CONVOLUTION PROPERTIES OF
A CLASS OF STARLIKE FUNCTIONS

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Abstract. Let $R$ denote the class of functions $f(z) = z + a_2 z^2 + \cdots$ that are analytic in the unit disc $E = \{z : |z| < 1\}$ and satisfy the condition $\Re(f'(z) + z f''(z)) > 0$, $z \in E$. It is known that $R$ is a subclass of $S_f$, the class of univalent starlike functions in $E$. In the present paper, among other things, we prove (i) for every $n \geq 1$, the $n$th partial sum of $f \in R$, $s_n(z, f)$, is univalent in $E$, (ii) $R$ is closed with respect to Hadamard convolution, and (iii) the Hadamard convolution of any two members of $R$ is a convex function in $E$.

1. Introduction

Let $A$ denote the class of functions $f$ that are regular in the unit disc $E = \{z : |z| < 1\}$ and satisfy the conditions $f(0) = f'(0) - 1 = 0$. We denote by $S$ the subclass of $A$ consisting of univalent functions and by $K$, $S_f$, and $C$ the usual subclasses of $S$ whose members are convex, starlike (w.r.t. the origin) and close-to-convex, respectively. Finally, denote by $R$ the family of functions $f \in A$ which satisfy the condition $\Re(f'(z) + z f''(z)) > 0$, $z \in E$. Chichra [1] proved that if $f \in R$, then $\Re f'(z) > 0$, $z \in E$, and hence $f$ is univalent in $E$. R. Singh and S. Singh [8] showed that if $f \in R$ then $f$ is also starlike in $E$.

In the present paper we improve Chichra's result and show that the assertion of Singh and Singh holds under a much weaker hypothesis. We also prove that for every integer $n \geq 1$, the $n$th partial sum of $f \in R$, $s_n(z, f)$, is close-to-convex in $E$. Finally, we prove that $R$ is closed with respect to Hadamard convolution and that if $f, g \in R$, then their Hadamard convolution is convex in $E$. The significance of the last two results will be made clear later on at the appropriate place.

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2. Preliminaries

We shall need the following definitions and results. If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) are analytic in \( |z| < \rho \), then their Hadamard product/convolution, \( f * g \), is the function defined by the power series

\[
(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.
\]

The function \( f * g \) is also analytic in \( |z| < \rho \).

A sequence \( \{c_n\}_0^\infty \) of non-negative numbers is said to be a convex null sequence if \( c_n \to 0 \) as \( n \to \infty \) and

\[
c_0 - c_1 \geq c_1 - c_2 \geq \cdots \geq c_n - c_{n+1} \geq 0.
\]

If \( f \) is analytic in \( |z| < \rho \), \( g \) is analytic and univalent in \( |z| < \rho \) and \( f(0) = g(0) \), then we say that \( f \) is subordinate to \( g \) in \( |z| < \rho \), in symbols, \( f \prec g \) in \( |z| < \rho \), if \( f(\mathbb{D}) \subseteq g(\mathbb{D}) \).

Lemma 1. Let \( \{c_n\}_0^\infty \) be a convex null sequence. Then the function

\[
q(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n
\]

is analytic in \( \mathbb{D} \) and \( \text{Re} q(z) > 0 \), \( z \in \mathbb{D} \).

Lemma 2. Suppose that \( w \) is a nonconstant function analytic in \( |z| < \rho \) and \( w(0) = 0 \). Then, if \( |w(z)| \) attains its maximum value on the circle \( |z| = r < \rho \) at the point \( z_0 \), we can write \( z_0 w'(z_0) = kw(z_0) \), where \( k \geq 1 \).

Lemma 3. For \( 0 \leq \theta \leq \pi \),

\[
\frac{1}{2} + \sum_{k=1}^{n} \frac{\cos k\theta}{k+1} \geq 0.
\]

Lemma 4. If \( P(z) \) is analytic in \( \mathbb{D} \), \( P(0) = 1 \), and \( \text{Re} P(z) > \frac{1}{2} \), \( z \in \mathbb{D} \), then for any function \( F \), analytic in \( \mathbb{D} \), the function \( P * F \) takes values in the convex hull of the image of \( \mathbb{D} \) under \( F \).

