ABSTRACT. Given a homology theory $H_n(A)$ on rings, based on a natural chain complex, one can form a new theory $H_n^0(A)$ which is universal with respect to the homotopy property $H_n(A) \approx H_n(A[t])$. We show that the homotopy theories $HH_n^0$ and $HC_n^0$ associated to Hochschild and cyclic homology are both zero. On the other hand, if $HC_n^-$ denotes Goodwillie's variant of cyclic homology, and $A$ contains a field of characteristic 0, we show that $(HC^-)_n^0 A$ is Connes' periodic cyclic homology $HP_n(A)$.

A functor $F$ from the category of associative rings with unit to an abelian category is called a homotopy functor if $F(A) \to F(A[t])$ is an isomorphism for every ring $A$. If $[F]A$ denotes the coequalizer of

$$F(A[t]) \overset{t=0}{\to} F(A),$$

then $[F]$ is a homotopy functor, and $F \to [F]$ is universal with respect to maps from $F$ to homotopy functors.

One way to measure how far $F$ is from a homotopy functor is to study the groups $N^kF(A)$ of [Bass, XII], defined by

$$N^kF(A) = \bigcap_{i=1}^n \ker \{ F(A[t_1, \ldots, t_n]) \overset{t_i=0}{\to} F(A[t_1, \ldots, \hat{t_i}, \ldots, t_n]) \}.$$ 

The map $A[t_1, \ldots, t_n] \to A[t_1, \ldots, t_{n-1}]$ sending $t_n$ to $1 - \sum t_j$ induces a map $d_0 : N^kF(A) \to N^{k-1}F(A)$, and these fit together to form a chain complex

$$N^*F(A) : 0 \leftarrow F(A) \overset{t=1}{\leftarrow} NF(A) \overset{d_0}{\to} N^2F(A) \overset{d_0}{\to} N^3F(A) \leftarrow \cdots.$$
We call \( N'F(A) \) the "Moore Complex" of \( F(A) \). In §2, we show that, if \( A \) is a \( k \)-algebra, \( HH_m(A) \) denotes Hochschild homology, and \( \Omega^m_{A/k} \) denotes Kähler differentials (for commutative \( A \)), then \( N'HH_m(A) \) and \( N'\Omega^m_{A/k} \) are exact. The corresponding result for the cyclic homology group \( HC_m \) is not true for \( m \neq 0 \). (See §3.)

When \( F_*(A) \) arises as the homology of a natural chain complex \( C(A) \), as is the case for Hochschild and cyclic homology, there is a more subtle notion of homotopy. Following \([W1, 2.2]\), we define a new homotopy functor \( F^h_* \) in §1, arising as the homology of the total chain complex of \( N'C(A) \), which is universal in the appropriate sense. We then show that \( HH^h_*(A) = HC^h_*(A) = 0 \), which shows just how far Hochschild and cyclic homology are from being homotopy functors. Finally, we apply these ideas in §4 to Goodwillie's variant \( HC^-_* \) of cyclic homology. If \( A \) contains \( Q \), we show that \( (HC^-)_*(A) \) is \( HP^-_*(A) \), Connes' periodic cyclic homology. This emphasizes the central role of \( HC^-_* \) in cyclic homology.

1. The Moore complex

In this section, we discuss ways of measuring how much a functor \( F \) fails to be a homotopy functor. We use the simplicial ring \( \Delta A \) of \( A \)-valued algebraic functions on the standard \( p \)-simplex (denoted \( P(A) \) in \([And]\)). Thus,

\[
\Delta_n A = A[t_0, \ldots, t_n]/(\sum t_j = 1) \simeq A[t_1, \ldots, t_n],
\]

while the face and degeneracy maps are given by

\[
d_j(t_i) = 0, \quad s_i(t_i) = t_i + t_{i+1},
\]

\[
d_j(t_i) = t_j \text{ or } t_{j-1}, \quad \text{and } s_i(t_j) = t_j \text{ or } t_{j+1}
\]

if \( i \neq j \), depending on whether \( j < i \) or \( j > i \).

