PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 106, Number 2, June 1989

SPECTRAL PROPERTIES FOR OPERATORS IN A LIE ALGEBRA

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(Communicated by Paul S. Muhly)

ABSTRACT. Given an exponentiable Lie algebra $\mathscr{L}$ of operators on a Hilbert space $\mathscr{H}$, we study the spectrum of those self-adjoint, non-adnilpotent operators $-iA$, with A in $\mathscr{L}$, for a certain class of solvable Lie algebras $\mathscr{L}$.

1. INTRODUCTION

The results of this note are motivated by the study of both contractive representations, and by virtual representations, π of a Lie group G with a Lie algebra $\mathfrak{g}$. In the results of [5, Appendix C], and those of [6, Theorem 3], it was assumed that, for some elements x in the Lie algebra $\mathfrak{g}$, the infinitesimal generator $d\pi(x)$ have a positive spectrum. Even though now it is known that this condition can be removed completely ([2] and [3]), we have seen that, in many of the applications, the hypothesis of positivity is satisfied. In this note we present a certain class of solvable Lie algebras where the positivity of the infinitesimal generators is not satisfied. Moreover, we are able to compute the spectrum. Semiboundedness is shown to be equivalent to a purely algebraic condition.

2. PRELIMINARIES

We assume that $\mathfrak{g}$ is a solvable Lie algebra of dimension greater than or equal to three with a nilradical $\mathscr{n}$ properly contained in $\mathfrak{g}$. We denote by aff($\mathbf{R}$) the non-Abelian two-dimensional Lie algebra spanned by $\{x, y\}$ such that $[x, y] = y$, this is the Lie algebra of the Aff($\mathbf{R}$) group (or the $ax + b$ group). For each real number α different from zero, let $\mathfrak{g}(\alpha)$ be the Lie algebra spanned by $\{x, y, z\}$ such that

$$[x, y] = y - \alpha z,$$
$$[x, y] = \alpha y + z,$$
$$[y, z] = 0.$$

Received by the editors June 17, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 22E27, 22E45, 47A67; Secondary 46N05, 46L45.

0002-9939/89 \$1.00 + \$.25 per page

Then $\mathfrak{g}(\alpha)$ is the Lie algebra of the group $G(\alpha)$ that can be realized as the semidirect product $\mathbb{R} \times \mathbb{C}$. Let $\mathscr{E}(2)$ be the Lie algebra spanned by $\{x, y, z\}$ and with commutation relations

$$[x, y] = -z, \qquad [x, z] = y, \qquad [z, y] = 0.$$

This is the Lie algebra of the Euclidean group $E(2)$.

Lemma 2.1. *Let $\mathfrak{g}$ be a solvable Lie algebra such that all its roots have nonzero real part. Then every non-adnilpotent element x of $\mathfrak{g}$ can be embedded either into the $\mathrm{aff}(\mathbb{R})$ Lie algebra, or there is a real number α different from zero such that x can be embedded into the Lie algebra $\mathfrak{g}(\alpha)$.*

Proof. Let $x \in \mathfrak{g} - \mathfrak{n}$. Then there is a root λ with $\mathrm{Re}(\lambda) \neq 0$ and an element $\xi = y + iz$ in $\mathfrak{g}_{\mathbb{C}}$, ξ different from zero and such that

$$\mathrm{ad}_x(\xi) = \lambda\xi. \tag{1}$$

Let $\mathrm{Re}(\lambda) = a$ and $\mathrm{Im}(\lambda) = b$, then from (1) we get

$$[x, y] = ay - bz, \tag{2}$$

$$[x, z] = by + az. \tag{3}$$

Then from equations (2) and (3) it follows that

$$[x, [y, z]] = 2a[y, z].$$

Hence if $[y, z]$ is different from zero then x is in $\mathrm{aff}(\mathbb{R})$. Otherwise by rescaling equations (2) and (3) we deduce that x is in $\mathfrak{g}(\alpha)$.

Remarks. (1) Note that if all the roots of $\mathfrak{g}$ are real then from equation (2) above we obtain that every non-adnilpotent element x of $\mathfrak{g}$ is in $\mathrm{aff}(\mathbb{R})$. We reach the same conclusion if $\mathfrak{g}$ is a solvable split Lie algebra, since in this case all the roots are real.

(2) We also note that if $\mathfrak{g}$ is solvable of exponential type then the assumptions of Lemma 2.1 are satisfied.

Let us recall that a solvable Lie algebra is said to be of type R if all its roots are purely imaginary.

Lemma 2.2. *Let $\mathfrak{g}$ be a type R Lie algebra with nilradical Abelian. Then every non-adnilpotent element x of $\mathfrak{g}$ can be embedded in $\mathscr{E}(2)$.*

Proof. There is $\xi \in \mathfrak{g}_{\mathbb{C}} - \{0\}$ and a nonzero real number α such that

$$\mathrm{ad}_x(\xi) = i\alpha\xi. \tag{1}$$

Now let $\xi = y + iz$ then equation (1) implies that y and z belong to $\mathfrak{n}$.