Lemmas 1, 2, and 3 are due to Fejér [2], Jack [3], and Rogosinski and Szegö [6], respectively. The assertion of Lemma 4 readily follows by using the Herglotz’ representation for \( P(z) \).

3. Theorems and their proofs

Theorem 1. Let \( f \in R \). Then we have

(a) \( \text{Re} f'(z) > -1 + 2 \log 2 = 0 \cdot 39 \ldots \), \( z \in \mathbb{D} \). The constant \( -1 + 2 \log 2 \) cannot be replaced by any larger one.

(b) \( \text{Re} \frac{f'(z)}{z} > \frac{1}{2} \), \( z \in \mathbb{D} \).

(c) For every \( n \geq 1 \), the \( n \)-th partial sum of \( f \), \( s_n(z,f) \), satisfies \( \text{Re} s_n'(z,f) > 0 \), \( z \in \mathbb{D} \), and hence \( s_n(z,f) \) is univalent in \( \mathbb{D} \).
(d) For every $n \geq 1$

$$\text{Re} \frac{s_n(z, f)}{z} > \frac{1}{3} \quad (z \in E).$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Since $\text{Re}(f'(z) + z f''(z)) > 0, \ z \in E$, we have

$$\text{Re} \left[ 1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1} \right] > 0 \quad (z \in E),$$

and hence

$$\text{Re} \left[ 1 + \frac{1}{2} \sum_{n=2}^{\infty} n^2 a_n z^{n-1} \right] > \frac{1}{2} \quad (z \in E).$$

Consider the function

$$P(z) = 1 + 2 \sum_{n=2}^{\infty} \frac{1}{n} z^{n-1}.$$ 

Clearly $P(z)$ is analytic in $E$, $P(0) = 1$ and

$$\text{Re} P(z) = \text{Re} \left( -1 + 2 \log 2 \right) \quad (z \in E).$$

Now, since we can write

$$f'(z) = \left[ 1 + \frac{1}{2} \sum_{n=2}^{\infty} n^2 a_n z^{n-1} \right] \ast \left[ 1 + \sum_{n=2}^{\infty} \frac{1}{n} z^{n-1} \right],$$

it follows, in view of (2), (3) and Lemma 4, that $\text{Re} f'(z) > -1 + 2 \log 2, \ z \in E$. That the constant $-1 + 2 \log 2$ cannot be replaced by any larger one follows from the fact that the function $f_0$ defined by $z f'_0(z) = -z - 2 \log(1 - z)$ is in the class $R$.

(b) We observe that since the sequence $\{c_n\}_{n=0}^{\infty}$ defined by $c_0 = 1$, $c_n = 2/(n + 1)^2, \ n \geq 1$, is a convex null sequence, we have, in view of Lemma 1.

$$\text{Re} \left[ 1 + 2 \sum_{n=2}^{\infty} \frac{1}{n^2} z^{n-1} \right] > \frac{1}{2} \quad (z \in E).$$

Writing $f(z)/z$ as

$$\frac{f(z)}{z} = \left[ 1 + \frac{1}{2} \sum_{n=2}^{\infty} n^2 a_n z^{n-1} \right] \ast \left[ 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} z^{n-1} \right]$$

and making use of (2), (4) and Lemma 4, we conclude that $\text{Re}(f(z)/z) > \frac{1}{2}, \ z \in E$.