Now let \( F \) be a functor from rings to an abelian category. It is well-known \([May, \S 22]\) that the simplicial homotopy groups of the simplicial object \( F(\Delta A) \) are the same as the homology groups of either the chain complex

\[
0 \leftarrow F(A) \xrightarrow{d_0} F(A[t_1]) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} F(A[t_1, \ldots, t_n]) \leftarrow \cdots
\]

or of the associated "Moore complex" \( N'F(A) \) described in the introduction. For example, it is easy to see that

\[
\pi_0 F(\Delta A) = H_0(N'F(A)) = [F]A.
\]

We may think of the rest of the \( \pi_* F(\Delta A) = H_* N'(F(A)) \) as left derived functors of \([F]\).

Now suppose that we are given a functorial chain complex \( C(A) \), and set \( F_n(A) = H_n(C(A)) \). In addition to the homotopy functors \([F_n]\), we have the functors \( F^h_n(A) = H_n(|C(\Delta A)|) \), where \(|C(\Delta A)|\) means the total complex of the associated double complex \( N'C(A) \). As in \([W1, 2.4]\), the functors \( F^h_n(A) \) are
homotopy functors. If $C$ is bounded below, the maps $[F_n] \to F_n^h$ are the edge maps of a convergent spectral sequence

$E_{pq}^2 = H_p(N_q F(A)) \Rightarrow F_{p+q}^h(A).$

(Cf. [CE, XV.6], [And, 2.3].) We draw attention to an immediate consequence of this formalism:

Lemma 1.2. Suppose $F_*(A) = H_*(C(A))$ for a natural complex $C(A)$.

(i) If $F_m(A) = F_m(A[i_1, \ldots, i_n])$ for all $m$, then $F_m(A) = F_m^h(A)$ for all $m$.

(ii) If the Moore complexes $N'F_m(A)$ are exact for all $m$, then $F_m^h(A) = 0$ for all $m$.

2. Hochschild homology

The formalism of the last section makes sense if we replace the category of rings by a category $\mathcal{R}$ such that, whenever $A$ is in $\mathcal{R}$, so are the objects and maps in $\Delta A$. For example if $k$ is a commutative ring, we can consider either the category of $k$-algebras and the Hochschild homology functors $HH^m_m(A)$, or the category of commutative $k$-algebras and the Kähler differential functors $\Omega^m_{A/k}$. Note that $HH^m_m(A)$ is $H_m$ of the natural chain complex of $k$-modules

$C(A): 0 \leftarrow A \leftarrow A \otimes_k A \leftarrow A \otimes_k A \otimes_k A \leftarrow \ldots$

(see [CE, IX.6], [Mac, X.4]). We can now construct the functors $HH^h_m$ as in §1.

Theorem 2.1. For every $m$ and every $k$-algebra $A$,

(i) $HH^h_m(A) = [HH^h_m]A = 0$.

(ii) The Moore complex $N'HH^m(A)$ is exact, where $N'HH^m(A)$ is:

$0 \leftarrow HH^m(A) \leftarrow NHH^m(A) \leftarrow N^2HH^m(A) \leftarrow \ldots$

Theorem 2.2. For every $m$, let $\Omega^m_m(A) = \Omega^m_{A/k}$ denote the Kähler differential functor. Then for every commutative ring $A$ the Moore complex

$N'\Omega^m_m(A): 0 \leftarrow \Omega^m_m(A) \leftarrow N\Omega^m_m(A) \leftarrow N^2\Omega^m_m(A) \leftarrow \ldots$

is split exact. (The splitting is not natural.)

Proof that Theorem 2.2 implies Theorem 2.1. The Künneth formula [Mac, V.10.1 and X.7] yields

$HH^m_m(A[t_1, \ldots, t_n]) \simeq \bigoplus_{i+j=m} HH^i_i(A) \otimes HH^j_j(k[t_1, \ldots, t_n]),$

so that $N^nHH^m_m(A) \simeq \bigoplus HH^i_i(A) \otimes N^nHH^j_j(k)$. Thus it is enough to show that $N'HH^j_j(k)$ is split exact for all $j$. But this is Theorem 2.2, since by [HKR, 5.1] we have

$HH^j_j(k[t_1, \ldots, t_n]) \simeq \Omega^j_{k[t_1, \ldots, t_n]/k}$. 