Note that if the hypotheses of Lemma 2.2 are satisfied then, for every x in $\mathfrak{g}, x$ not adnilpotent, the one parameter subgroup $\exp(tx)$, $t \in \mathbb{R}$, is compact.

3. Main results

In this section we assume that there is a representation ρ of the Lie algebra $\mathfrak{g}$ in a separable Hilbert space $\mathscr{H}$, and that the operator Lie algebra $\mathscr{L} = \rho(\mathfrak{g})$

exponentiates to a strongly continuous unitary representation π of a connected Lie group whose Lie algebra is $\mathfrak{g}$ (i.e., $\mathscr{L} = d\pi(\mathfrak{g})$). We assume that $\mathscr{L}$ is not nilpotent.

Theorem 3.1. *Let $\mathscr{L}$ be a solvable Lie algebra where all roots have nonzero real part. Then, for every A in $\mathscr{L}$, A not adnilpotent, the spectrum of the self-adjoint operator $-iA$ is the whole real line.*

Proof. Since A is not adnilpotent then there is x in $\mathfrak{g}$, x not in $\mathfrak{n}$, such that $A = d\pi(x)$. Now from Lemma 2.1, x is either in aff($\mathbf{R}$) or in $\mathfrak{g}(\alpha)$. If x is in aff($\mathbf{R}$), let $(\mathscr{H}, U)$ be the restriction of the unitary representation $(\mathscr{H}, \pi)$ to the Aff($\mathbf{R}$) group. Hence $(\mathscr{H}, U)$ decomposes into a direct integral of irreducible components such that

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1,$$

where $\mathscr{H}_0$ decomposes into the one-dimensional representation of the group and $\mathscr{H}_1$ decomposes into the infinite-dimensional representation U^+ and U^-; these representations are defined on $(L^2(0, +\infty), dt/t)$. On the other hand we have that

$$idU^{\pm}(x)(f)(s) = sf(s)$$

for $f \in C_c^{\infty}(0, +\infty)$. Then by a Mellin transform argument it follows that spec$(-idU^{\pm}(x))$ is the interval $(-\infty, +\infty)$. Therefore the direct integral of these operators will have the same spectrum. Now if x belongs to $\mathfrak{g}(\alpha)$ then the preceding argument applies the same. Hence by denoting $(\mathscr{H}, U)$ the restriction of $(\mathscr{H}, \pi)$ to the group $G(\alpha)$ we have that $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$ where $\mathscr{H}_0$ decomposes into one-dimensional pieces and $\mathscr{H}_1$ decomposes into the infinite-dimensional representations $U^s (1 \le s < e^{2\pi/\alpha})$ [7]. Now let us recall that the operators U_g^s act on $L^2(\mathbf{R})$ and if $g = \exp(x) = (p, 0)$, then $U_g^s(f)(t) = f(p+t)$. Therefore the spectrum of $-idU^s(x)$ is $(-\infty, +\infty)$.

It follows from the remarks of §2 that if $\mathfrak{g}$ is a solvable Lie algebra that is either split or has only real roots then we reach to the same conclusions as in Theorem 3.1.

Corollary 3.1. *Let $\mathscr{L}$ be a solvable Lie algebra that satisfies either the hypothesis of Theorem 3.1 or the condition of the above remark. Then if A belongs to $\mathscr{L}$ and if A is such that the spectrum of $-iA$ is semibounded then A is adnilpotent.*

If $\mathfrak{g}$ is a type R Lie algebra, with Abelian nilradical then it follows from Lemma 2.2 that there are elements x, y, z in $\mathfrak{g}$ with x adnilpotent such that $\{x, y, z\}$ generates the Lie algebra $\mathscr{E}(2)$. From the remark that follows Lemma 2.2 the one-parameter group generated by x is in $SO(2)$. This implies the representation $(\mathscr{H}, \pi)$ defines by restriction a unitary representation of $E(2)$. Thus $(\mathscr{H}, \pi)$ when restricted to $E(2)$ decomposes, where the irreducible components are $(L^2(SO(2)), U^s)$, $0 \le s < \infty$. Now when we restrict U^s to $SO(2)$ then this is just the regular representation of $SO(2)$. Hence, by an application of the Peter–Weyl theorem, the spectrum of $-idU^s(x)$ is the set of all integers. Thus we arrive at the following:

Theorem 3.2. *Let $\mathscr{L}$ be a solvable Lie algebra of type R with abelian nilradical. Then for C in $\mathscr{L}$, C not adnilpotent, the spectrum of $-iC$ is* $\{\dots, -2, -1, 0, 1, 2, \dots\}$.

Under the assumptions of Theorem 3.2 we conclude that there are linearly independent operators A, B, C in $\mathscr{L}$ that satisfy the commutation relations of $\mathscr{E}(2)$, in particular $[A, B] = 0$. Now from the representation theory of $E(2)$ it follows that A and B belong to $\mathscr{B} \cap Z(\mathscr{L})$ where $\mathscr{B}$ is the ideal [4] of bounded operators in $\mathscr{L}$. This observation implies the next corollary.

Corollary 3.2. *If the hypotheses of Theorem* 3.2 *are satisfied then the dimension of $\mathscr{B} \cap Z(\mathscr{L})$ is greater than or equal to two.*

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