(c) We can write

$$s_n'(z, f) = \left[ 1 + \sum_{k=2}^{\infty} k^2 a_k z^{k-1} \right] \ast \left[ 1 + \sum_{k=2}^{n} \frac{1}{k} z^{k-1} \right].$$
Putting \( z = re^{i\theta} \), \( 0 \leq r < 1 \), \( 0 \leq |\theta| \leq \pi \), and making use of the minimum principle for harmonic functions along with Lemma 3, we obtain

\[
\text{Re} \left[ 1 + \sum_{k=2}^{n} \frac{1}{k^2} z^{k-1} \right] = \text{Re} \left[ 1 + \sum_{k=1}^{n-1} \frac{z^k}{k+1} \right] \\
= 1 + \sum_{k=1}^{n-1} \frac{r^k \cos k\theta}{(k+1)} \quad (0 \leq \theta \leq \pi)
\]

(6)

In view of (1), (6), (5) and Lemma 4, we deduce that \( \text{Re} s'_n(z,f) > 0 \), \( z \in E \), and so \( s_n(z,f) \) is close-to-convex in \( E \) for every \( n \geq 1 \).

(d) Let

\[
q(z) = 1 + \sum_{k=2}^{n} \frac{1}{k^2} z^{k-1}.
\]

Then by (6), we have \( \text{Re} q(z) > \frac{1}{2} \) in \( E \). An application of Lemma 2 readily provides that

\[
\text{Re} S(z) = \text{Re} \left[ \frac{1}{z} \int_0^z q(t) \, dt \right] \\
= \text{Re} \left[ 1 + \sum_{k=2}^{n} \frac{1}{k^2} z^{k-1} \right] > \frac{2}{3} \quad (z \in E).
\]

Writing \( s_n(z,f)/z \) as

\[
\frac{s_n(z,f)}{z} = \left[ 1 + \sum_{k=2}^{\infty} k^2 a_k z^{k-1} \right] * \left[ 1 + 2 \sum_{k=2}^{n} \frac{1}{k^2} z^{k-1} \right] \\
= \left[ 1 + \sum_{k=2}^{\infty} k^2 a_k z^{k-1} \right] * \left[ 1 + 2 \left( 1 + \sum_{k=2}^{n} \frac{1}{k^2} z^{k-1} \right) \right],
\]

and making use of (2), (7) and Lemma 4, the conclusion (d) follows at once.

Remark 1. It is clear that \( f \in R \) if and only if \( \text{Re} g'(z) > 0 \), \( z \in E \), where \( g(z) = zf'(z) \). From this it follows that if \( f \in R \), then \( f \) has the integral representation

\[
f'(z) = -1 - \int_{|x|=1} \frac{2}{xz} \log(1 - xz) \, d\mu(x),
\]

where \( \mu \) is a probability measure on \( |x| = 1 \).

Since \( \text{Re}[(-2/xz) \log(1 - xz)] > 2 \log 2[5] \), part (a) also follows from (8).

If \( f \in R \), then as seen in Theorem 1, part (c), each \( s_n(z,f) \) is univalent in \( E \). It is, therefore, natural to ask for the largest number \( \lambda_n \), \( 0 < \lambda_n < 1 \),
such that $\lambda_n s_n(z, f) < s_{n+1}(z, f)$, $z \in E$. Using part (d) of the above theorem along with the fact that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R$, then $|a_n| \leq 2/n^2$, $n \geq 2$, we readily obtain the following result which provides a lower bound for $\lambda_n$.

**Corollary 1.** Let $f \in R$. Then

(i) $\frac{1}{2} z = \frac{1}{2} s_1(z, f) < s_2(z, f)$, $z \in E$, 

(ii) $\frac{(n+1)^2-6}{(n+1)^2} s_n(z, f) < s_{n+1}(z, f)$, $z \in E$, $n \geq 2$.

The constant $\frac{1}{2}$ in (i) is the best possible one.

Our next result shows that the assertion of R. Singh and S. Singh mentioned in the Introduction holds under a much weaker hypothesis.

**Theorem 2.** If $f \in A$ and

$$\text{Re}(f'(z) + zf''(z)) > -\frac{1}{4} \quad (z \in E),$$

then $f \in S_r$.