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Proof of Theorem 2.2. For every commutative \( k \)-algebra \( A \),

\[
\Omega_{A[t_1, \ldots, t_n]/k} \cong \left( \Omega_{A/k} \otimes_A A[t_1, \ldots, t_n] \right) \oplus \left( A \otimes \Omega_{k[t_1, \ldots, t_n]/k} \right)
\]

by [Mat, 26.1], so that

\[
\Omega^m(A[t_1, \ldots, t_n]) = \bigoplus_{i+j=m} \Omega^i(A) \otimes \Omega^j(k[t_1, \ldots, t_n]).
\]

Therefore it is enough to prove that \( N'\Omega^m(k) \) is split exact for all \( m \). But each \( k \)-module \( N^p\Omega^m(k) \) is projective, since by [Bass, XII.7.3] it is a direct summand of the free \( k \)-module \( \Omega^m(k[t_1, \ldots, t_n]) \). Therefore it suffices to prove that \( N'\Omega^m(k) \) is exact for all \( m \).

We proceed by induction on \( m \). When \( m = 0 \), \( \Omega^0(A) = A \), and \( N^p\Omega^0(k) \) is the ideal \( (t_1 \cdots t_n)k[t_1, \ldots, t_n] \) of the ring \( k[t_1, \ldots, t_n] \). The differential \( N^0\Omega^0(k) \to N^{-1}\Omega^0(k) \) sends \( t_j \) to \( t_j \) for \( j \neq n \), and sends \( t_n \) to \( 1 - \sum_{j=1}^{n-1} t_j \).

The proof of exactness of \( N'\Omega^0(k) \) is now a pleasant exercise.

Let us analyze the groups \( N^p\Omega^m(k) \) slightly, introducing some notation, before continuing. We write \( k[T_n] \) for \( k[t_1, \ldots, t_n] \). Since \( \Omega^m(k[T_n]) = 0 \) for \( m > n \), \( N^p\Omega^m(k) = 0 \) for \( n < m \). Thus the complex \( N'\Omega^m(k) \) starts in degree \( m \) with

\[
N^m\Omega^m(k) = \Omega^m(k[T_m]) \cong k[T_m] \otimes dt_1 \wedge \cdots \wedge dt_m.
\]

In order to describe the rest of \( N'\Omega^m(k) \), let \( I = \{i_1, \ldots, i_m\} \) be an ordered \( m \)-element subset of \( \{1, \ldots, n\} \) (‘ordered’ means \( i_1 < \cdots < i_m \)). Set

\[
\omega^m_n(I) = \left( \prod_{j \notin I} t_j \right) dt_{i_1} \wedge \cdots \wedge dt_{i_m} \quad \text{in } N'\Omega^m(k).
\]

Then a straightforward computation shows that \( N^m\Omega^m(k) \) is the direct sum of copies of \( k[T_n] \) on generators \( \omega^m_n(I) \).

In order to take advantage of the inductive hypothesis, we filter the complex \( N'\Omega^m(k) \) by subcomplexes \( F_p \), where \( F_p^n \) is the \( k[T_n] \)-submodule of \( N^m\Omega^m(k) \) generated by those \( \omega^m_n(I) \) with \( I \subseteq \{1, \ldots, p\} \). Since \( N'\Omega^m(k) \) is the union of the \( F_p \), there is a convergent spectral sequence

\[
E^1_{pq} = H_{p+q}(F_p/F_{p+1}) \Rightarrow H_{p+q}(N'\Omega^m(k))
\]

attached to the filtration [Mac, XI.3.1 and 3.2]. Since \( F_{p+q}^p = 0 \) for \( q < 0 \) or \( p < m \), this is a first quadrant spectral sequence. To prove Theorem 2.2, we will show that \( E^2_{pq} = 0 \) for all \( p \), \( q \).

