A SEMIGROUP TREATMENT OF A ONE DIMENSIONAL NONLINEAR PARABOLIC EQUATION

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Abstract. In this paper we study the existence, uniqueness and differentiability of solutions (in a certain generalized sense) for the nonlinear parabolic equation

$$u_t = u_{xx} - F(u, u_x) \quad (0 < x < 1, t > 0),$$

under the maximal monotone boundary conditions:

$$(-1)^i u_x(t, i) \in \beta_i(u(t, i)), \quad t > 0, \; i = 0, 1.$$

1. Introduction and summary of results

We consider the following initial boundary value problem:

$$\begin{cases}
    u_t = u_{xx} - F(u, u_x), & t > 0, \; 0 < x < 1 \\
    u_x(t, 0) \in \beta_0(u(t, 0)), \quad -u_x(t, 1) \in \beta_1(u(t, 1)), \quad t > 0 \\
    u(0, x) = u_0(x), & 0 < x < 1,
\end{cases}$$

from the viewpoint of the theory of nonlinear semigroups.

Throughout this paper we assume that

1. $\beta_i$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta_i(0), \; i = 0, 1,$
2. $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $F(0, 0) = 0$ and $F(\cdot, p): \mathbb{R} \to \mathbb{R}$ is nondecreasing for each $p \in \mathbb{R}$.

Choose $C[0, 1]$ as the Banach space associated with (P) and define an operator $A$ in $C[0, 1]$ by

$$\begin{align*}
    D(A) &= \{ u \in C^2[0, 1]; \; (-1)^i u'(i) \in \beta_i(u(i)), \; i = 0, 1 \} \\
    Au &= u'' - F(u, u'), \quad u \in D(A).
\end{align*}$$

Then $A$ is dissipative in $C[0, 1]$ and the problem (P) can be viewed as an evolution equation:

$$\frac{du}{dt} = Au(t), \quad t > 0, \quad u(0) = u_0 \text{ in } C[0, 1].$$
If $A$ is $m$-dissipative in $C[0,1]$, then for every $u_0 \in D(A)^{C[0,1]}$ the Crandall–Liggett theorem [2] provides us with the unique solution (in a certain generalized sense) of (1.2) represented by the exponential formula:

\[(1.3) \quad u(t, \cdot) = \lim_{\lambda \to 0^+} (I - \lambda A)^{-1/\lambda} u_0(\equiv \exp(tA)u_0), \quad t \geq 0.\]

We shall call $u(t, \cdot)$ the semigroup solution of (P).

The purpose of this paper is to give conditions on $F(u, p)$ under which the operator $A$ defined by (1.1) is $m$-dissipative in $C[0,1]$.

**Theorem.** Let (b) and (FO) hold, and assume in addition that one of the following conditions holds:

- (F1) $|F(u, p)| \leq \zeta(u) \cdot \eta(|p|)$ for all $(u, p) \in R \times R$, where $\zeta(\cdot): R \to [0, \infty)$ is continuous, and $\eta(\cdot): [0, \infty) \to (0, \infty)$ is continuous, nondecreasing and $\int_0^\infty \frac{\eta(r)}{r} dr = \infty$.
- (F2) $F_r: R \times R \to R$ is continuous and $F(0, p) = 0$ for all $p \in R$.

Then the operator $A$ defined by (1.1) is $m$-dissipative in $C[0,1]$.

Our result is closely related to the work of Konishi [5], in which he treated the existence, uniqueness and differentiability of semigroup solutions for the equation $u_t = u_{xx} - F(u_x)$ under the periodic boundary conditions. By almost the same methods in [5], we can examine the differentiability of semigroup solutions of (P).