**Proof.** Letting $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, it follows from the hypothesis of the theorem that

$$\text{Re} \left[ 1 + \frac{2}{5} \sum_{k=2}^{\infty} k^2 a_k z^{k-1} \right] > \frac{1}{2}, \quad z \in E. \quad (9)$$

Also one can easily see that the sequence $\{c_n\}_{n=0}^{\infty}$, where $c_0 = 1$ and $c_n = 5/(n+1)^2$, $n \geq 1$, is a convex null sequence and as such

$$\text{Re} \left[ 1 + \frac{5}{2} \sum_{k=2}^{\infty} \frac{1}{k^2} z^{k-1} \right] > \frac{1}{2} \quad (z \in E). \quad (10)$$

From (9) and (10) and Lemma 4 we deduce that

$$\text{Re} \left( \frac{f(z)}{z} \right) = \text{Re} \left[ \left( 1 + \frac{2}{5} \sum_{k=2}^{\infty} k^2 a_k z^{k-1} \right) \ast \left( 1 + \frac{5}{2} \sum_{k=2}^{\infty} \frac{1}{k^2} z^{k-1} \right) \right]$$

$$> \frac{1}{2}, \quad z \in E. \quad (11)$$

An application of Lemma 2 readily yields that if $f$ satisfies the hypothesis of Theorem 2, then $\text{Re} f'(z) > 0$, $z \in E$, and hence $f$ is univalent in $E$. Define a function $w$ in $E$ by

$$\frac{zf'(z)}{f(z)} = \frac{1 + w(z)}{1 - w(z)} \quad (12)$$

Clearly $w$ so defined is meromorphic in $E$, $w(0) = 0$ and since $f$ is univalent in $E$, we have $w(z) \neq 1$ in $E$. From (12) we obtain

$$f'(z) + zf''(z) = \left( \frac{f(z)}{z} \right) \left[ \left( \frac{1 + w(z)}{1 - w(z)} \right)^2 + \frac{2zw'(z)}{(1 - w(z))^2} \right]. \quad (13)$$
We claim that \(|w(z)| < 1\) in \(E\). If possible, suppose that there exists a point \(z_0 \in E\) such that \(\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1\). Then from Lemma 2 it follows that \(zw'(z_0) = kw(z_0)\), where \(k \geq 1\) and \(w(z_0) = e^{i\theta}\), \(0 < \theta < 2\pi\). Putting \(z = z_0\) in (13), we get

\[
\text{Re}[f'(z_0) + z_0f''(z_0)] = \text{Re} \left[ \left( \frac{f(z_0)}{z_0} \right) \left( \frac{1 + e^{i\theta}}{(1 - e^{i\theta})^2} \right) \right]
\]

(14)

since \(k \geq 1\) and, in view of (11), \(\text{Re}(f(z)/z) > \frac{1}{2}\), \(z \in E\). As (14) contradicts our hypothesis, we conclude that \(|w(z)| < 1\) in \(E\). Equation (12) then implies that \(f\) must belong to \(S_I\).

**Corollary 2.** If \(g \in A\) and

\[
\text{Re}[g'(z) + 3zg''(z) + z^2g'''(z)] > -\frac{1}{4} \quad (z \in E),
\]

then \(g \in K\).

It is known [7] that if \(f \in S_I\) and \(g \in K\), then \(f * g \in S_I\) and that if \(f, g \in S_I\), then \(f * g\) need not be in \(S_I\). In the following theorem we prove that if \(f, g \in R\), a subclass of \(S_I\), then so does \(f \ast g\), i.e. \(R\) is closed with respect to Hadamard product.

**Theorem 3.** If \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\) and \(g(z) = z + \sum_{n=2}^{\infty} b_n z^n\) belong to \(R\), then so does their Hadamard product

\[
h(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.
\]

**Proof.** Since \(h(z) = (f * g)(z)\), we have

\[
zh'(z) = z f'(z) * g(z)
\]

and hence

(15) \[h'(z) + zh'''(z) = (f'(z) + zf''(z)) * g(z)/z.\]

Since \(\text{Re}(f'(z) + zf''(z)) > 0\), \(z \in E\), and by Theorem 1, part (b), \(\text{Re}(g(z)/z) > \frac{1}{2}\), \(z \in E\), the desired result follows at once from (15) and Lemma 4.