We first show that \( E^1_{pq} = 0 \) for \( q \neq 0 \). Since \( F_{m+1}^p = 0 \), \( E^1_{pq} = 0 \) for \( p < m \), and we may begin with \( E^1_{mq} \). Let \( \omega_r = \omega^m_r(1, 2, \ldots, m) \); the complex \( F_r^m \) has \( F_{r+1}^r = 0 \) for \( r < m \), and \( F_{r+1}^r \cong k[T_r] \) on generator \( \omega_r \). Since \( d^1: F_{r+1}^r \to F_{r}^r \) sends \( \omega_{r+1} \) to \( (1 - \sum t_j) \omega_r \), we can identify \( F_{r}^m \) with that portion of the exact
complex $N'\Omega^0(k)$ which lies in degrees $\geq m$. Hence $E_{mq}^1 = H_{m+q}(F_m^*) = 0$ for $q \neq 0$, and

$$E_{m0}^1 = H_m(F_m^*) = (k[T_m]/(1 - \sum t_j)k[T_m])\omega_m \cong k[T_{m-1}].$$

When $p > m$, the computation of $E_{pq}^1$ is slightly more complicated, because $F_p'/F_{p-1}' \cong \oplus_i k[T_r]\omega^m(I \cup \{p\})$, the sum running over all ordered $(m-1)$-tuples $I$ of $\{1, \ldots, p-1\}$. The boundary map $d^1$ sends $\omega^m(I \cup \{p\})$ to $(1 - \sum t_j)\omega^m_{r-1}(I \cup \{p\})$ if $r > p$. (If $r \leq p$, $d^1 = 0$ because $F_{p-1}' = 0$.) Thus $F_m^*$ is a direct sum (over $I$) of subcomplexes, each isomorphic to that portion of the exact complex $N'\Omega^0(k)$ lying in degrees $\geq p$. In this case $E_{pq}^1 = H_{p+q}(F_p'/F_{p-1}') = 0$ if $q \neq 0$, and

$$E_{p0}^1 = H_p(F_p'/F_{p-1}') = \bigoplus_I (k[T_p]/(1 - \sum t_j)k[T_p])\omega_p^m(I \cup \{p\})$$

$$\cong \bigoplus_I k[T_{p-1}][\omega_{p-1}^m(I)] dt_p.$$

Having shown that $E_{pq}^1 = 0$ for $q \neq 0$, we need only see that the complex $E_{*0}^1$ is exact in order to see that $E_{pq}^2 = 0$ for all $p, q$, and hence that $N'\Omega^m(k)$ is exact. But exactness of $E_{*0}^1$ follows from our inductive hypothesis and

**Lemma 2.3.** For each $m \geq 1$, the chain complex $E_{*0}^1$ is naturally isomorphic to the chain complex $N'\Omega^{m-1}(k)[-1]$, i.e., $N'\Omega^{m-1}(k)$ shifted to the left by one.

**Proof.** Letting $I$ run over all ordered $(m-1)$-tuples of $\{1, \ldots, p-1\}$, and writing $\omega(I)$ for $\omega_{p-1}^m(I)$, we have

$$N_{p-1}^{p-1} \Omega^{m-1}(k) = \bigoplus_I k[T_{p-1}]\omega(I),$$

$$E_{p0}^1 \cong \bigoplus_I k[T_{p-1}]\omega(I) \wedge dt_p.$$

Thinking of $k[T_{p-1}]\omega(I) \wedge dt_p$ as a subgroup of $F_p^p \subseteq \Omega^m(k[T_p])$, we can calculate $d^1$ by applying the differential $d_0: N_p^p \Omega^m(k) \to N_{p-1}^{p-1} \Omega^m(k)$ in the original Moore complex. If $f \in k[T_{p-1}]$, we have

$$d^1(f\omega(I) \wedge dt_p) = -f\omega(I) \wedge \sum_{j=1}^{p-1} dt_j \quad \text{mod}(F_{p-2} + d(F_{p-1}^p))$$