Let us define the operator $\tilde{A}$ in $L^\infty(0,1)$ by

\[(1.4) \quad D(\tilde{A}) = \{u \in C^1[0,1]; u'' \in L^\infty(0,1), (-1)^i u'(i) \in \beta_i(u(i)), i = 0,1\},
\]

\[\tilde{A}u = u'' - F(u, u'), u \in D(\tilde{A}).\]

**Proposition.** Let the hypothesis of the theorem be satisfied. For each $u_0 \in D(\tilde{A})$ let $u(t) = \exp(tA)u_0$, $t \geq 0$ be the semigroup solution of (P). Then we have the following:

- (i) $u(t) \in D(\tilde{A})$ for $t \geq 0$.
- (ii) $u \in C([0, \infty); C^1[0,1])$.
- (iii) $u: [0, \infty) \to L^\infty(0,1) = L^1(0,1)^*$ is weakly * continuously differentiable and

\[w^* - (d/dt)u(t) = \tilde{A}u(t) \text{ in } L^\infty(0,1) \text{ for } t \geq 0.\]

## 2. Preliminaries

In what follows, by $C[0,1]$ we mean the Banach space of all real valued continuous functions on $[0,1]$ with the supremum norm $\|\|$. $C[0,1]$ is a closed subspace of the Banach space $L^\infty(0,1)$ with the norm $\|u\|_\infty = \text{ess sup}\{|u(x)|; x \in (0,1)\}$, and $\|u\|_\infty = \|u\|$ for all $u \in C[0,1]$. Similarly, let $C^1[0,1]$ denote the Banach space of all continuously differentiable functions on $[0,1]$ with the norm $\|u\|_1 = \max\{|u|, |u'|\}$. 

An operator $A: D(A) \subset X \to X$ in a Banach space $(X, \| \cdot \|)$ is called dissipative if
\[
\|u - v\| \leq \|u - v - \lambda(Au - Av)\| \quad \text{for all } u, v \in D(A)
\]
and $\lambda > 0$.

A dissipative operator $A$ in $X$ is said to be $m$-dissipative if $R(I - \lambda A) = X$ for every, or equivalently, for some $\lambda > 0$; here $R(I - \lambda A)$ denotes the range of $I - \lambda A$.

A subset $\beta \subset R \times R$ is called monotone if $(u_1 - u_2)(v_1 - v_2) \geq 0$ whenever $v_i \in \beta(u_i), \ i = 1, 2$. A monotone set not properly contained in any other monotone set is called maximal monotone. For more details on this material see [1].

3. Proof of Theorem

We begin with the following lemmas.

**Lemma 1.** Let $(\beta)$ and $(F_0)$ hold. Then the operator $A$ defined by (1.1) is dissipative in $C[0, 1]$.

**Proof.** Let $u, v \in D(A)$ and let $\|u - v\| = |u(x_0) - v(x_0)|, \ x_0 \in [0, 1]$. Then we have
\[
(u'(x_0) - v'(x_0)) = 0, \quad (u(x_0) - v(x_0))(u''(x_0) - v''(x_0)) \leq 0,
\]
by maximum principle arguments and by the monotonicity of $\beta_i, \ i = 0, 1$. From this and $(F_0)$ the assertion follows easily.

The following lemma is a slight modification of [4, Chapter 12, Lemma 5.1].

**Lemma 2.** Let $\eta: [0, \infty) \to (0, \infty)$ be a continuous and nondecreasing function satisfying \(\int_0^\infty (r/\eta(r)) \, dr = \infty\), and let $R \geq 0, \ K \geq 0, \ C \geq 0$ be real numbers.

Then there exists a number $M$ (depending only on $\eta, \ R, \ K, \ C$) with the following property:

If $u \in C^2[0, 1], \ |u(x)| \leq R$ and $|u''(x)| \leq K + C\eta(|u'(x)|), \ x \in [0, 1]$, then $|u'(x)| \leq M \text{ for all } x \in [0, 1]$.

**Proof of Theorem.** By Lemma 1 it suffices to show $R(I - A) = C[0, 1]$. Define the operator $A_0$ in $C[0, 1]$ by $A_0u = u''$ for $u \in D(A)$. Then $A_0$ is $m$-dissipative in $C[0, 1]$ and $(I - A_0)^{-1}: C[0, 1] \to C^1[0, 1]$ is continuous and compact. (See [3, Lemma 4.3] or [7, Proposition 1].)

Fix an arbitrary $f \in C[0, 1]$ and solve the equation
\[
(3.2) \quad u = (I - A_0)^{-1}(f - F(u, u')).
\]
Then the solution of (3.2) satisfies $u - Au = f$.