From the proof of Theorem 3 it is clear that in fact the following more general result holds:

**Theorem 3'.** If \(f \in R\), \(g \in A\) and \(\text{Re}(g(z)/z) > \frac{1}{2}\), \(z \in E\), then \(f \ast g \in R\).

We observe that \(\text{Re}(g(z)/z) > \frac{1}{2}\), \(z \in E\), need not even imply the univalence of \(g\) in \(E\).
Corollary 3. If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R \), then so does
\[
f_k(z) = z + \sum_{n=1}^{\infty} a_{nk+1} z^{nk+1}, \quad k = 1, 2, 3, \ldots.
\]

In the next theorem we prove that if \( f, g \in R \), then \( f * g \in K \). Since the class \( K \) is closed with respect to Hadamard convolution [7], the significance of our result will be apparent only if we show that \( R \) (which has hitherto been shown to be a subset of \( S_t \)) is not contained in \( K \). To prove that \( R \notin K \), denote by \( P' \) the family of functions \( f \in A \) which satisfy the condition \( \text{Re} f'(z) > 0, z \in E \). Krzyż [4] has demonstrated a function \( f_0 \in P' \) such that \( f_0 \notin S_t \) (space does not permit us to carry out the construction of \( f_0 \)). Clearly the function \( f^n \), defined by \( f^n(z) = \int_0^z (f_0(\xi)/\xi) d\xi \), is a member of \( R \) which is not in \( K \), showing that \( R \notin K \).

**Theorem 4.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) are in \( R \), then
\[
h(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \in K.
\]

**Proof.** In view of Corollary 2 it suffices to show that
\[
\text{Re}[h'(z) + 3zh''(z) + z^2h'''(z)] > -\frac{1}{4}, \quad z \in E,
\]
or, equivalently,
\[
(16) \quad \text{Re} \left[ 1 + \sum_{n=2}^{\infty} n^3 a_n b_n z^{n-1} \right] > -\frac{1}{4}, \quad z \in E.
\]

Since \( f, g \in R \), we have
\[
\text{Re} \left[ 1 + \frac{1}{2} \sum_{n=2}^{\infty} n^2 a_n z^{n-1} \right] > \frac{1}{2}, \quad z \in E,
\]
and
\[
\text{Re} \left[ 1 + \frac{1}{2} \sum_{n=2}^{\infty} n^2 b_n z^{n-1} \right] > \frac{1}{2}, \quad z \in E.
\]

Therefore, in view of Lemma 4, it follows that
\[
(17) \quad \text{Re} \left[ 1 + \frac{1}{4} \sum_{n=2}^{\infty} n a_n b_n z^{n-1} \right] > \frac{1}{2}, \quad z \in E.
\]

Now, we can write
\[
(18) \quad \left[ 1 + \sum_{n=2}^{\infty} n^3 a_n b_n z^{n-1} \right] = \left[ 1 + \frac{1}{4} \sum_{n=2}^{\infty} n^4 a_n b_n z^{n-1} \right] \ast \left[ 1 + 4 \sum_{n=2}^{\infty} \frac{1}{n} z^{n-1} \right].
\]
Since

\[
\text{Re}\left[1 + 4 \sum_{n=2}^{\infty} \frac{1}{n} z^{n-1}\right] = \text{Re}\left[-3 - \frac{4}{z} \log(1 - z)\right]
\]

(19)

\[
> -3 + 4 \log 2[5]
\]

\[
\Rightarrow -0.231 > -\frac{1}{4} \quad (z \in E),
\]

it follows from (17), (18), (19) and Lemma 4 that (16) holds for all \( z \in E \).

The proof of Theorem 4 is, therefore, complete.

**References**


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