if \( |\beta| = m - 1, \) the above equation is equivalent to

\[
D^{(\beta, j)} \xi^i = h^i_{(\beta, j)}(x, D^\gamma \xi: |\gamma| \leq m - 1), \quad |\beta| = m - 1,
\]

which is a complete system of order \( m. \) Q.E.D.

**Proof of Theorem 1.1.** Let (1.2) be the complete system for the infinitesimal automorphism of \( P \) and let \( \Omega^k \) be the \( k \)th contact system on \( \Delta^k \subseteq J^k \) for each \( k = 1, \ldots, m - 1. \) For each multi-index \( \beta \) with \( |\beta| = m - 1 \) and each \( i = 1, \ldots, n, \) let \( \Omega^i_\beta \equiv dp^i_\beta - H^i_{(\beta, j)} dx^j, \) where \( H \) are the same as in the complete system (1.2). Let \( D \) be the distribution as in Lemma 1.4. We will put tilde on the corresponding notions on \( \bar{\mathcal{O}}: \bar{J}^{m-1} = \bar{\mathcal{O}} \times \mathbb{R}^{(m-1)} = (\bar{x}, \bar{\xi}, \bar{p}), \) etc. A \( C^{m+1} \) diffeomorphism \( f: \mathcal{O} \to \bar{\mathcal{O}} \) naturally defines a \( C^1 \) diffeomorphism \( F^k: J^k \to \bar{J}^k \) for each \( k = 1, \ldots, m - 1 \) as follows: Let \( F^k(x, \xi, p) = (\bar{x}, \bar{\xi}, \bar{p}). \)

Then

\[
\begin{align}
\bar{x}^i(x, \xi, p) &= f^i(x) \\
\bar{\xi}^i(x, \xi, p) &= \xi^\lambda \frac{\partial f^i}{\partial x^\lambda}, \quad i = 1, \ldots, n
\end{align}
\]
and define $\tilde{P}(x, \xi, p)$ by chain rule, namely

\begin{equation}
\tilde{p}^i_j(x, \xi, p) = \frac{\partial \tilde{\xi}^i}{\partial \tilde{x}^j} = \frac{\partial \xi^i}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tilde{x}^j}
\end{equation}

substitute (1.4) for $\xi^i$ and $p^\mu_\mu$ for $\frac{\partial \xi^i}{\partial x^\mu}$

\[ p^i_j(x, \xi, p) = \left( p^\lambda_\mu \frac{\partial f^i}{\partial x^\lambda} + \xi^\lambda \frac{\partial^2 f^i}{\partial x^\mu \partial x^\lambda} \right) \frac{\partial x^\mu}{\partial \tilde{x}^i} \]

each $\frac{\partial x^\mu}{\partial \tilde{x}^j}$ is an entry of

\[ \left[ \frac{\partial f^i}{\partial x^j} \right]_{i,j=1, \ldots, n}^{-1} \]

therefore a $C^\infty$ function of $\frac{\partial f^i}{\partial x^j}$, $i, j = 1, \ldots, n$, so

\[ \frac{\partial^2 f^i}{\partial x^\mu \partial x^\lambda} \frac{\partial x^\mu}{\partial \tilde{x}^i} + a^\mu_\lambda p^\lambda_\mu, \]

where $a^\mu_\lambda$ are $C^\infty$ functions of $(D^\gamma f: |\gamma| \leq 1)$.

Now let $\beta = (\beta_1, \ldots, \beta_n)$ be a multi-index and $(j_1, \ldots, j_{|\beta|})$ denotes the sequence $(1, \ldots, 1, 2, \ldots, 2, \ldots, n, \ldots, n)$. Then by induction on $|\beta|$ we get

\begin{equation}
\tilde{p}^i_\beta(x, \xi, p) = \xi^\lambda \left[ \frac{\partial |\beta|+1 f^i}{\partial x^\lambda \partial x^\lambda} \frac{\partial x^\lambda}{\partial \tilde{x}^i} \ldots \frac{\partial x^\lambda_{|\beta|}}{\partial \tilde{x}^i} + a^\lambda_\beta \right] \\
+ a^\gamma_\beta p^\lambda_\gamma, |\gamma| \leq |\beta|,
\end{equation}

where $a$ are $C^\infty$ functions of $(D^\gamma f: |\gamma| \leq |\beta|)$. Then we claim

1. $F^{m-1}(\Delta^{m-1}) = \tilde{\Delta}^{m-1}$ and
2. $F^{m-1}(\mathcal{D}) = \tilde{\mathcal{D}}$.