$$= \begin{cases} 
-f(1 - \sum t_j)\omega_{p-2}^m(I) \wedge dt_{p-1} & \text{if } p - 1 \notin I, \\
+ f\omega_{p-2}^m(I - \{p - 1\}) \wedge \sum dt_j \wedge dt_{p-1} & \text{if } p - 1 \in I,
\end{cases}$$
where \( \overline{f} \) denotes \( f(t_1, \ldots, t_{p-2}, 1 - \sum t_j) \in k[T_{p-2}] \). Since this is \((-1)^p\) times the differential of \( f \omega(I) \in N^{p-1} \Omega^{m-1}(k) \), it follows that the map from \( N^{p-1} \Omega^{m-1}(k) \) to \( E_{p0}^1 \) sending \( f \omega(I) \) to \((-1)^p f \omega(I) \wedge dt_p \) is an isomorphism of chain complexes, whence the lemma.

**Corollary 2.4.** If \( I \) is an ideal of a \( k \)-algebra \( A \), the relative Moore complex \( N'HH_m(A, I) \) is split exact for all \( m \). If \( A \to B \) sends \( I \) isomorphically onto an ideal of the \( k \)-algebra \( B \), then the double relative Moore complex \( N'HH_m(A, B, I) \) is also split exact for all \( m \).

**Proof.** As in the proof of Theorem 2.1, the Künneth formula yields

\[
HH_m(A[T], I[T]) \cong \bigoplus_{i+j=m} HH_i(A, I) \otimes HH_j(k[T]).
\]

The complex \( N'HH_m(A, I) \) is therefore the direct sum of the complexes \( HH_i(A, I) \otimes N'HH_j(k) \), each of which is split exact. A similar argument applies in the double relative case.

### 3. Cyclic homology

We can now apply the results for Hochschild homology to get similar, but weaker, results for cyclic homology. For example, the complex \( N'HC_0(A) \) is exact for every \( k \)-algebra \( A \) because \( HC_0 = HH_0 \). On the other hand, since \( HC_2(k) \cong HC_2(k[t]) \cong k \), it is easy to see that \( N'HC_2(k) \) is not exact.

Recall from [LQ] that there is a double complex \( B(A) \) for which \( HC_*(A) \) is the homology of \( \text{Tot}(B(A)) \). We can now construct the functors \( HC_m \) as in §1.

**Theorem 3.1.** For every \( k \)-algebra \( A \),

(i) The Moore complex \( N'HC_0(A) \) is exact.

(ii) The Moore complex

\[
0 \to HC_1(A) \to NHC_1(A) \to N^2HC_1(A) \to N^3HC_1(A)
\]

is exact at \( HC_1(A) \) and \( NHC_1(A) \), but is not always exact at \( N^2HC_1(A) \).

(iii) \( HC_m(A) = 0 \) for every \( m \).

**Proof.** As in [LQ, 1.6], one has an exact sequence of simplicial complexes

\[
0 \to C(\Delta A) \to B(\Delta A) \to B(\Delta A)[-2] \to 0.
\]

This gives a long exact sequence, part of which is

\[
HH_m^h(A) \to HC_m^h(A) \to HC_{m-2}^h(A) \to HH_{m-1}^h(A).
\]

By Theorem 2.1, this makes the groups \( HC_m^h(A) \) periodic in \( m \). However, by construction we have \( HC_m^h(A) = 0 \) for \( m < 0 \), proving (iii).
To prove ii), we use the spectral sequence
\[ E_1^{pq} = N^pHC_q(A) \Rightarrow HC_{p+q}^h(A) = 0. \]
Now \( HC_q(A) = 0 \) for \( q < 0 \), and the row \( q = 0 \) is exact by i). The row \( q = 1 \) is the Moore complex of ii), and we have \( E_0^{11} = E_1^{11} = 0 \), \( E_2^{11} \cong E_0^{2} = [HC_2]A \).
This proves ii) since, for example, \([HC_2]k = HC_2(k) \cong k\).