For $m = 1, 2, \ldots$ let
\[
B_m v = \begin{cases}  \ f - F(v, v') & \text{if } \|v\|_1 \leq m, \\  f - F(mv/\|v\|_1, mv'/\|v\|_1) & \text{if } \|v\|_1 > m. \end{cases}
\]
$B_m : C^1[0, 1] \to C[0, 1]$ is continuous and uniformly bounded. Consequently $T_m = (I - A_0)^{-1} B_m : C^1[0, 1] \to C^1[0, 1]$ is continuous, compact and uniformly bounded. Hence $T_m$ maps some closed ball in $C^1[0, 1]$ into itself, and so by the Schauder’s fixed point theorem we get a fixed point

$$u_m \in D(A); \quad u_m = (I - A_0)^{-1} B_m u_m, \quad m = 1, 2, \ldots.$$  

If there is an $m_0$ such that $\|u_{m_0}\|_1 \leq m_0$ then $u_{m_0}$ satisfies (3.2), and we complete the proof. Suppose, for contradiction, that $\|u_m\|_1 > m$ for all $m$. Then (3.3) can be rewritten as

$$u_m - u'' + F(mu_m/\|u_m\|_1, mu_m'/\|u_m\|_1) = f,$$

$m = 1, 2, \ldots$.

Let $x_0$ be any point in $[0, 1]$ satisfying $\|u_m\| = |u_m(x_0)|$. Then, using (3.1) with $u = u_m$ and $v = 0$ we get $u''(x_0) = 0$, $u_m(x_0)u''(x_0) \leq 0$. Plugging this into (3.4) gives

$$\|u_m\|^2 + F(mu_m(x_0)/\|u_m\|_1, 0)u_m(x_0) \leq \|f\| \|u_m\|.$$  

From this and (F0) we obtain

$$\|u_m\| \leq \|f\| \text{ for all } m.$$  

On the other hand, if $F$ satisfies the condition (F1) then it follows from (3.4) and (3.5) that

$$|u_m''(x)| \leq 2\|f\| + Cn(|u_m'(x)|) \quad \text{for all } x \in [0, 1],$$

where $C = \max\{\xi(u); |u| \leq \|f\|\}$. Hence, it follows from Lemma 2 that there exists a constant $M$ such that

$$\|u'_m\| \leq M \quad \text{for all } m.$$  

Thus, by (3.5) and (3.6), we have

$$\|u_m\|_1 \leq \max\{\|f\|, M\} \quad \text{for all } m,$$

which is the desired contradiction.

Next suppose that $F$ satisfies the condition (F2). Multiplying (3.4) by $u''_m$ and integrating over $[0, 1]$ we obtain

$$- [u'_m u'_m]_0 + \int_0^1 u'_m^2 dx + \int_0^1 u''_m^2 dx - \int_0^1 F(mu_m/\|u_m\|_1, mu_m'/\|u_m\|_1)u''_m dx = - \int_0^1 f u''_m dx.$$  

Let $G(u, p) = \int_0^p F(u, r) dr$. Then, by using (F0) and (F2), we can check easily that $G(u, p)u_p \geq 0$ and $G_u(u, p)p \geq 0$ for all $u, p \in R$. Hence by ($\beta$) we
have
\[ \int_{0}^{1} F\left( m \frac{u_m'}{\|u_m\|_1}, m \frac{u_m'}{\|u_m\|_1} \right) u_m'' \, dx = m^{-1} \left\| u_m \right\|_1 \left[ G\left( m \frac{u_m'}{\|u_m\|_1}, m \frac{u_m'}{\|u_m\|_1} \right) \right]_0^1 \]
\[ - \int_{0}^{1} G_u\left( m \frac{u_m'}{\|u_m\|_1}, m \frac{u_m'}{\|u_m\|_1} \right) u_m' \, dx \leq 0. \]

From this and the monotonicity of \( \beta_i, \ i = 0, 1 \), we see that
\[ \int_{0}^{1} u_m''^2 \, dx \leq - \int_{0}^{1} f u_m'' \, dx \leq \frac{1}{2} \int_{0}^{1} (f^2 + u_m''^2) \, dx, \]
and hence
\[ \int_{0}^{1} u_m''^2 \, dx \leq \int_{0}^{1} f^2 \, dx \leq \|f\|^2. \]

Since \( u_m'(x) = \int_{x_0}^{x} u_m''(r) \, dr \), we can conclude that
\[ (3.7) \quad \|u_m'\| \leq \|f\|. \]

Thus, by (3.5) and (3.7), we have
\[ \|u_m\|_1 \leq \|f\| \quad \text{for all} \ m, \]
which is also the desired contradiction. This completes the proof.