Proof of claim.

1. A $C^1$ vector field $X = \xi^i (\partial / \partial x_i)$ is an infinitesimal automorphism of $P$ if and only if $f^i X$ is an infinitesimal automorphism of $\tilde{P}$. This implies that $F^1(\Delta^1) = \tilde{\Delta}^1$. Then it is clear that $F^k(\Delta^k) = \tilde{\Delta}^k$ for $k = 2, \ldots, m - 1$.

2. For each $k = 1, \ldots, m - 1$, we have $(F^k)^*(\tilde{\Omega}^k) = \Omega^k$ which is immediate from the definition of the contact system. In particular
\[(F^{m-1})^* (\tilde{\Omega}^{m-1}) = \Omega^{m-1}\]. Thus we have

\[v \in \mathcal{D} \iff v \in T(\Delta^{m-1}) \text{ and } v \text{ annihilates } \Omega^{m-1}\]
\[\iff F_* v \in T(\tilde{\Delta}^{m-1}) \text{ and } F_* v \text{ annihilates } \tilde{\Omega}^{m-1}\]
\[\iff F_+^{m-1} v \in \mathcal{D}, \quad \text{Q.E.D.}\]

Now we compute \(F^* \tilde{\Omega}_\beta^i\), \(|\beta| = m - 1\):

\[(1.7) (F^{m-1})^* \tilde{\Omega}_\beta^i = (F^{m-1})^* (d\tilde{p}^i - \tilde{H}_\beta^i(x, \tilde{\zeta}, \tilde{p}) d\tilde{x}^i)\]

substitute (1.3)–(1.6) for \(\tilde{x}, \tilde{\zeta}\) and \(\tilde{p}\), respectively,

\[
\begin{align*}
&= \left[ d\xi^\lambda \frac{\partial^m f^i}{\partial x^\lambda_1 \cdots \partial x_1 \partial x^\lambda_{m-1}} \cdots \frac{\partial x^\lambda_{m-1}}{\partial \tilde{x}_1} + a^\beta_\lambda \right] d\xi^\lambda \\
&+ a^\gamma_\lambda d\tilde{p}^j, \quad |\gamma| \leq m - 1 \\
&+ \left[ d\xi^\lambda \frac{\partial^{m+1} f^i}{\partial x^\lambda_1 \cdots \partial x_1 \partial x^\lambda_{m-1}} \cdots \frac{\partial x^\lambda_{m-1}}{\partial \tilde{x}_1} + b^i_{\beta, \lambda} \right] d\xi^\lambda
\end{align*}
\]

where \(a\) are \(C^\infty\) functions of \((D^\gamma f: |\gamma| \leq m - 1)\) and \(b\) are \(C^\infty\) functions of \((x, \xi, p, D^\gamma f: |\gamma| \leq m)\).

By the proof of Lemma 1.4, \(\mathcal{D}\) on \(\tilde{\Delta}^{m-1}\) is given by

\[
\left\{ \begin{array}{l}
\tilde{\Omega}^{m-1} = 0 \\
\tilde{\Omega}_\beta^i = 0, \quad \forall i = 1, \ldots, n, \quad \forall \beta \text{ with } |\beta| = m - 1.
\end{array} \right.
\]