Remark 3.2. When \( Q \subseteq A \), it follows from [G, II.4.6] that
\[ 0 \to N^pHC_{m-1}(A) \to N^pHH_m(A) \to N^pHC_m(A) \to 0 \]
is exact for \( p \geq 1 \). Using 3.1(i), we see that the Moore complex \( N'HC_1(A) \) is exact except at \( N^2HC_1(A) \). Moreover,
\[ HC_0(A) \xrightarrow{S} HC_2(A) \leftarrow NHC_2(A) \leftarrow N^2HC_2(A) \]
is exact, \([HC_m]A\) is the image of \( S: HC_m(A) \to HC_{m-2}(A) \), and \( N'HC_m(A) \) is exact at \( N^pHC_m(A) \) for all \( p > m \).

Remark 3.3. The proof of Theorem 3.1 makes it clear that, if one cites Corollary 2.4 instead of 2.1, both 3.1 and 3.2 remain valid with \( HC_m(A) \) replaced by \( HC_m(A, I) \) or even \( HC_m(A, B, I) \). We content ourselves with an application to double relative \( K \)-theory that we shall need in [GW2].

Corollary 3.4. Let \( A \to B \) be a ring map, and \( I \) an ideal of \( A \) mapped isomorphically onto an ideal of \( B \). Then the following are exact sequences:

(i) the entire Moore complex for \( K_1(A, B, I) \):
\[ 0 \to K_1(A, B, I) \leftarrow NK_1(A, B, I) \leftarrow \cdots \leftarrow N^pK_1(A, B, I) \leftarrow \cdots. \]

(ii) The beginning of the Moore complex for \( K_2(A, B, I) \):
\[ 0 \to K_2(A, B, I) \leftarrow NK_2(A, B, I) \leftarrow N^2K_2(A, B, I). \]

Proof. The calculations of [GW1] show that \( K_1(A, B, I) \) is naturally isomorphic to \( HC_0(A, B, I) \), so i) follows from 3.3. Using the Gersten-Anderson spectral sequence [W3,3.5]
\[ E_1^{pq} = N^pK_q(A, B, I) \Rightarrow 0 \quad (p \geq 0, q \geq 1). \]
We see by i) that \( E_2^{pq} = 0 \) for all \( p \). Therefore \( E_2^{02} = E_2^{12} = 0 \), proving ii).

4. Periodic cyclic homology
In a similar spirit, one can study the theories \( HC_\cdot(A) \) and \( HP_\cdot(A) \) of [G2]. These functors are the homology groups of natural double chain complexes \( B_\cdot(A) \) and \( BP(A) \), respectively.
Theorem 4.1. Let $A$ be a $k$-algebra. Then for all $m$,

(i) $(HC^{-})_{m}^{h}(A) \cong HP_{m}^{h}(A)$

(ii) There is a natural periodicity isomorphism

$$S: HP_{m}^{h}(A) \rightarrow HP_{m-2}^{h}(A)$$

(iii) If $Q \subset A$, then $HP_{m}(A) \cong HP_{m}(A)$, and thus

$$(HC^{-})_{m}^{h}(A) \cong HP_{m}(A).$$

Proof. As in [G2], there is a short exact sequence

$$0 \rightarrow \text{Tot} B^{-}(A) \rightarrow \text{Tot} BP(A) \rightarrow \text{Tot} B(A)[-2] \rightarrow 0.$$ 

Replacing $A$ with $\Delta A$, we obtain a short exact sequence of simplicial chain complexes, and the corresponding long exact homology sequence is

$$\cdots HC_{m-1}^{h}(A) \rightarrow (HC^{-})_{m}^{h}(A) \rightarrow (HP)_{m}^{h}(A) \rightarrow HC_{m-2}^{h}(A) \cdots.$$ 

Thus i) follows from 3.1. The natural shift isomorphism $S: BP(A) \rightarrow B(P)(A)[-2]$ induces the shift isomorphism of ii). Finally, if $Q \subset A$ then $HP_{m}(A) \cong HP_{m}(A[t])$ for all $m$ by [K, 4.3]; iii) follows from Lemma 1.2 (cf. [W2, 5.3]).

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References


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