Remark. The Theorem is true if we assume \( (\beta), (F0) \) and one of the following conditions:

(F3) \( F_u: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous, \( F(u, p) \geq 0 \) for all \( (u, p) \in \mathbb{R} \times \mathbb{R} \) and \( R(\beta_i) \subset [0, \infty), i = 0, 1 \).

(F4) \( F_u: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous, \( F(u, p) \leq 0 \) for all \( (u, p) \in \mathbb{R} \times \mathbb{R} \) and \( R(\beta_i) \subset (-\infty, 0], i = 0, 1 \).

(F5) \( F_u: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous and \( \beta_i = 0, i = 0, 1 \).

4. Proof of proposition

In a similar way as in [5] we can prove the proposition. However, for completeness, we give here an outline of the proof.

The operator \( \tilde{A} \) defined by (1.4) is dissipative in \( L^\infty(0, 1) \) and satisfies the range condition:
\[ R(I - \lambda \tilde{A}) \supset R(I - \lambda A) = C[0, 1] \supset \overline{D(A)}^C[0, 1] = \overline{D(A)} \]
\[ = \{ u \in C[0, 1]; u(i) \in \overline{D(\beta_i)}, i = 0, 1 \} (\equiv D), \quad \lambda > 0. \]

Thus \( \tilde{A} \) generates a nonlinear contraction semigroup \( \exp(t\tilde{A}), t \geq 0 \) on \( D \) in the sense of Crandall–Liggett [2], which together with (1.3) implies
\[ \exp(t\tilde{A})u_0 = \exp(tA)u_0 \quad \text{for all} \ t \geq 0, \ u_0 \in D. \]
Let $u_0 \in D(\tilde{A})$. Then
\[
\| (I - \lambda A)^{-[t/\lambda]} u_0 \| \leq \| u_0 \|,
\]
\[
\| \tilde{A}(I - \lambda A)^{-[t/\lambda]} u_0 \|_\infty \leq \| \tilde{A} u_0 \|_\infty
\]
for each $\lambda > 0$ and $t \geq 0$.

Define the operator $A$ by
\[
D(A) = \{ u \in C[0,1]; u' \in L^\infty(0,1) \},
\]
\[
Au = u' \quad \text{for } u \in D(A).
\]

Then, by the same way as in the proofs of (3.6) and (3.7), we obtain the estimate:
\[
\| A(I - \lambda A)^{-[t/\lambda]} u_0 \| \leq M,
\]
where $M$ is a constant independent of $\lambda > 0$ and $t \geq 0$. Hence it follows that
\[
\| A^2(I - \lambda A)^{-[t/\lambda]} u_0 \|_\infty \leq \| \tilde{A} u_0 \|_\infty + \max\{|F(u,p)|; |u| \leq \| u_0 \|, |p| \leq M\}.
\]

From these estimates we see that
\[
\lim_{\lambda \to 0+} \Lambda(I - \lambda A)^{-[t/\lambda]} u_0 = \Lambda \exp(tA)u_0 \quad \text{in } C[0,1],
\]
\[
w^* - \lim_{\lambda \to 0+} \tilde{A}(I - \lambda A)^{-[t/\lambda]} u_0 = \tilde{A} \exp(tA)u_0 \quad \text{in } L^\infty(0,1)
\]
for each $t \geq 0$.

Furthermore, we can check that
\[
\exp(\cdot A)u_0 \in C([0,\infty); C^1[0,1]) \text{ and the function}
\]
\[
t \in [0,\infty) \mapsto \tilde{A} \exp(tA)u_0 \in L^\infty(0,1)
\]
is weakly* continuous. Thus, by using the estimate due to Ôharu [6, (8)], we can conclude that
\[
\exp(tA)u_0 - u_0 = w^* - \int_0^t \tilde{A} \exp(sA)u_0 \, ds \quad \text{in } L^\infty(0,1),
\]
$t \geq 0$, from which the assertion follows.

**References**

5. Y. Konishi, *On $u_t = u_{xx} - F(u_x)$ and the differentiability of the nonlinear semigroup associated with it*, Proc. Japan Acad. 48 (1972), 281–286.

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