Recall that \(\tilde{\Omega}^{m-1}\) is the contact system and \(\tilde{\Omega}_\beta^i\) are the 1 forms defined by the complete system. Since \(F_+^{m-1}(\mathcal{D}) = \mathcal{D}\), \((F^{m-1})^* \tilde{\Omega}_\beta^i\) is a linear combination of \(\{\omega^i, \omega_\gamma^i, \Omega_\delta^i: i = 1, \ldots, n, \ |\gamma| < m - 1, \ |\delta| = m - 1\}\), where \(\omega\) are contact forms. So we set

\[(1.8) (F^{m-1})^* \tilde{\Omega}_\beta^i = c_{\beta, \lambda}^i \omega^\lambda + c_{\beta, \gamma}^i \omega_\gamma^\lambda + c_{\beta, \delta}^i \Omega_\delta^i = c_{\beta, \lambda}^i (d\xi^\lambda - p_\lambda^k d\tilde{x}^k) + c_{\beta, \gamma}^i (d\tilde{p}^\gamma - p_\gamma^k d\tilde{x}^k) + c_{\beta, \delta}^i (d\tilde{p}_\delta^k - H_{(\delta, k)} d\tilde{x}^k),\]

where \(c\) are \(C^1\) functions on \(\Delta^{m-1}\), \(|\gamma| \leq m - 2\) and \(|\delta| = m - 1\).

By equating the components of \(d\xi^\lambda\) and \(d\tilde{p}\) in (1.7) and (1.8) we get \(c\)’s as \(C^\infty\) functions of \((x, \xi, p, D^\gamma f: |\gamma| \leq m)\) for \((x, \xi, p) \in \Delta^{m-1}\). Substitute this in (1.8) and equate the components of \(d\tilde{x}^k\) in (1.7) and (1.8) to get

\[(1.9) \begin{align*}
\xi^\lambda &= \frac{\partial^{m+1} f^i}{\partial x^\lambda_1 \cdots \partial x_1 \partial x^\lambda_{m-1}} \cdots \frac{\partial x^\lambda_{m-1}}{\partial \tilde{x}_1} \\
&= \text{C}^\infty \text{ function of } (x, \xi, p, D^\gamma f: |\gamma| \leq m),
\end{align*}\]
where \((x, \xi, p) \in \Delta^{m-1}\). Since (1.1) is a system of linear partial differential equations of first order obtained from the structure equations of the Lie Algebra \(\mathcal{G}\), we see that \(dx^1 \wedge \ldots \wedge dx^n \wedge d\xi^1 \ldots d\xi^n \neq 0\) on \(\Delta^1 \subseteq J^1\) and therefore on \(\Delta^{m-1} \subseteq J^{m-1}\). Thus there exists a \(C^\infty\) function \(p(x, \xi)\) such that \((x, \xi, p(x, \xi)) \in \Delta^{m-1}, \forall (x, \xi)\). For each \(j = 1,\) the restriction of (1.9) to the submanifold \(\{(x, \xi, p(x, \xi)): \xi = (0, \ldots , 0, 1, 0, \ldots , 0)\}\) is

\[
\frac{\partial^{m+1} f^i}{\partial x^1 \partial x^i \ldots \partial x^{\lambda_1} \partial x^k \partial \tilde{x}^i \partial \tilde{x}^{\lambda_{m-1}}} = C^\infty \text{ function of } (x, D^j f: |\gamma| \leq m).
\]

Here \(i, j, j_1, \ldots , j_{m-1}\) and \(k\) are arbitrary. Since the matrix

\[
\left( \frac{\partial x^i}{\partial \tilde{x}^j} \right)_{i,j=1,\ldots,n}
\]

is nonsingular and each \(\frac{\partial x^i}{\partial \tilde{x}^j}\) is a \(C^\infty\) function of \((Df)\), from (1.10) we get

\[
\frac{\partial^{m+1} f^i}{\partial x^1 \partial x^i \ldots \partial x^{\lambda_1} \partial x^k \partial \tilde{x}^i \partial \tilde{x}^{\lambda_{m-1}}} = C^\infty \text{ function of } (x, D^j f: |\gamma| \leq m).
\]

This completes the proof.

2. \(G\)-STRUCTURES OF FINITE ORDER

Let \(G\) be a Lie subgroup of \(GL(n; \mathbb{R})\) and \(\mathcal{G}\) be the associated Lie algebra. The \(k\)th prolongation \(\mathcal{G}^{(k)}\) of \(\mathcal{G}\) is the space of symmetric multilinear mappings

\[
t: \bigotimes_{(k+1) \text{ times}} \mathbb{R}^n \to \mathbb{R}^n
\]

such that, for each fixed \(v_1, \ldots , v_k \in \mathbb{R}^n\), the linear transformation

\[
v \in \mathbb{R}^n \mapsto t(v, v_1, \ldots , v_k) \in \mathbb{R}^n
\]

belongs to \(\mathcal{G}\).

\(G\) is said to be of finite order \(k\) if \(\mathcal{G}^{(k)} = 0\) and \(\mathcal{G}^{(k-1)} \neq 0\). Riemannian structures and conformal structures (when dimension \(\geq 3\)) are of finite order 1 and 2, respectively (cf. [5 or 6]).

**Theorem 2.1.** Let \(M\) be a \(C^\infty\) manifold of dimension \(n\) and \(P\) be a \(G\)-structure on \(M\). If \(G\) is of finite order \(m-1\) \((m \geq 1)\), then \(P\) is of Frobenius type of order \(m\).

**Proof.** Since \(\mathcal{G}\) is a linear subspace of \(gl(n; \mathbb{R})\), it is defined by \(\mathcal{G} = \{(y^i_j) \in gl(n; \mathbb{R}): \sum_{i,j=1}^n c_{i\lambda}^j y^i_j = 0, \lambda = 1, \ldots , N\}\), where the \(c_{i\lambda}^j\) are constants and \(N\) is the codimension of \(\mathcal{G}\) in \(gl(n; \mathbb{R})\). Therefore, as a linear space, \(\mathcal{G}^{(m-1)}\) is isomorphic to the subspace of

\[
\mathbb{R}^{nm+1} = (y_{j_1,\ldots,j_m}^i), \quad \text{each } i, j \in \{1, \ldots , n\},
\]
which is defined by the following system of linear equations:

\[
\sum_{i,j_1=1}^{n} c^{ij}_{i_1j_1...j_m} y_{j_1,...,j_m}^i = 0, \quad \lambda = 1, \ldots, N
\]

and

\[
\begin{align*}
\begin{cases}
 y_{j_1j_2,...,j_m}^i - y_{j_2j_1,...,j_m}^i = 0 \\
y_{j_1,...,j_{m-1}j_m}^i - y_{j_1,...,j_{m-1},j_m}^i = 0
\end{cases}
\end{align*}
\]

symmetry in subscripts

Since the only solution of (2.11) is \( y = 0 \), there exists \( n^{m+1} \) independent equations in (2.11). Let

\[
g^1(y) = 0, \ldots, g^{n^{m+1}}(y) = 0.
\]

Now let \( M \) be a \( C^\infty \) manifold of dimension \( n \) with a G-structure \( P \). We fix a frame \( (e_1, \ldots, e_n) \) belonging to \( P \). Let \( X = \sum_{i=1}^{n} \xi^i e_i \) be an infinitesimal automorphism of \( P \).

Define \( \xi^i_j \) by \( [e_j, X] = \sum_{i=1}^{n} \xi^i_j e_i \). Then the matrix \( [\xi^i_j] \) belongs to \( \mathcal{G} \). For any sequence \( (j_2, \ldots, j_k) \), each \( j \in \{1, \ldots, n\} \), we denote by \( \xi^i_{j_1...j_k} \) the derivative of \( e_{j_1} \cdots e_{j_k}(\xi^i_j) \). Then in the Jacobi identity

\[
[e_k, [e_j, X]] - [e_j, [e_k, X]] = [[e_k, e_j], X]
\]

substitute \( \sum_{i=1}^{n} \xi^i_j e_i \) and \( \sum_{i=1}^{n} \xi^i_k e_i \) for \( [e_j, X] \) and \( [e_k, X] \), respectively, we get \( \xi^i_{j_k} - \xi^i_{k_j} = \langle \xi^i_1, \xi^i_k \rangle \), where \( \langle \; , \; \rangle \) denotes a linear combination of the variables inside with \( C^\infty \) coefficients. By induction on the number of the subscripts we see that a transposition for any two subscripts in \( \xi^i_{j_1...j_k} \) makes a difference by a linear combination of \( \{\xi^i_j: |J| < k, \lambda = 1, \ldots, n\} \), where \( J = (j_1, j_2, \ldots) \) is a sequence of subscripts and \( |J| \) is the size of \( J \). Moreover, since \( \mathcal{G} \) is a linear space, for each fixed \( j_2 \cdots j_k \), the matrix of the derivatives

\[
[\xi^i_{j_1j_2...j_k}]_{i,j_1=1,...,n}
\]

belongs to \( \mathcal{G} \).

Now at each point \( x \in M \), consider

\[
\sum_{i,j_1=1}^{n} c^{ij}_{i_1j_1...j_m} \xi^i_{j_1j_2...j_m} = 0, \quad \lambda = 1, \ldots, N
\]

and

* is the symmetry in the subscripts modulo lower order terms \( \delta \), where each \( \delta \) is a linear combination of \( \{\xi^i_j: |J| < m - 1, \lambda = 1, \ldots, n\} \) with \( C^\infty \) coefficients. Let

\[
g^1(x, \xi) = 0, \ldots, g^{n^{m+1}}(x, \xi) = 0,
\]
be the equations corresponding to (2.12). Since the last \( n^{m+1} \) columns of the Jacobian matrix \((\partial g(x, \xi) / \partial \xi)\) is equal to \((\partial g / \partial y)\), which is nonsingular, we can solve (2.14) to get \( \xi_{j_1, \ldots, j_m}^{i} = \) a linear combination of \( \{\xi_j^i : |J| \leq m - 1, \ t = 1, \ldots, n\} \) with \( C^\infty \) coefficients, for each \( i, j_1, \ldots, j_m \). This completes the proof.

3. Remarks on CR structures

A CR structure \( P \) of complex dimension \( n \) and CR codimension \( d \) on a \( C^\infty \) manifold \( M \) if dimension \( 2n + d \) is a \( G \)-structure, where \( G \) is the group of all the matrices of the form

\[
\begin{bmatrix}
A & * \\
0 & B
\end{bmatrix}, \quad \text{where} \ A \in GL(n; \mathbb{C}) \subset GL(2n; \mathbb{R}) \text{ and } B \in GL(d; \mathbb{R}).
\]

Let \( (e_1, \ldots, e_{2n+d}) \) be a frame field belonging to \( P \). Let \( H(M) \) be the sub-bundle of the tangent bundle \( T(M) \) spanned by \( (e_1, \ldots, e_{2n}) \). We assume the integrability condition on \( H(M) \), as usual (cf. [2]). This group \( G \) is of infinite order, for the associated Lie algebra of \( G \) contains a matrix or rank 1 (Proposition 1.4 of [5]). However, under certain conditions on the Levi form \( P \) is of Frobenius type. When \( d = 1 \) the following is well known: If \( M \) and \( \widetilde{M} \) are \( C^\infty \) CR manifolds with nondegenerate Levi forms, and \( f: M \to \widetilde{M} \) is a CR diffeomorphism then

1. \( f \) is locally determined by a finite number of constants, and
2. \( f \in C^7 \) implies that \( f \in C^\infty \).

These are consequences of the existence of the invariant Cartan connection on the bundle of pseudoconformal frames over \( M \) (cf. [2, 7]). From the viewpoint of this paper the above (1) and (2) follow from the fact that a nondegenerate CR structure of CR codimension 1 is of Frobenius type of order 3. This can be easily shown by the method used in [3] and [4], where direct construction of complete system of mappings have been treated.

When \( M \) and \( \widetilde{M} \) are real analytic hypersurfaces in \( \mathbb{C}^n \), regularity of CR mappings has been studied in [1] and [7] in relation to the problem of holomorphic extension of CR mappings. For abstract CR manifolds, the greater the CR codimension \( d \) is, the more conditions on \( P \) are required in order for mappings to be regular. Our further problem is to find conditions for given \( n \) and \( d \), under which CR structure \( P \) becomes Frobenius type, or more generally, to find conditions on a nonelliptic \( G \)-structure (cf. [5]) which imply the regularity of transformations.

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